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On The Alpha Power Transformed Quasi Aradhana Distribution: Properties and Applications

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Abstract: In this study, a new three-parameter lifetime model is proposed with the purpose to obtain more flexible model in terms of hazard rate function. The new model is called the Alpha Power Transformed Quasi Aradhana distribution. The new model has the advantage of being capable modeling various shapes of failure criteria. The distribution has the ability to model both monotone and non-monotone failure rates. It has been proved from the study that the proposed model provides a better fit to the data having monotone as well as non-monotone behavior. Several statistical properties of the model has been derived such as reliability function, hazard function, moments, moment generating function, characteristic function, order statistics, entropies, Bonferroni and Lorenz curves. The parameters of the model are investigated by the method of maximum likelihood estimation technique. The usefulness of the model is demonstrated by using real-life data.

Keywords: Alpha Power Transformation Model, Moments, Moment Generating Function, Parameter Estimation.

1 Introduction

In the past many years, numerous distributions have been suggested and discussed along with their applicability for various disciplines. However, most of these are inappropriate for modeling heavy-tailed data. Consequently, for implementation, it is necessary to have extensive forms of distributions for different fields. This becomes the major interest of researchers to develop new distributions by incorporating parameters into the standard distribution. The newly developed distributions are more flexible in modeling data. The current growth and expansion in distribution theory emphasizes the problem-solving challenges that researchers face and presents several models to better assess and investigate lifetime data in various applicable areas.

The Quasi Aradhana distribution was proposed by Rama Shanker and Ravi Shanker [1]. The statistical properties of the Quasi Aradhana distribution were discussed by the author(s). The usefulness of the model was demonstrated by numerical application and was revealed that the Quasi Aradhana distribution provides a better fit than gamma, Weibull, Lognormal, Aradhana, exponential, Lindley distributions.

A random variable Z is said to follow a Quasi Aradhana distribution with parameters η *and* β, if its probability density function (pdf) is given by

$$f(z;\eta,\beta) = \frac{\eta}{\beta^2+2\beta+2}(\beta+\eta z)^2 e^{-\eta z} \; ; \quad z>0, \beta^2+2\beta+2>0 \tag{1}$$

The cumulative distribution function (cdf) of the Quasi Aradhana distribution is given by

$$F(z;\eta,\beta) = 1-\left(1+\frac{\eta z\,(\eta z+2\eta+2)}{\beta^2+2\beta+2}\right)e^{-\eta z} \; ; \quad z>0,\ \eta>0,\ \beta^2+2\beta+2>0 \tag{2}$$

The Quasi Aradhana distribution is a two-parameter distribution which is a combination of gamma (3,θ) distribution, exponential (θ) distribution and gamma distribution (2,θ). The hazard function of the model is monotonically increasing for the selected values of parameters.

To overcome this situation the generalization of the Quasi Aradhana distribution is required to bring more flexibility to the model. Mahdavi and Kundu [2] proposed the Alpha power transformation method (APT) to incorporate the additional (shape) parameter to the continuous distributions in order to increase their flexibility. According to Mahdavi and Kundu, for a given cumulative distribution function (cdf) and probability

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density function (pdf) of a baseline distribution, defined the cumulative distribution function (cdf) of Alpha power transformation family of distribution as follows

$$F(z;\alpha)=\begin{bmatrix} \dfrac{\alpha^{G(z)}-1}{\alpha-1} & ; \quad \alpha>0\,,\alpha\neq 1 \\ \\ G(z) & ; \quad \alpha=1 \end{bmatrix} \tag{3}$$

Where, $F(z)$ and $f(z)$ are the cdf and pdf of the baseline distribution.

Mahdavi and Kundu applied the APT approach to the exponential distribution to obtain two-parameter exponential distribution and is called the alpha power exponential distribution and also studied its several properties. Based on the APT model, some new distributions are considered. Alpha power Weibull was proposed by Nasaar et al. [3]. The properties of the model are derived and discussed such as reliability function, hazard function, mean residual function, quantiles, moments, moment generating function, entropy, order statistics, stress-strength parameter. The simulation study was performed by using Mathcad to illustrate the behavior of model parameters. The maximum likelihood estimation is established for calculating the parameters of the model. Two real data sets are used to illustrate the importance of the proposed model. Alpha Power Log-Logistic distribution was suggested by Aldahlan [4] for modeling the data set of carbon fibres. The statistical properties of the model are discussed. The parameters of the model are obtained by the maximum likelihood estimation method. The simulation study was also conducted to know the behavior of the parameters of the model. Alpha Power Transformed Frechet distribution was presented and discussed by Nasiru et al. [5]. Some statistical properties of the model are obtained. The Monte Carlo simulation was carried out to examine the finite properties of the maximum likelihood estimators of the model. Alpha power transformed Aradhana distribution was introduced by Maryam and Kannan [6] for modeling real-life data. Some properties of the model were investigated along with their parameter estimates. Alpha power transformed Garima distribution was proposed by Maryam and Kannan [7] with application to real-life data. Some properties of the model were investigated. These include survival function, hazard rate function, moments, moment generating function, characteristic function, entropy, order statistics, mean waiting time, mean residual life, stress-strength reliability. The maximum likelihood estimation and least-square estimation was investigated for the assessment of parameters.

Many more generalizations on the APT method have been proposed by the researchers like APT Power Lindley suggested by Hassan [8], APT extended exponential introduced by Fayomi [9], APT Lindley proposed by Ghosh [10], APT Weibull Frechet presented by Thomas et al. [11], APT Exponentiated Inverse Rayleigh distribution developed by Noor et al. [12], Khalil et al. [13] proposed APT Pareto distribution. Alpha power inverse Weibull distribution was suggested by Basheer [14]. The properties of the model are also examined. These include reliability function, hazard function, reversed hazard function, mean residual life, mean inactivity time, and stress-strength reliability. The parameters of the model were investigated by using maximum likelihood estimation for reliability. The statistical properties of the model were also obtained such as quantile function, moments, moment generating function, renyi entropy, Shannon entropy, stochastic ordering and order statistics. The simulation study was also presented for the model parameters. The real data was used to illustrate Alpha power inverse distribution for comparing with many known distributions such as exponentiated (generalized) inverse Weibull (GIW), Kumaraswamy inverse Weibull (KIW) and inverse Weibull (IW) distributions.

Alpha Power Inverted Topp Leone distribution was obtained by Amal et al. [15]. The parameters of the model were investigated by using Bayesian and non-Bayesian methods. The maximum likelihood, weighted least-square, maximum product of spacing estimators for the parameters have been obtained. It has been observed that the Bayesian estimates provide more accurate results for the model parameters. The performance of the model was demonstrated by using the data sets related to reliability, engineering and medicine. All these researchers discussed useful characteristics of the distributions and demonstrated their performance using real-life data sets. Their findings revealed that the models outperformed the baseline distributions.

Following the idea of Mahdavi and Kundu, we introduce a new three parameter model which is a more flexible model and is called the Alpha power transformed Quasi Aradhana distribution. The aim of the new model is that the addition of shape parameter can give several desirable properties and more flexibility in the form of density and hazard function. The improvement in the flexibility of the existing distributions by using APT-G, we hope that the models will provide better fits than the competing modified models.

The rest of the paper is organized as follows. In section 2, the new model is proposed called alpha power transformed Quasi Aradhana (APTQA) distribution along with reliability function, hazard function,. In section 3, various statistical properties of the APTQA distribution are derived along with moments, moment generating function. In section 4, order statistics of the APTQA distribution are also obtained. Entropies, Bonferroni and Lorenz curves are also derived and discussed in sections 5 and 6. In section 7, the method of maximum likelihood technique is employed for the parameter estimation. The usefulness and applicability of the APTQA distribution are discussed by the illustration provided in section 8. Finally, the study is concluded in section 9.

