Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/160506

The Modified Extended Chen Distribution: Properties and Application to Rainfall Data

Abdulzeid Yen Anafo[1,], Lewis Brew[1] and Suleman Nasiru[2]*

[1]Department of Mathematical Sciences, Faculty of Engineering, University of Mines and Technology, Ghana
[2]Department of Statistics, School of Mathematical Sciences, C. K. Tedam University of Technology and Applied Sciences, Ghana

Received: 14 Sep. 2021, Revised: 21 Oct. 2021, Accepted: 21 Dec. 2021
Published online: 1 Sep. 2022

Abstract: The Burr-Hatke differential equation is used in this study to generate a novel three parameter modified extended Chen distribution. The statistical features of the developed distribution are derived, which include the quantile function, moment, incomplete moment, inequality measures, moment generating function, characteristic function, stress-strength reliability, order statistic, Rényi entropy, and stochastic ordering. Furthermore, the functions of reliability, survival, reversed hazard rate, and hazard rate are developed. The parameters of the new distribution are estimated using the maximum likelihood, ordinary, and weighted least squares methods. A simulation experiment is carried out to examine the performance of the several estimating techniques for the parameters of the MEC distribution. Rainfall data from three locations in Ghana are used to demonstrate the applicability of the modified extended Chen distribution. Finally, the study estimates the return levels and periods for the three locations.

Keywords: Burr-differential, extreme values, hazard function, estimation methods, and rainfall data

1 Introduction

There have been a substantial growth of the volume, variety and velocity of data sets over the last few decades due to improved technological data collection systems. Analysis of data is mostly determined by its distributional pattern. As a result, there is a continuing push to broaden the development of statistical distributions with diverse properties for modeling data.
In recent times, many statistical distributions have been proposed using varied methods such as differential equation [1,2], transformation [3] and quantile approach [4]. Additional methods also used include method of generating skew distribution [5], beta generated method [6], including one or more additional parameter [7] and, transformed-transformer method [8]. However, some of these continuous distributions proposed are not flexible enough for modelling certain certain data sets from meteorology, life testing, finance, geology, biometry and reliability due to the complex traits they exhibit. Hence, the modification of these existing distributions are very necessary.

Chen [9] developed a new two-parameter lifetime statistical distribution with monotonically increasing, decreasing, or constant hazard rates features. The proposed two-parameter distribution by [9] has varied features in modelling lifetime data. Consequently, various extensions of the Chen distribution has been developed through the use of varied methods. Xie et al. [10] developed an extension through the approach of adding a parameter to the Chen distribution and named it the extended-Weibull distribution.

Additional extensions were developed by [11] and [12], in their work they named the new distributions as the Marshall-Olkin extended Chen's distribution and Exponentiated Chen (EC) distribution respectively. Recently, [14] proposed another extension of the Chen distribution through the use of the Burr-Hatke differential equation method and named it the Extended Chen distribution. Although the Extended Chen exhibits some level of flexibility towards modelling life-time data, the distribution lacks the ability to appropriately model data that are heavy tailed in nature and exhibit upside-down bathtub shapes. In this article we apply the Burr-Hatke differential approach suggested by [1] to define a new model which modifies the Extended Chen distribution proposed by [14], and call the modified distribution as the Modified Extended Chen (MEC) distribution.

* Corresponding author e-mail: ayanafo@st.umat.edu.gh

The article is organised as follows: Section 2 introduces the cumulative distribution function (CDF) and the corresponding probability density function (PDF) of the MEC distribution. Section 3 presents a series representation of the MEC distribution. Section 4 presents several statistical properties. The maximum likelihood, ordinary and weighted least square approach are presented in Section 5. In Sections 6, we obtain the best method of estimating the parameters of the MEC distribution by obtaining the maximum likelihood, ordinary and weighted least square estimates through the use of a simulation study. We will analyze three maximum annual rainfall data sets from three locations in Ghana in Section 7 and the results are compared with different known distributions. Finally, in Section 8, the conclusions of the study is presented.