2 The Proposed Model

A random variable Z is said to have a three parameter Alpha Power Transformed Quasi Aradhana (APTQA) distribution with parameters α, β and η if its probability density function (pdf) is of the form

$$f(z;\alpha,\beta,\eta)=\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\eta}{\beta^2+2\beta+2}\right)(\beta+\eta z)^2 \alpha^{1-\left(1+\frac{\eta z(\eta z+2\beta+2)}{\beta^2+2\beta+2}\right)e^{-\eta z}} e^{-\eta z};\ z>0,\alpha\neq 1,\beta^2+2\beta+2>0 \quad (4)$$

The cumulative distribution function (cdf) of the APTQA model is given by

$$F(z;\alpha,\beta,\eta)=\frac{\alpha^{1-\left(1+\frac{\eta z(\eta z+2\beta+2)}{\beta^2+2\beta+2}\right)e^{-\eta z}}-1}{\alpha-1}; \quad (5)$$

$$\alpha\neq 1,\ \alpha,\ \beta,\ \eta>0,\ \beta^2+2\beta+2>0$$

The reliability function $S(z)$ of the APTQA distribution is given by

$$S(z;\alpha,\beta,\eta)=\frac{\alpha-\alpha^{1-\left(1+\frac{\eta z(\eta z+2\beta+2)}{\beta^2+2\beta+2}\right)e^{-\eta z}}}{\alpha-1};\ \alpha\neq 1,\ \alpha,\ \beta,$$

$$\eta>0,\ \beta^2+2\beta+2>0 \quad (6)$$

The hazard function $h(z)$ of the APTQA distribution is defined by

$$h(z;\alpha,\beta,\eta)=\frac{\log\alpha\,\eta\,(\beta+\eta z)^2\alpha^{1-\left(1+\frac{\eta z(\eta z+2\beta+2)}{\beta^2+2\beta+2}\right)e^{-\eta z}}}{(\beta^2+2\beta+2)\left(\alpha-\alpha^{1-\left(1+\frac{\eta z(\eta z+2\beta+2)}{\beta^2+2\beta+2}\right)e^{-\eta z}}\right)}; \quad (7)$$

$$\alpha\neq 1,\alpha,\beta,\ \eta>0,\beta^2+2\beta+2>0$$

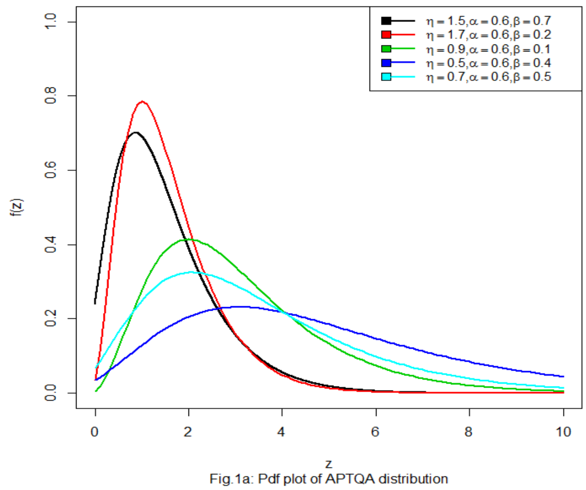

Fig. 1a: pdf plot of APTQA distribution

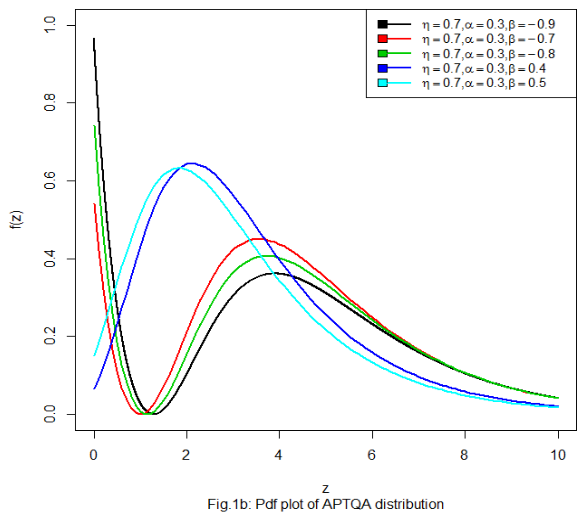

Fig. 1b: pdf plot of APTQA distribution

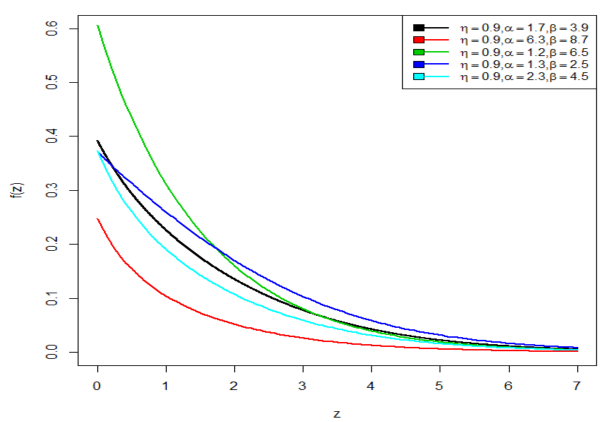

Fig. 1c: pdf plot of A PTQA distribution

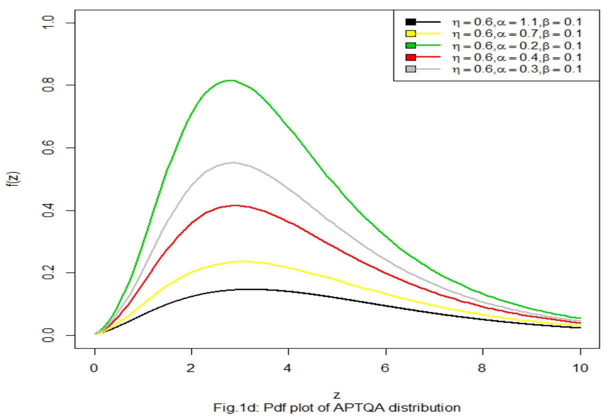

Fig. 1d: pdf plot of APTQA distribution

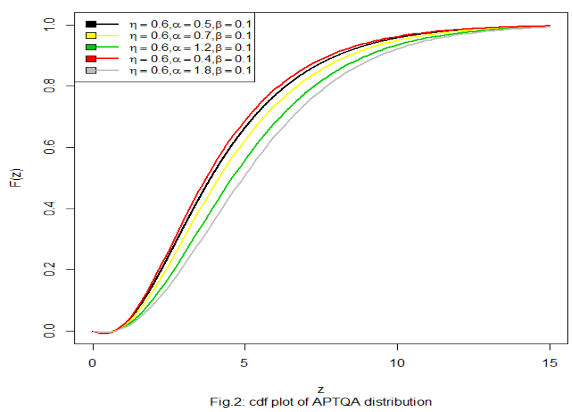

Fig. 2: cdf plot of APTQA distribution

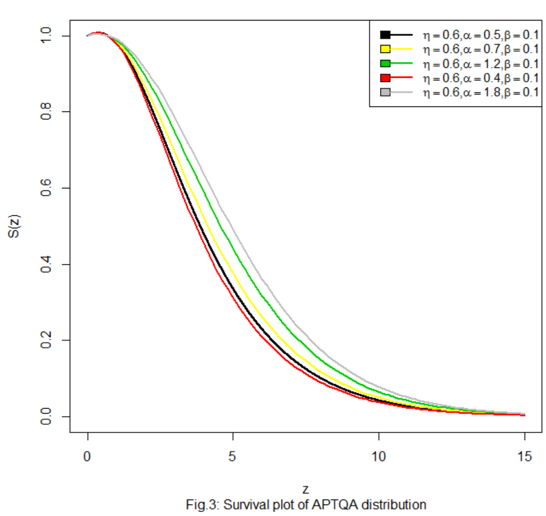

Fig. 3: Survival plot of APTQA distribution

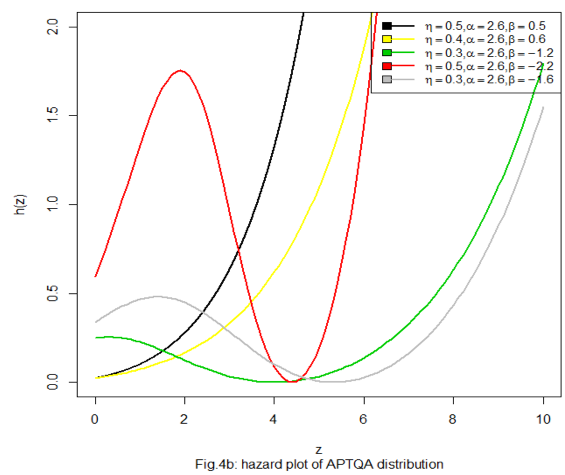

Fig. 4: hazard plot of APTQA distribution

The pdf plot of the APTQA distribution is decreasing, if $\alpha > 1,\ \beta > 1$ and $\eta < 1$ and it is also increasing if $\alpha > 1,\ \beta > 1$ and $\eta > 1$. The model shows decreasing-increasing-decreasing probability density function when, $\alpha < 1,\ \beta < 1$ and $\eta < 1$. The plots display various shapes for the hazard function including, J-shaped, increasing-decreasing-increasing for different combinations of the parametric values.

3 Statistical Properties

In this section, some properties of the APTQA distribution, including moments, moment generating function, characteristic function, order statistics, Entropies, Bonferroni and Lorenz curves are obtained.

3.1 Moments

The n^{th} order moment of a random variable Z having APTQA distribution is defined by

$$E(Z^n) = \mu'_n = \int_0^\infty z^n f(z)\,dz$$

Substituting equation (4) in the above expression, then we have

$$\mu'_n = \int_0^\infty \left(\begin{array}{c} z^n \left(\dfrac{\log\alpha}{\alpha-1}\right)\left(\dfrac{\eta}{\beta^2+2\beta+2}\right)(\beta+\eta z)^2 \\ \alpha^{1-\left(1+\frac{\eta z(\eta z+2\beta+2)}{\beta^2+2\beta+2}\right)e^{-\eta z}} e^{-\eta z} \end{array} \right) dz \qquad (8)$$

The equation (8) can be re-written as

$$\mu_n' = \left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\eta\ \alpha}{\beta^2+2\beta+2}\right)$$

$$\int_0^\infty z^n\left(\beta+\eta z\right)^2 \alpha^{-\left(1+\frac{\eta z(\eta z+2\beta+2)}{\beta^2+2\beta+2}\right)e^{-\eta z}} e^{-\eta z}\ dz$$

$$\alpha^{-t} = \sum_{i=0}^{\infty}\frac{(-\log\alpha)^i}{i\ !}t^i \tag{9}$$

Using (9) to equation (8), equation (8) will be

$$\mu_n' = \left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\eta\ \alpha}{\beta^2+2\beta+2}\right)\sum_{i=0}^{\infty}\frac{(-\log\alpha)^i}{i\ !}$$

$$\int_0^\infty z^n\left(\beta+\eta z\right)^2\left(1+\frac{\eta z(\eta z+2\beta+2)}{\beta^2+2\beta+2}\right)^i e^{-\eta i z}e^{-\eta z}\ dz \tag{10}$$

Using the binomial expansion to (10), we get

$$\mu_n' = \left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\eta\ \alpha}{\beta^2+2\beta+2}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{i}{j}\frac{(-\log\alpha)^i}{i\ !}$$

$$\int_0^\infty z^n\left(\beta+\eta z\right)^2\left(\frac{\eta z(\eta z+2\beta+2)}{\beta^2+2\beta+2}\right)^j e^{-\eta(1+i)z}\ dz$$

$$\mu_n' = \left(\frac{\alpha\log\alpha}{\alpha-1}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{i}{j}\frac{(-\log\alpha)^i}{i\ !}\left(\frac{\eta}{\beta^2+2\beta+2}\right)^{j+1}$$

$$\int_0^\infty z^{n+j}\left(\beta+\eta z\right)^2\left(\eta z+2\beta+2\right)^j e^{-\eta(1+i)z}\ dz$$

$$\mu_n' = \left(\frac{\alpha\log\alpha}{\alpha-1}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{i}{j}\frac{(-\log\alpha)^i}{i\ !}\left(\frac{\eta}{\beta^2+2\beta+2}\right)^{j+1}\eta^j$$

$$\int_0^\infty z^{n+2j}\left(\beta+\eta z\right)^2\left(1+\frac{2\beta}{\eta z}+\frac{2}{\eta z}\right)^j e^{-\eta(1+i)z}\ dz$$

$$\mu_n' = \left(\frac{\alpha\log\alpha}{\alpha-1}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\binom{i}{j}\binom{j}{k}\frac{(-\log\alpha)^i}{i\ !}\left(\frac{\eta}{\beta^2+2\beta+2}\right)^{j+1}$$

$$\eta^j\int_0^\infty z^{n+2j}\left(\beta+\eta z\right)^2\left(\frac{2\beta}{\eta z}+\frac{2}{\eta z}\right)^k e^{-\eta(1+i)z}\ dz$$

$$\mu_n' = \left(\frac{\alpha\log\alpha}{\alpha-1}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\binom{i}{j}\binom{j}{k}\frac{(-\log\alpha)^i}{i\ !}\left(\frac{\eta}{\beta^2+2\beta+2}\right)^{j+1}$$

$$\eta^{j-k}2^k\beta^k\int_0^\infty z^{n+2j-k}\left(\beta+\eta z\right)^2\left(1+\frac{1}{\beta}\right)^k e^{-\eta(1+i)z}\ dz$$

$$\mu_n' = \left(\frac{\alpha\log\alpha}{\alpha-1}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{i}{j}\binom{j}{k}\binom{k}{l}\frac{(-\log\alpha)^i}{i\ !}\left(\frac{\eta}{\beta^2+2\beta+2}\right)^{j+1}$$

$$\eta^{j-k}2^k\beta^k\int_0^\infty z^{n+2j-k}\left(\beta+\eta z\right)^2\left(\frac{1}{\beta}\right)^l e^{-\eta(1+i)z}\ dz$$

$$\mu_n' = \left(\frac{\alpha\log\alpha}{\alpha-1}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{i}{j}\binom{j}{k}\binom{k}{l}\frac{(-\log\alpha)^i}{i\ !}\left(\frac{\eta}{\beta^2+2\beta+2}\right)^{j+1}$$

$$\eta^{j-k}2^k\beta^{k-l}\int_0^\infty z^{n+2j-k}\left(\beta^2+\eta^2z^2+2\eta\beta z\right)e^{-\eta(1+i)z}\ dz$$

$$\mu_n' = \left(\frac{\alpha\log\alpha}{\alpha-1}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{i}{j}\binom{j}{k}\binom{k}{l}\frac{(-\log\alpha)^i}{i\ !}$$

$$\left(\frac{\eta}{\beta^2+2\beta+2}\right)^{j+1}\eta^{j-k}2^k\beta^{k-l}$$

$$\left(\begin{array}{l}\beta^2\int_0^\infty z^{n+2j-k}e^{-\eta(1+i)z}\ dz+\eta^2\int_0^\infty z^{n+2j-k+2}e^{-\eta(1+i)z}\ dz\\ +2\eta\beta\int_0^\infty z^{n+2j-k+1}\ e^{-\eta(1+i)z}\ dz\end{array}\right)$$

$$\mu_n' = \left(\frac{\alpha\log\alpha}{\alpha-1}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{i}{j}\binom{j}{k}\binom{k}{l}\frac{(-\log\alpha)^i}{i\ !}$$

$$\left(\frac{\eta}{\beta^2+2\beta+2}\right)^{j+1}\eta^{j-k}2^k\beta^{k-l}$$

$$\left(\begin{array}{l}\beta^2\int_0^\infty z^{(n+2j-k+1)-1}e^{-\eta(1+i)z}dz+\eta^2\int_0^\infty z^{(n+2j-k+3)-1}e^{-\eta(1+i)z}dz\\ +2\eta\beta\int_0^\infty z^{(n+2j-k+2)-1}e^{-\eta(1+i)z}dz\end{array}\right)$$

The n^{th} order moment of the APTQA distribution is given as

$$\mu_n' = \left(\frac{\alpha\log\alpha}{\alpha-1}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{i}{j}\binom{j}{k}\binom{k}{l}\frac{(-\log\alpha)^i}{i\,!}$$

$$\left(\frac{\eta}{\beta^2+2\beta+2}\right)^{j+1}\eta^{j-k}2^k\beta^{k-l}$$

$$\frac{\beta^2\,\Gamma(n+2j-k+1)}{(\eta\,(1+i))^{(n+2j-k+1)}}+\frac{\eta^2\,\Gamma(n+2j-k+3)}{(\eta\,(1+i))^{(n+2j-k+3)}}$$

$$+\frac{2\eta\,\beta\,\Gamma(n+2j-k+2)}{(\eta\,(1+i))^{(n+2j-k+2)}} \tag{11}$$

Put, $n = 1,2,3$ in equation (11), we obtain the first moment or mean, second moment, third moment, so on of the proposed distribution.