2 Modified Extended Chen Distribution

Let the random variable Y follow the MEC distribution with positive parameters $a > 0$, $b > 0$ and $\rho > 0$. Let $y \in \mathbb{R}^+$, consider the general solution of the Burr-Hatke differential equation,

$$F(y) = \left[\exp\left\{-\int w[y, F(y)]\,dy\right\}+1\right]^{-1}. \quad (1)$$

Suppose the function $w[y, F(y)]$ is given by

$$\frac{ab\rho y^{-b-1}e^{y^{-b}}\left[1+\rho\left(e^{y^{-b}}-1\right)\right]^{a-1}}{1-\left[1+\rho\left(e^{y^{-b}}-1\right)\right]^{a}}, \quad (2)$$

the CDF of the random variable Y is obtained by substituting equation (2) into equation (1) and simplifying the integral. Hence, the CDF of the MEC distribution is given by

$$F(y) = \left[\rho\left(e^{y^{-b}}-1\right)+1\right]^{-a}, y \geq 0, \quad (3)$$

where $a > 0$ and $b > 0$ are the shape parameters and $\rho > 0$ is a scale parameter. The PDF of the MEC distribution is obtained by differentiating equation (3) with respect to y. Hence, the PDF is given by

$$f(y) = ab\rho y^{-b-1}e^{y^{-b}}\left[\rho\left(e^{y^{-b}}-1\right)+1\right]^{-a-1}, y > 0. \quad (4)$$

The PDF and CDF plots of the MEC distribution are displayed in Figure 2. The density plots for various parameter values exhibit different kinds of shape, most of the plots are uni-modal in shape with different degrees of kurtosis. For the various parameter values that were used, it can be observed that the density plots exhibits right skewed shape. Also the CDF of the addition for some parameter values the density exhibits a reverse J shape. Also, for the CDF of the MEC distribution for different parameter values, it can be observed that for some parameter values, as y gets closer to zero the CDF approaches zero and as y approaches infinity the CDF approaches one. The survival function, hazard function

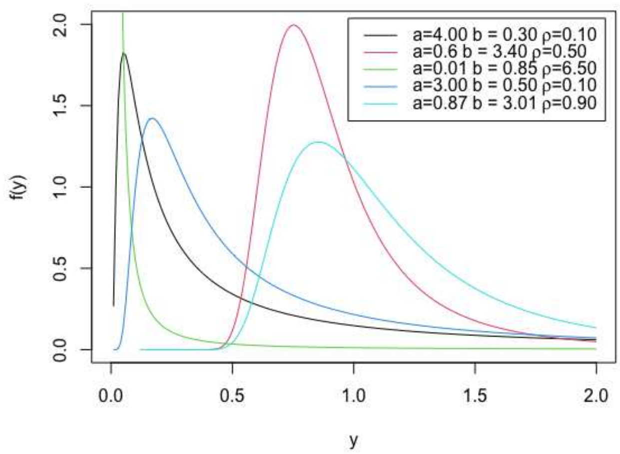


Fig. 1: PDF plots the MEC distribution

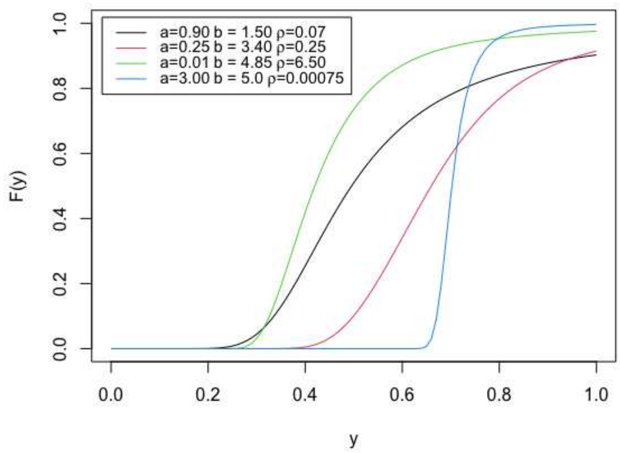


Fig. 2: CDF plots of the MEC distribution.

and reversed hazard rate (RHR) of the MEC distribution are given by Equations (5), (6) and (7) respectively;

$$S(y) = 1-\left(\rho\left(e^{y^{-b}}-1\right)+1\right)^{-a}, y > 0, \quad (5)$$

$$h(y) = \frac{f(y)}{S(y)} = \frac{ab\rho y^{-b-1}e^{y^{-b}}\left[\rho\left(e^{y^{-b}}-1\right)+1\right]^{-a-1}}{1-\left(\rho\left(e^{y^{-b}}-1\right)+1\right)^{-a}}, y > 0, \quad (6)$$

$$RHR(y) = \frac{f(y)}{F(y)} = \frac{ab\rho e^{y^{-b}} y^{-b-1}}{\rho\left(e^{y^{-b}}-1\right)+1}, y > 0. \quad (7)$$

Figures 3 and 4 exhibits the hazard and reversed hazard function plots of the MEC distribution for different parameter values. It can be observed that in Figure **??** that the hazard function exhibits a unimodal and reverse J shape. The RHR function decays faster towards 2.0 on the axis. Also, it can be clearly observed that the RHR exhibits an upside down bathtub feature.