The mean of the distribution is

$$\mu_1' = \left(\frac{\alpha\log\alpha}{\alpha-1}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{i}{j}\binom{j}{k}\binom{k}{l}\frac{(-\log\alpha)^i}{i\,!}$$

$$\left(\frac{\eta}{\beta^2+2\beta+2}\right)^{j+1}\eta^{j-k}2^k\beta^{k-l}$$

$$\frac{\beta^2\,\Gamma(2j-k+2)}{(\eta(1+i))^{(2j-k+2)}}+\frac{\eta^2\,\Gamma(2j-k+4)}{(\eta(1+i))^{(2j-k+4)}}+\frac{2\eta\,\beta\,\Gamma(2j-k+3)}{(\eta\,(1+i))^{(2j-k+3)}}$$

The second moment of the distribution is

$$\mu_2' = \left(\frac{\alpha\log\alpha}{\alpha-1}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{i}{j}\binom{j}{k}\binom{k}{l}\frac{(-\log\alpha)^i}{i\,!}$$

$$\left(\frac{\eta}{\beta^2+2\beta+2}\right)^{j+1}\eta^{j-k}2^k\beta^{k-l}$$

$$\frac{\beta^2\,\Gamma(2j-k+3)}{(\eta\,(1+i))^{(2j-k+3)}}+\frac{\eta^2\,\Gamma(2j-k+5)}{(\eta\,(1+i))^{(2j-k+5)}}+\frac{2\eta\,\beta\,\Gamma(2j-k+4)}{(\eta\,(1+i))^{(2j-k+4)}}$$

The third moment of the proposed model is

$$\mu_3' = \left(\frac{\alpha\log\alpha}{\alpha-1}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{i}{j}\binom{j}{k}\binom{k}{l}\frac{(-\log\alpha)^i}{i\,!}$$

$$\left(\frac{\eta}{\beta^2+2\beta+2}\right)^{j+1}\eta^{j-k}2^k\beta^{k-l}$$

$$\frac{\beta^2\,\Gamma(2j-k+4)}{(\eta(1+i))^{(2j-k+4)}}+\frac{\eta^2\,\Gamma(2j-k+6)}{(\eta(1+i))^{(2j-k+6)}}+\frac{2\eta\,\beta\,\Gamma(2j-k+5)}{(\eta(1+i))^{(2j-k+5)}}$$

The forth moment of the distribution is

$$\mu_4' = \left(\frac{\alpha\log\alpha}{\alpha-1}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{i}{j}\binom{j}{k}\binom{k}{l}\frac{(-\log\alpha)^i}{i\,!}$$

$$\left(\frac{\eta}{\beta^2+2\beta+2}\right)^{j+1}\eta^{j-k}2^k\beta^{k-l}$$

$$\frac{\beta^2\,\Gamma(2j-k+5)}{(\eta(1+i))^{(2j-k+5)}}+\frac{\eta^2\,\Gamma(2j-k+4)}{(\eta(1+i))^{(2j-k+4)}}+\frac{2\eta\,\beta\,\Gamma(2j-k+3)}{(\eta(1+i))^{(2j-k+3)}}$$

3.2 Moment Generating Function

If $Z \sim APTQA\ (\alpha\ ,\beta,\eta)$, then the expression for the moment generating function (mgf) is given by

$$M_z(t) = E(e^{tz}) = \int_0^{\infty} e^{tz}\ f(z)\,dz$$

Using Taylor's series expansion of e^{tz}, we get

$$M_z(t) = \int_0^{\infty}\left(1+tz+\frac{(tz)^2}{2!}+\ldots\right)f(z)\,dz$$

$$M_z(t) = \sum_{p=0}^{\infty}\frac{t^p}{p!}\int_0^{\infty} z^p f(z)\,dz$$

$$M_z(t) = \sum_{p=0}^{\infty}\frac{t^p}{p!}\ \mu_p'$$

$$M_z(t) = \sum_{p=0}^{\infty}\frac{t^p}{p!}\ \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{i}{j}\binom{j}{k}\binom{k}{l}\frac{(-\log\alpha)^i}{i\,!}$$

$$\left(\frac{\eta}{\beta^2+2\beta+2}\right)^{j+1}\left(\frac{\alpha\log\alpha}{\alpha-1}\right)\eta^{j-k}2^k\beta^{k-l}$$

$$\frac{\beta^2\,\Gamma(p+2j-k+1)}{(\eta\,(1+i))^{(p+2j-k+1)}}+\frac{\eta^2\,\Gamma(p+2j-k+3)}{(\eta\,(1+i))^{(p+2j-k+3)}}+\frac{2\eta\,\beta\,\Gamma(p+2j-k+2)}{(\eta\,(1+i))^{(p+2j-k+2)}}$$

3.3 Characteristic Function

If Z is a random variable with pdf (4) and cdf (5), then the characteristic function of the distribution is given by

$$\psi(z) = M_z(it)$$

$$M_z(it)=\sum_{p=0}^{\infty}\frac{(it)^p}{p!}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{i}{j}\binom{j}{k}\binom{k}{l}\frac{(-\log\alpha)^i}{i!}$$

$$\left(\frac{\eta}{\beta^2+2\beta+2}\right)^{j+1}\left(\frac{\alpha\log\alpha}{\alpha-1}\right)\eta^{j-k}2^k\beta^{k-l}\frac{\beta^2\,\Gamma(p+2j-k+1)}{(\eta(1+i))^{(p+2j-k+1)}}$$

$$+\frac{\eta^2\,\Gamma(p+2j-k+3)}{(\eta(1+i))^{(p+2j-k+3)}}+\frac{2\eta\beta\,\Gamma(p+2j-k+2)}{(\eta(1+i))^{(p+2j-k+2)}}$$

4 Order Statistics

Let $Z_{(1)},Z_{(2)},Z_{(3)}...Z_{(n)}$ be the order statistics of a random variable $Z_1,Z_2,Z_3...Z_n$ drawn from the continuous population with pdf $f_z(z)$ and cdf $F_z(z)$, then the probability density function of k^{th} order statistic $Z_{(k)}$ is given by

$$f_{Z(k)}(z)=\frac{n!}{(k-1)!\,(n-k)!}f_Z(z)\left[F_Z(z)\right]^{k-1}\left[1-F_Z(z)\right]^{n-k} \quad (12)$$

Substituting equations (4) and (5) in equation (12), the probability density function of k^{th} order statistic $Z_{(k)}$ of the APTQA distribution is given by

$$f_{Z(k)}(z)=\frac{n!}{(k-1)!\,(n-k)!}\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\eta}{\beta^2+2\beta+2}\right)(\beta+\eta z)^2$$

$$\alpha^{1-\left(1+\frac{\eta z(\eta z+2\beta+2)}{\beta^2+2\beta+2}\right)e^{-\eta z}}e^{-\eta z}\left(\frac{\alpha^{1-\left(1+\frac{\eta z(\eta z+2\beta+2)}{\beta^2+2\beta+2}\right)e^{-\eta z}}-1}{\alpha-1}\right)^{k-1}$$

$$\left(1-\frac{\alpha^{1-\left(1+\frac{\eta z(\eta z+2\beta+2)}{\beta^2+2\beta+2}\right)e^{-\eta z}}-1}{\alpha-1}\right)^{n-k}$$

The probability density function of the higher order statistic $Z_{(n)}$ is obtained by setting $k=n$ in equation (12).

$$f_{Z(n)}(z)=n\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\eta}{\beta^2+2\beta+2}\right)(\beta+\eta z)^2$$

$$\alpha^{1-\left(1+\frac{\eta z(\eta z+2\beta+2)}{\beta^2+2\beta+2}\right)e^{-\eta z}}e^{-\eta z}\left(\frac{\alpha^{1-\left(1+\frac{\eta z(\eta z+2\beta+2)}{\beta^2+2\beta+2}\right)e^{-\eta z}}-1}{\alpha-1}\right)^{n-1}$$

The probability density function of the first order statistic $Z_{(1)}$ is obtained by setting $k=1$ in equation (12).

$$f_{Z(1)}(z)=n\left(\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\eta}{\beta^2+2\beta+2}\right)(\beta+\eta z)^2\;\alpha^{1-\left(1+\frac{\eta z(\eta z+2\beta+2)}{\beta^2+2\beta+2}\right)e^{-\eta z}}e^{-\eta z}\right)$$

$$\left(1-\frac{\alpha^{1-\left(1+\frac{\eta z(\eta z+2\beta+2)}{\beta^2+2\beta+2}\right)e^{-\eta z}}-1}{\alpha-1}\right)^{n-1}$$

5 Entropy Measures

In terms of probability, we define entropy as the degree of uncertainty in a random variable or the outcome of a random process, which is measured in terms of entropy measure. In science and technology, entropies have been used in a variety of situations. It is used to figure out how much of a random variable is uncertain. There exist many entropy metrics, however, the most prominent and extensively used are as follows

5.1 Renyi Entropy

The Renyi entropy [16] of a random variable represents a measure of the variation of uncertainty. The Renyi entropy for given pdf (4) is defined by

$$R(\omega)=(1-\omega)^{-1}\log\int_0^{\infty}f^{\omega}(z)\,dz$$

Substituting equation (4) in above equatin as

$$R(\omega)=(1-\omega)^{-1}\log\int_0^{\infty}\left(\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\eta}{\beta^2+2\beta+2}\right)e^{-\eta z}\;(\beta+\eta z)^2\alpha^{1-\left(1+\frac{\eta z(\eta z+2\beta+2)}{\beta^2+2\beta+2}\right)e^{-\eta z}}\right)^{\omega}dz$$

$$R(\omega)=(1-\omega)^{-1}\log\left(\frac{\alpha\log\alpha}{\alpha-1}\right)^{\omega}\left(\frac{\eta}{\beta^2+2\beta+2}\right)^{\omega}$$

$$\sum_{i=0}^{\infty}\frac{(-\omega\log\alpha)^i}{i!} \quad (13)$$

$$\int_0^{\infty}(\beta+\eta z)^{2\omega}\left(1+\frac{\eta z(\eta z+2\beta+2)}{\beta^2+2\beta+2}\right)^i e^{-\eta(i+\omega)z}\,dz$$

Using equation (9) to (13), then we have

$$R(\omega)=(1-\omega)^{-1}\log\left(\frac{\alpha\log\alpha}{\alpha-1}\right)^{\omega}\left(\frac{\eta}{\beta^2+2\beta+2}\right)^{\omega}$$

$$\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{i}{j}\frac{(-\omega\log\alpha)^i}{i!} \tag{14}$$

$$\int_0^{\infty}(\beta+\eta z)^{2\omega}\left(\frac{\eta z\,(\eta z+2\beta+2)}{\beta^2+2\beta+2}\right)^{j}e^{-\eta(i+\omega)z}\,dz$$

On simplification, we get

$$R(\omega)=(1-\omega)^{-1}\log\left(\frac{\alpha\log\alpha}{\alpha-1}\right)^{\omega}\left(\frac{\eta}{\beta^2+2\beta+2}\right)^{\omega+j}$$

$$\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{s=0}^{\infty}\binom{2\omega}{s}\binom{i}{j}\binom{j}{k}\binom{k}{l}\frac{(-\omega\log\alpha)^i}{i!}$$

$$\eta^{j-k+s}2^k\beta^{2\omega+k-l-s}\frac{\Gamma(2j-k+s+1)}{(\eta(\omega+i))^{(2j-k+s+1)}}$$

5.2 Tsallis Entropy

The Tsallis entropy [17] of order (ν) associated with a random variable Z is given by

$$S(\nu)=\frac{1}{\nu-1}\left(1-\int_0^{\infty}f^{\nu}(z)\,dz\right)$$

Substituting equation (4) in above expression as

$$S(\nu)=\frac{1}{\nu-1}\left(1-\int_0^{\infty}\left(\begin{array}{c}\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\eta}{\beta^2+2\beta+2}\right)e^{-\eta z}\\ (\beta+\eta z)^2\alpha^{1-\left(1+\frac{\eta z\,(\eta z+2\beta+2)}{\beta^2+2\beta+2}\right)e^{-\eta z}}\end{array}\right)^{\nu}dz\right)$$

$$S(\nu)=\frac{1}{\nu-1}\left(\begin{array}{c}1-\left(\frac{\alpha\log\alpha}{\alpha-1}\right)^{\nu}\left(\frac{\eta}{\beta^2+2\beta+2}\right)^{\nu}\\ \int_0^{\infty}(\beta+\eta z)^{2\nu}\alpha^{-\nu\left(1+\frac{\eta z\,(\eta z+2\beta+2)}{\beta^2+2\beta+2}\right)e^{-\eta z}}e^{-\eta\nu z}\,dz\end{array}\right)$$

On simplifying the resulting equation, the Tsallis entropy of the APTQA distribution is

$$S(\nu)=1-\left(\frac{\alpha\log\alpha}{\alpha-1}\right)^{\nu}\left(\frac{\eta}{\beta^2+2\beta+2}\right)^{\nu+j}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{j}{k}$$

$$\sum_{s=0}^{\infty}\binom{2\nu}{s}\binom{i}{j}\binom{k}{l}\frac{(-\nu\log\alpha)^i}{i!}\eta^{j-k+s}2^k\beta^{2\nu+k-s}$$

$$\frac{\Gamma(2j-k+s+1)}{(\eta(\nu+i))^{(2j-k+s+1)}}$$

6 Bonferroni Curve

For analyzing and illustrating the income disparity of a country, Bonferroni and Lorenz curves are extensively used. The Lorenz curve is the proportion of total income volume accumulated by units with income less than or equal to the volume z. The Bonferroni curve was proposed by Bonferroni [18] is defined by a random variable Z. The Bonferroni curve of the APTQA distribution is given by

$$B(p)=\frac{1}{p\mu_1'}\int_0^{q}z\,f(z)\,dz\;;\quad\text{where, } q=F^{-1}(p)$$

$$B(p)=\frac{1}{p\mu_1'}\int_0^{q}\left(\begin{array}{c}z\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\eta}{\beta^2+2\beta+2}\right)(\beta+\eta z)^2\\ \alpha^{1-\left(1+\frac{\eta z(\eta z+2\beta+2)}{\beta^2+2\beta+2}\right)e^{-\eta z}}e^{-\eta z}\end{array}\right)dz$$

$$B(p)=\frac{1}{p\mu_1'}\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\eta\,\alpha}{\beta^2+2\beta+2}\right) \tag{15}$$

$$\int_0^{q}z(\beta+\eta z)^2\alpha^{-\left(1+\frac{\eta z(\eta z+2\beta+2)}{\beta^2+2\beta+2}\right)e^{-\eta z}}e^{-\eta z}\,dz$$