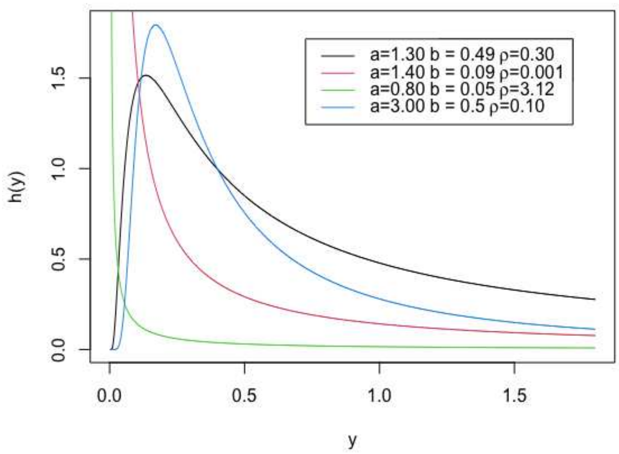


Fig. 3: The hazard function plots of the MEC distribution

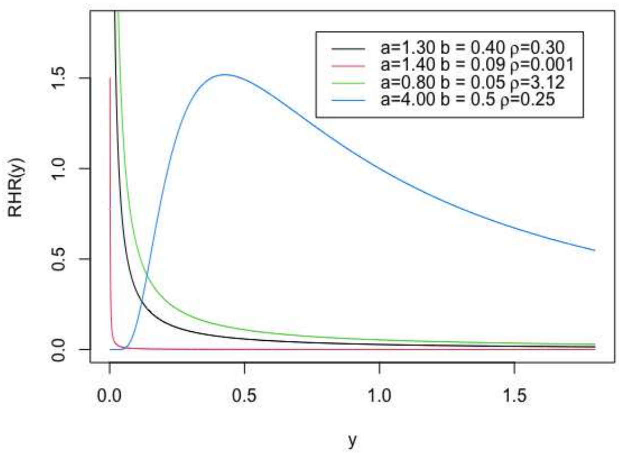


Fig. 4: The reversed hazard function plots of the MEC distribution.

3 Mixture Representation

We introduce an alternative representation for the PDF of the MEC distribution.

Lemma 31 *The density function of the MEC distribution can be expressed in a mixture form as*

$$f(y) = ab\rho \sum_{i=0}^{\infty} \sum_{j=0}^{0} \varpi_{ij} y^{-b-1} e^{-(j-i-1)y^{-b}}, \quad (8)$$

where $\varpi_{ij} = \frac{(-1)^{i+j}\rho^i (a+1)^{(i)}}{i} \binom{i}{j}$.

Proof. Given the density function

$$f(y) = ab\rho y^{-b-1} e^{y^{-b}} \left[\rho\left(e^{y^{-b}}-1\right)+1\right]^{-a-1} \quad y > 0.$$

Using the Taylor series expansion,

$$Z^{-\alpha} = \sum_{i=0}^{\infty} \frac{(-1)^i \alpha^{(i)}}{i!} (Z-1)^i,$$

where $\alpha^{(i)} = \alpha(\alpha+1)(\alpha+2)....(\alpha+i-1)$ is the rising factorial. Thus,

$$\left[1+\rho(e^{y^{-b}}-1)\right]^{-(a+1)} = \sum_{i=0}^{\infty} \frac{(-1)^i (a+1)^{(i)}}{i!} \rho^i (e^{y^{-b}}-1)^i.$$

Further,

$$\sum_{i=0}^{\infty} \frac{(-1)^i (a+1)^{(i)}}{i!} \rho^i (e^{y^{-b}}-1)^i = \sum_{i=0}^{\infty} \frac{(-1)^i \rho^i (a+1)^{(i)}}{i!} e^{iy^{-b}} (1-e^{-y^{-b}})^i.$$

Employing the fact that $0 < (1-e^{-y^{-b}})^i < 1$ and applying the binomial expansion, we have

$$\sum_{i=0}^{\infty} \frac{(-1)^i (a+1)^{(i)}}{i!} \rho^i (e^{y^{-b}}-1)^i = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \frac{(-1)^{i+j} \rho^i (a+1)^{(i)}}{i!} \binom{i}{j} e^{-(j-i)y^{-b}}.$$

The density function can therefore be written as

$$f(y) = ab\rho \sum_{i=0}^{\infty} \sum_{j=0}^{i} \frac{(-1)^{i+j} \rho^i (a+1)^{(i)}}{i!} \binom{i}{j} y^{-b-1} e^{-(j-i-1)y^{-b}}.$$

This completes the proof.