Applying expansion (9) to (15), (15) becomes,

$$B(p)=\frac{1}{p\mu_1'}\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\eta\,\alpha}{\beta^2+2\beta+2}\right)\sum_{i=0}^{\infty}\frac{(-\log\alpha)^i}{i\,!}$$

$$\int_0^{q}z(\beta+\eta z)^2\left(1+\frac{\eta z(\eta z+2\beta+2)}{\beta^2+2\beta+2}\right)^{i}e^{-\eta iz}e^{-\eta z}\,dz \tag{16}$$

Using the binomial expansion to (16), we get

$$B(p)=\frac{1}{p\mu_1'}\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\eta\ \alpha}{\beta^2+2\beta+2}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{i}{j}\frac{(-\log\alpha)^i}{i\ !}$$

$$\int_0^q z\left(\beta+\eta z\right)^2\left(\frac{\eta z(\eta z+2\beta+2\ )}{\beta^2+2\beta+2}\right)^j e^{-\eta(1+i)z}\,dz$$

$$B(p)=\frac{1}{p\mu_1'}\left(\frac{\alpha\log\alpha}{\alpha-1}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{i}{j}\frac{(-\log\alpha)^i}{i\ !}\left(\frac{\eta}{\beta^2+2\beta+2}\right)^{j+1}$$

$$\int_0^q z^{1+j}\left(\beta+\eta z\right)^2(\eta z+2\beta+2\ )^j e^{-\eta(1+i)z}\,dz$$

$$B(p)=\frac{1}{p\mu_1'}\left(\frac{\alpha\log\alpha}{\alpha-1}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{i}{j}\frac{(-\log\alpha)^i}{i\ !}\left(\frac{\eta}{\beta^2+2\beta+2}\right)^{j+1}$$

$$\eta^j\int_0^q z^{1+2j}(\beta+\eta z)^2\left(1+\frac{2\beta}{\eta z}+\frac{2}{\eta z}\right)^j e^{-\eta(1+i)z}\,dz$$

$$B(p)=\frac{1}{p\mu_1'}\left(\frac{\alpha\log\alpha}{\alpha-1}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\binom{i}{j}\binom{j}{k}$$

$$\frac{(-\log\alpha)^i}{i\ !}\left(\frac{\eta}{\beta^2+2\beta+2}\right)^{j+1}\eta^j$$

$$\int_0^q z^{1+2j}\left(\beta+\eta z\right)^2\left(\frac{2\beta}{\eta z}+\frac{2}{\eta z}\right)^k e^{-\eta(1+i)z}\,dz$$

$$B(p)=\frac{1}{p\mu_1'}\left(\frac{\alpha\log\alpha}{\alpha-1}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\binom{i}{j}\binom{j}{k}\frac{(-\log\alpha)^i}{i\ !}$$

$$\left(\frac{\eta}{\beta^2+2\beta+2}\right)^{j+1}\eta^{j-k}2^k\beta^k$$

$$\int_0^q z^{1+2j-k}\left(\beta+\eta z\right)^2\left(1+\frac{1}{\beta}\right)^k e^{-\eta(1+i)z}\,dz$$

$$B(p)=\frac{1}{p\mu_1'}\left(\frac{\alpha\log\alpha}{\alpha-1}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{i}{j}\binom{j}{k}\binom{k}{l}\frac{(-\log\alpha)^i}{i\ !}$$

$$\left(\frac{\eta}{\beta^2+2\beta+2}\right)^{j+1}\eta^{j-k}2^k\beta^k\int_0^q z^{1+2j-k}\left(\beta+\eta z\right)^2\left(\frac{1}{\beta}\right)^l e^{-\eta(1+i)z}\,dz$$

$$B(p)=\frac{1}{p\mu_1'}\left(\frac{\alpha\log\alpha}{\alpha-1}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{i}{j}\binom{j}{k}\binom{k}{l}\frac{(-\log\alpha)^i}{i\ !}$$

$$\left(\frac{\eta}{\beta^2+2\beta+2}\right)^{j+1}\eta^{j-k}2^k\beta^{k-l}\int_0^q z^{1+2j-k}\left(\beta^2+\eta^2z^2+2\eta\beta z\right)e^{-\eta(1+i)z}\,dz$$

$$B(p)=\frac{1}{p\mu_1'}\left(\frac{\alpha\log\alpha}{\alpha-1}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{i}{j}\binom{j}{k}\binom{k}{l}$$

$$\left(\frac{(-\log\alpha)^i}{i\ !}\right)\left(\frac{\eta}{\beta^2+2\beta+2}\right)^{j+1}\eta^{j-k}2^k\beta^{k-l}$$

$$\left(\begin{array}{l}\beta^2\int_0^q z^{2j-k+1}e^{-\eta(i+1)z}\,dz+\eta^2\int_0^q z^{1+2j-k+2}e^{-\eta(i+1)z}\,dz\\ +2\eta\beta\int_0^q z^{1+2j-k+1}e^{-\eta(i+1)z}\,dz\end{array}\right)$$

Put $\eta(1+i)z=t\Rightarrow\eta\ (1+i)\ dz=dt\ \Rightarrow dz=\dfrac{dt}{\eta\ (1+i)}$

As $z\to0\ \Rightarrow\ t\to0,\ z\to q\ \Rightarrow\ t\to\eta(i+1)q$

Using lower incomplete gamma function and the resulting equation will be

$$B(p)=\frac{1}{p\mu_1'}\left(\frac{\alpha\log\alpha}{\alpha-1}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{i}{j}\binom{j}{k}\binom{k}{l}$$

$$\frac{(-\log\alpha)^i}{i\ !}\left(\frac{\eta}{\beta^2+2\beta+2}\right)^{j+1}\eta^{j-k}2^k\beta^{k-l}$$

$$\left(\begin{array}{l}\beta^2\int_0^{\eta(i+1)q}\left(\frac{t}{\eta\ (i+1)}\right)^{2j-k+1}e^{-t}\frac{dt}{\eta\ (i+1)}+\eta^2\\ \int_0^{\eta(i+1)q}\left(\frac{t}{\eta\ (i+1)}\right)^{2j-k+3}e^{-t}\frac{dt}{\eta\ (i+1)}\\ +2\eta\beta\int_0^{\eta(i+1)q}\left(\frac{t}{\eta\ (i+1)}\right)^{2j-k+2}e^{-t}\frac{dt}{\eta\ (i+1)}\end{array}\right)$$

$$B(p)=\frac{1}{p\mu_1'}\left(\frac{\alpha\log\alpha}{\alpha-1}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{i}{j}\binom{j}{k}\binom{k}{l}$$

$$\frac{(-\log\alpha)^i}{i\,!}\left(\frac{\eta}{\beta^2+2\beta+2}\right)^{j+1}\eta^{j-k}2^k\beta^{k-l}$$

$$\left(\begin{array}{l}\beta^2\left(\frac{1}{\eta(i+1)}\right)^{2j-k+2}\int_0^{\eta(i+1)q}t^{2j-k+1}e^{-t}\,dt+\eta^2\\ \left(\frac{1}{\eta(i+1)}\right)^{2j-k+4}\int_0^{\eta(i+1)q}t^{2j-k+3}e^{-t}\,dt+2\eta\beta\\ \left(\frac{1}{\eta(i+1)}\right)^{2j-k+3}\int_0^{\eta(i+1)q}\left(\frac{t}{\eta(i+1)}\right)^{2j-k+2}e^{-t}\,dt\end{array}\right)$$