4 Statistical Properties

This section provides some mathematical properties of the MEC distribution.

4.1 Quantile Function

Given a random variable Y with continuous and strictly monotonic PDF $f(y)$, a quantile function $Q(f)$ assigns to each probability p attained by f the value y for which $Pr(Y \leq y) = u$. Expressed mathematically as,

$$Q_f(u) = \{y : Pr(Y \leq y) = u\}, \quad (9)$$

where $u \in (0,1)$.

Proposition 41 *Given a random variable Y with CDF $F(y;a,b,\rho)$, The quantile function for the MEC distribution is given by*

$$Q(u)=\left[\log\left(\frac{u^{-\frac{1}{a}}-1}{\rho}+1\right)\right]^{-\frac{1}{b}},\quad 0<u<1. \tag{10}$$

*Proof.*Suppose the random variable U follows the standard uniform distribution then $0<u<1$.
Let $u=\left[1+\left(e^{y^{-a}}-1\right)\rho\right]^{-b}$. Solving for y yields the quantile function of the MEC distribution.

4.2 Moments

Through the applications of moments, important features such as measures of central tendency, dispersion, skewness and kurtosis of a distribution can be derived and studied.

Proposition 42 *The r^{th} moment of the MEC distributed random variable Y is given as*

$$\mu_r^{'}=a\rho\sum_{i=0}^{\infty}\sum_{j=0}^{i}\varpi_{ij}\frac{\Gamma(1-\frac{r}{b})}{(j-i-1)^{1-\frac{r}{b}}},r<b,r=1,2,\ldots, \tag{11}$$

where $\Gamma(\beta)=\int_0^{\infty}t^{\beta-1}e^{-t}dt$ is the gamma function.

Proof. By definition, the r^{th} non-central moment is given by

$$\mu_r^{'}=\int_0^{\infty}y^r f(y)dy$$

Using the mixture representation of the PDF in Lemma (31), we have

$$\mu_r^{'}=ab\rho\sum_{i=0}^{\infty}\sum_{j=0}^{i}\varpi_{ij}\int_0^{\infty}y^{r-b-1}e^{-(j-i-1)y^{-b}}dy.$$

Let $u=(j-i-1)y^{-b}$. As $y\to 0,u\to\infty$ and as $y\to\infty,u\to 0$. Also, $y=\left(\frac{u}{j-i-1}\right)^{-1/b}$ and $dy=\frac{-du}{b(j-i-1)y^{-b-1}}$.
Thus,

$$\mu_r^{'}=a\rho\sum_{i=0}^{\infty}\sum_{i=0}^{j}\varpi_{ij}\frac{\Gamma(1-\frac{r}{b})}{(j-i-1)^{1-\frac{r}{b}}},r<b.$$

This completes the proof.

4.3 Incomplete Moment

Incomplete moments play a vital role in the field of economics such as the income quintiles, the Lorenz curve, Pietra and Gini measures of inequality.

Proposition 43 *The incomplete moment of the MEC distribution is*

$$m_r(y)=a\rho\sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{\varpi_{ij}\gamma(1-\frac{r}{b},(j-i-1)y^{-b})}{(j-i-1)^{1-\frac{r}{b}}},r<b,r=1,2,\ldots, \tag{12}$$

where $\gamma(s,x)=\int_x^{\infty}t^{s-1}e^{-t}dt$ is the lower incomplete gamma function.

Proof. The incomplete moment of a random variable is defined as

$$m_r(y)=\int_0^{y}x^r f(x)dx.$$

Thus, substituting the mixture representation of the density function of the MEC distribution into the definition of the incomplete moment, we have

$$m_r(y)=ab\rho\sum_{i=0}^{\infty}\sum_{j=0}^{i}\varpi_{ij}\int_0^{y}x^{r-b-1}e^{-(j-i-1)x^{-b}}dx.$$

Let $u=(j-i-1)x^{-b}$. As $x\to 0,u\to\infty$ and as $x\to y,u\to(j-i-1)y^{-b}$. Also, $x=\left(\frac{u}{j-i-1}\right)^{-1/b}$ and $dx=\frac{-du}{b(j-i-1)x^{-b-1}}$. Hence,

$$m_r(y)=a\rho\sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{\varpi_{ij}\gamma(1-\frac{r}{b},(j-i-1)y^{-b})}{(j-i-1)^{1-\frac{r}{b}}},r<b. \tag{13}$$

This completes the proof.