On simplification, one will obtain,

$$B(p)=\frac{1}{p\mu_1'}\left(\frac{\alpha\log\alpha}{\alpha-1}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{i}{j}\binom{j}{k}\binom{k}{l}$$
$$\frac{(-\log\alpha)^i}{i\,!}\left(\frac{\eta}{\beta^2+2\beta+2}\right)^{j+1}\eta^{j-k}2^k\beta^{k-l}$$

$$\left(\begin{array}{l}\beta^2\left(\frac{1}{\eta(i+1)}\right)^{2j-k+2}\int_0^{\eta(i+1)q}t^{(2j-k+2)-1}e^{-t}\,dt+\eta^2\\ \left(\frac{1}{\eta(i+1)}\right)^{2j-k+4}\int_0^{\eta(i+1)q}t^{(2j-k+4)-1}e^{-t}\,dt+\\ 2\eta\beta\left(\frac{1}{\eta(i+1)}\right)^{2j-k+3}\int_0^{\eta(i+1)q}\left(\frac{t}{\eta(i+1)}\right)^{(2j-k+3)-1}e^{-t}\,dt\end{array}\right)$$

$$B(p)=\frac{1}{p\mu_1'}\left(\frac{\alpha\log\alpha}{\alpha-1}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{i}{j}\binom{j}{k}\binom{k}{l}$$

$$\left(\frac{\eta}{\beta^2+2\beta+2}\right)^{j+1}\eta^{j-k}2^k\beta^{k-l}\left(\frac{1}{\eta(i+1)}\right)^{2j-k+2}\frac{(-\log\alpha)^i}{i\,!}$$

$$\left(\begin{array}{l}\beta^2\gamma\left((2j-k+2),\eta(i+1)q\right)+\eta^2\left(\frac{1}{\eta(i+1)}\right)^2\\ \gamma\left((2j-k+4),\eta(i+1)q\right)+2\eta\beta\left(\frac{1}{\eta(i+1)}\right)\\ \gamma\left((2j-k+3),\eta(i+1)q\right)\end{array}\right)$$

7 Lorenz Curve

The Lorenz curve is a graphical representation of the degree of economic disparity in a country. Max O. Lorenz [19] was an American economist who proposed the Lorenz curve as an approach for comparing changes in the population's income distribution over time. For given pdf (4) the Lorenz curve is defined by

$$L(p)=\frac{1}{\mu_1'}\int_0^q z\,f(z)\,dz$$

$$L(p)=p\,B(p)$$

$$L(p)=\frac{1}{\mu_1'}\left(\frac{\alpha\log\alpha}{\alpha-1}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{i}{j}\binom{j}{k}\binom{k}{l}\frac{(-\log\alpha)^i}{i\,!}$$

$$\left(\frac{\eta}{\beta^2+2\beta+2}\right)^{j+1}\left(\frac{1}{\eta(i+1)}\right)^{2j-k+2}\eta^{j-k}2^k\beta^{k-l}$$

$$\left(\begin{array}{l}\beta^2\gamma\left((2j-k+2),\eta(i+1)q\right)+\eta^2\\ \left(\frac{1}{\eta(i+1)}\right)^2\gamma\left((2j-k+4),\eta(i+1)q\right)+\\ 2\eta\beta\left(\frac{1}{\eta(i+1)}\right)\gamma\left((2j-k+3),\eta(i+1)q\right)\end{array}\right)$$

8 Methods of Estimation

In this section, the estimation of parameters is investigated by using the method of maximum likelihood technique for complete data.

8.1 Maximum Likelihood Estimation

To obtain the ML estimators of the APTQA distribution with a set of parameters (β,α,η), Suppose Z_1, $Z_2,\ldots,Z_n$ be the observed values from APTQA distribution. Hence, the likelihood function for the vector $\{\alpha,\beta,\eta\}$ parameters can be written as

$$f(z;\alpha,\beta,\eta)=\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\eta}{\beta^2+2\beta+2}\right)(\beta+\eta z)^2$$

$$\alpha^{1-\left(1+\frac{\eta z(\eta z+2\beta+2)}{\beta^2+2\beta+2}\right)e^{-\eta z}}e^{-\eta z}$$

$$L(z;\alpha,\beta,\eta)=\prod_{i=1}^{n} f(z_i,\alpha,\beta,\eta)$$

$$L(z;\alpha,\beta,\eta)=\prod_{i=1}^{n}\left(\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\eta}{\beta^2+2\beta+2}\right)(\beta+\eta z_i)^2 \; \alpha^{1-\left(1+\frac{\eta z_i(\eta z_i+2\beta+2)}{\beta^2+2\beta+2}\right)e^{-\eta z_i}} e^{-\eta z_i}\right)$$

$$L(z;\alpha,\beta,\eta)=\left(\frac{\log\alpha}{\alpha-1}\right)^n\left(\frac{\eta}{\beta^2+2\beta+2}\right)^n \tag{17}$$

$$\prod_{i=1}^{n}(\beta+\eta z_i)^2\,\alpha^{1-\left(1+\frac{\eta z_i(\eta z_i+2\beta+2)}{\beta^2+2\beta+2}\right)e^{-\eta z_i}} e^{-\eta z_i}$$

Taking logarithm to equation (17), then we have

$$\log L = n\left(\begin{array}{l}\log(\log\alpha)-\log(\alpha-1)+\log\alpha+\log\eta\\ -\log(\beta^2+2\beta+2\end{array}\right)$$

$$-\eta z_i+2\sum_{i=1}^{n}\log(\beta+\eta z_i)-\log\alpha$$

$$\sum_{i=1}^{n}\left(1+\frac{\eta z_i(\eta z_i+2\beta+2)}{\beta^2+2\beta+2}\right)e^{-\eta z_i}$$

Differentiating the model parameters $\{\alpha,\beta,\eta\}$ with their estimators $\hat{\alpha}$, $\hat{\beta}$, $\hat{\eta}$, we get

$$\frac{\partial}{\partial\alpha}L=\frac{n}{\alpha\log\alpha}-\frac{n}{\alpha-1}-\frac{1}{\alpha}\sum_{i=1}^{n}\left(1+\frac{\eta z_i(\eta z_i+2\beta+2)}{\beta^2+2\beta+2}\right)e^{-\eta z_i} \tag{18}$$

$$\frac{\partial}{\partial\beta}L=-\frac{n(2\beta+2)}{\beta^2+2\beta+2}+\sum_{i=1}^{n}\left(\frac{1}{\beta+\eta z_i}\right)- \tag{19}$$

$$\log\alpha\sum_{i=1}^{n}\left(\frac{e^{-\eta z_i}\eta z_i(-2\eta\beta z_i-2\eta z_i-2\beta^2+4\beta)}{(\beta^2+2\beta+2)^2}\right)$$

$$\frac{\partial}{\partial\eta}L=\frac{n}{\eta}+\sum_{i=1}^{n}\left(\frac{z_i}{\beta+\eta z_i}\right)-\log\alpha \tag{20}$$

$$\sum_{i=1}^{n}\left(\frac{-e^{-\eta z_i}\eta^2 z_i^3-2\beta\eta z_i e^{-\eta z_i}-\beta^2 e^{-\eta z_i} z_i)}{\beta^2+2\beta+2}\right)$$

The equation (18), equation (19) and equation (20) are in closed form so they cannot be solved numerically. Therefore, the parameters of the model can be obtained by using R, MATLAB, Mathematica Software [20].

9 Numerical Results

In this section, we use real-life data to demonstrate the applicability of the APTQA distribution by using model selection techniques.

To determine the performance of the Alpha Power Transformed Quasi Aradhana distribution, a model selection techniques has bee applied to a data set and the APTQA distribution has been compared with Quasi Aradhana distribution, A new Quasi Aradhana distribution, Sujatha, Amarendra, two-parameter Weibull, Rayleigh, Rama and Exponential distributions. The model which possesses minimum AIC, BIC, AICc are considered the best model for the fitted data set.

9.1 Data Set 1

We have taken the data set consisting of the remission time of 128 bladder cancer patients to demonstrate the performance of the proposed model Alpha power Quasi Aradhana distribution. The data were also studied by Zeal et al. Lee and Wang [21].

Data Set I								
3.7	3.11	4.42	3.28	3.75	2.96	3.39	3.31	3.15
2.76	3.19	1.59	2.17	3.51	1.84	1.61	1.57	1.89
2.41	3.09	2.43	2.53	2.81	3.31	2.35	2.77	2.68
2.00	1.17	2.17	0.39	2.79	1.08	2.88	2.73	2.87
2.95	2.67	4.20	2.85	2.55	2.17	2.97	3.68	0.81
1.69	3.68	4.70	2.03	2.82	2.50	1.47	3.22	3.15
3.33	2.56	2.59	2.83	1.36	1.84	5.56	1.12	2.48
2.03	1.61	2.05	3.60	3.11	1.69	4.90	3.39	3.22
2.38	1.92	0.98	1.59	1.73	1.71	1.18	4.38	0.85
3.65.	1.41	3.27	1.57	1.87	5.08	2.93	2.48	3.56
2.81	2.74	4.91	3.19	1.22	2.97	1.25	2.55	1.80
2.12								

9.2 Data Set 2

The data set provides tensile strength (measured in GPA) of 100 carbon fibers reported by Flaih et al. [22] initially used by Nicholas et al. [23] in his paper are as follows

Data Set II						
0.08	0.20	0.40	0.50	0.51	0.81	0.90
1.26	1.35	1.40	1.46	1.76	2.02	2.02
2.23	2.26	2.46	2.54	2.62	2.64	2.69
2.83	2.87	3.02	3.25	3.31	3.36	3.36
3.57	3.64	3.70	3.82	3.88	4.18	4.23
4.34	4.40	4.50	4.51	4.87	4.98	5.06
5.32	5.32	5.34	5.41	5.41	5.49	5.62
6.25	6.54	6.76	6.93	6.94	6.97	7.09
7.32	7.39	7.59	7.62	7.63	7.66	7.87
8.37	8.53	8.65	8.66	9.02	9.22	9.47
10.34	10.66	10.75	11.25	11.64	11.79	11.98
12.07	12.63	13.11	13.29	13.80	14.24	14.76
15.96	16.62	17.12	17.14	17.36	18.10	19.13
22.69	23.63	25.74	25.82	26.31	32.15	34.26
46.12	79.05	1.19	2.09	2.75	3.52	4.33
5.85	7.28	8.26	10.06	12.03	14.83	21.73
1.05	2.07	2.69	3.48	4.26	5.09	5.71
7.26	7.93	9.74	12.02	14.77	20.28	36.66
5.17	43.01					

9.3 Akaike Information Criterion

Akaike information criterion (AIC) is based on information theory and was named after the Japanese Statistician Hirotugu Akaike [24]. The AIC is an estimator of prediction error and thereby the relative quality of the statistical models for the given data set. The AIC estimates the quality of each of the other models. It estimates the amount of information lost by the given model: the less information a model loses, the higher the quality of the model.

$$AIC = 2s - 2\log \hat{L}$$

9.4 Bayesian Information Criterion

The Bayesian information criterion was developed by Schwarz [25]. Bayesian information criterion (BIC) or Schwarz information is a criterion for model selection among a finite set of models; models with the lowest BIC are preferred.

$$BIC = s \ln n - 2 \log \hat{L}$$

9.5 Corrected Akaike Information Criterion

It is an information score of the model. The extra c indicates that the value has been calculated from the AIC test corrected for small sample sizes. The smaller the AIC value, the better the model fit.

$$AICc = AIC + \frac{2s\,(s+1)}{n-s-1}$$

Where, s is the number of estimated parameters in the model and L is the maximum value of the likelihood function of the model. The model which possesses a minimum Akaike information criterion (AIC), Bayesian information criterion (BIC) and Corrected Akaike information criterion (AICc) is considered the best model for the considered data set.

The results observed from table 1 and 2 reveals that the new model viz Alpha power transformed Quasi Aradhana distribution is the better model for considered data sets as it provides a close fit and possesses minimum values for the goodness of fit criteria among the competing models include Quasi Aradhana distribution, New Quasi Aradhana distribution, Sujatha distribution, Amarendra distribution, two-parameter Weibull distribution, Rayleigh distribution, Rama distribution and Exponential distribution.

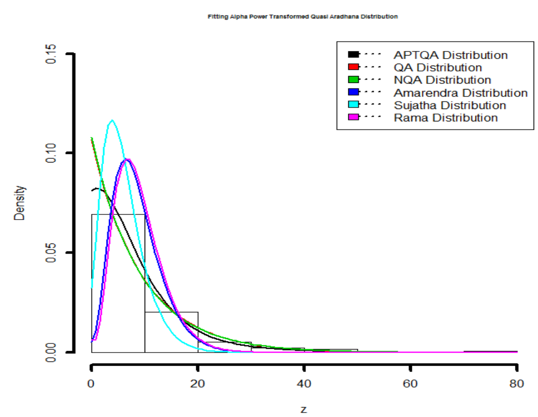

Fig. 5: Data set 1 fitted to the APTQA distribution

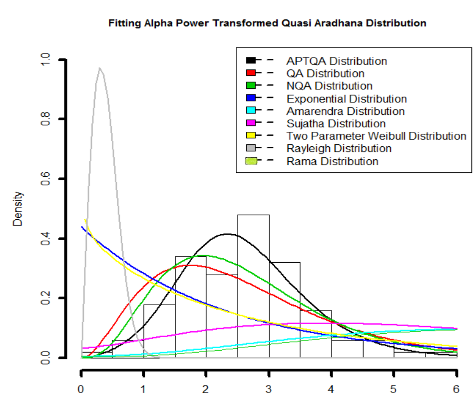

Fig. 6: Data set 2 fitted to the APTQA distribution

Table 1: Goodness of fit Criterion for Data Set 1

Model	ML Estimates	AIC	BIC	AICc	-2 ln L
APTQAD	$\hat{\theta}$ = 0.10544 $\hat{\alpha}$ = 0.02043 $\hat{\beta}$ = 0.97466	829.3204	837.8765	829.5139	831.5555
QAD	$\hat{\theta}$ = 0.01069 $\hat{\beta}$ = 2.09308	832.6838	838.3879	832.8774	834.1739
NQAD	$\hat{\theta}$ = 0.12194 $\hat{\alpha}$ = 0.91044	832.4739	838.1780	832.6675	833.9641
Sujatha	$\hat{\theta}$= 0.29896	881.8248	884.6768	882.0183	879.9113
Amarendra	$\hat{\theta}$ = 0.40669	942.8512	945.7032	942.7349	940.8512
Rama	$\hat{\theta}$ =0.42313	961.0615	963.9135	961.0774	959.0615

Table 2: Goodness of fit Criterion for Data Set 2.

Model	ML Estimates	AIC	BIC	AICc	-2 ln L
APTQAD	$\hat{\theta}$ = 1.79281 $\hat{\alpha}$ = 49.73527 $\hat{\beta}$ = 0.00100	288.4901	296.3056	288.7401	282.4901
QAD	$\hat{\theta}$ = 1.14849 $\hat{\beta}$ = 0.00100	309.7526	314.963	310.0026	305.7527
NQAD	$\hat{\theta}$ = 1.53163 $\hat{\alpha}$ = 0.00100	296.7137	301.924	296.9637	292.7137
Sujatha	$\hat{\theta}$ = 1.24975	345.0524	347.6576	345.0932	343.0524
Amarendra	$\hat{\theta}$ = 1.18949	330.3315	332.9366	330.3723	328.3315
Two Parameter Weibull	$\hat{k}$ =2.61141 $\hat{\lambda}$ =0.92107	395.2824	400.4928	395.5324	391.2825
Rayleigh	$\hat{\sigma}$ =2.79746	438.6466	441.2518	441.2926	436.6466
Rama	$\hat{\theta}$ =1.24974	335.4366	338.0418	335.4774	333.4366
Exponential	$\hat{\theta}$ =0.38293	393.9773	396.5825	394.0182	391.9773

10 Conclusion

In this study, a new generalization of Quasi Aradhana distribution is introduced and is referred to as Alpha Power Transformed Quasi Aradhana Distribution. The proposed model is obtained by the technique provided by Mahdavi and Kundu. Various properties of the model are studied. The plots of the density function, specify that model has desired shapes depending on the value of the parameters such as increasing, decreasing, left-skewed and uni-model. The proposed model can be used to fit the data with monotone and non-monotone failure rates. The model parameters are investigated by the maximum likelihood estimation technique. An illustration is carried out to know the performance of the model by using the goodness of fit criteria and has been observed that the alpha power transformed Quasi Aradhana distribution provides a close fit against the other competing distributions in terms of data fitting.

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