4.4 Inequality Measures

Lorenz and Bonferroni curves are income inequality measures that are widely useful and applicable to some other areas including reliability, demography, medicine and insurance [13].

Proposition 44 *The Lorenz curve for a random variable having the MEC distribution is,*

$$L_F(y)=\frac{a\rho}{\mu}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{\varpi_{ij}\gamma(1-\frac{1}{b},(j-i-1)y^{-b})}{(j-i-1)^{1-\frac{1}{b}}},\quad b>1. \tag{14}$$

Proof. The Lorenz curve of the distribution of a random variable is defined as

$$L_F(y) = \frac{1}{\mu}\int_0^y xf(x)dx.$$

From the definition, the Lorenz curve is simply the product of the first incomplete moment and the reciprocal of the mean of the random variable. Hence,

$$L_F(y) = \frac{a\rho}{\mu}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{\varpi_{ij}\gamma(1-\frac{1}{b},(j-i-1)y^{-b})}{(j-i-1)^{1-\frac{1}{b}}},\ b>1.$$

The proof is therefore complete.

Proposition 45 *The Bonferroni curve of a random variable having the MEC distribution is*

$$B_F(y) = \frac{a\rho}{\mu F(y)}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{\varpi_{ij}\gamma(1-\frac{1}{b},(j-i-1)y^{-b})}{(j-i-1)^{1-\frac{1}{b}}},\ b>1. \quad (15)$$

Proof. The Bonferroni curve by definition is given as

$$B_F(y) = \frac{L_F(y)}{F(y)}.$$

Thus, substituting the Lorenz curve into the definition of the Bonferroni curve completes the proof.

4.5 Moment Generating Function

The moment generating function if it exist is a special functions used to find the moments.

Proposition 46 *Given a random variable Y having the MEC distribution, the moment generating function for the MEC distribution is given by*

$$M_Y(t) = a\rho\sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{t^r\varpi_{ij}}{r!(j-i-1)^{1-\frac{r}{b}}}\Gamma\left(1-\tfrac{r}{b}\right), r<b, \quad (16)$$

where $\Gamma(\beta) = \int_0^{\infty} t^{\beta-1}e^{-t}dt$ *is the gamma function.*

Proof. The MGF of Y is defined as,

$$M_Y(t) = E(e^{tY}) = \int_{-\infty}^{\infty} e^{ty}f(y)dy.$$

Using Taylor series expansion, the MGF can be rewritten as,

$$M_Y(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r'$$

substituting the r^{th} moment in Proposition (1.2) yields

$$M_Y(t) = a\rho\sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{t^r\varpi_{ij}}{r!(j-i-1)^{1-\frac{r}{b}}}\Gamma\left(1-\tfrac{r}{b}\right).$$

The proof is therefore complete.

4.6 Entropy

The entropy of a random variable Y is a measure of variation of uncertainty. Here, we derive the explicit form of the Rényi entropy for the MEC distribution .

Proposition 47 *Given a random variable Y having the MEC distribution, then the Rényi entropy of the MEC distribution is given by,*

$$(1-\psi)^{-1}\log\left\{(a\rho b)^{\psi}\sum_{i=0}^{\infty}\sum_{i=0}^{j}\eta_{ij}\frac{\Gamma\left(\frac{\psi-1}{b}+\psi\right)}{b(j-i-\psi)^{\frac{\psi-1}{b}+\psi}}\right\}, \quad (17)$$

where $\psi>0,\ \psi\neq 0$ *and* $\eta_{ij} = \frac{(-1)^{i+j}\psi^{(i)}\rho^{i}(a+1)^{(i)}}{i}\binom{i}{j}$.

Proof. [15] proposed the Rényi entropy which is denoted by $I_R(\psi)$ for Y is defined as,

$$I_R(\psi) = (1-\psi)^{-1}\log\left\{\int_0^{\infty} f(y)^{\psi}dy\right\}, \quad (18)$$

where $\psi>0$ and $\psi\neq 0$. Substituting the density of the MEC distribution into equation (18), we obtain

$$I_R(\psi) = (1-\psi)^{-1}\log\left\{\int_0^{\infty}\left[ab\rho y^{-b-1}e^{y^{-b}}\left[\rho\left(e^{y^{-b}}-1\right)+1\right]^{-a-1}\right]^{\psi}dy\right\}.$$

Let

$$v(y) = \int_0^{\infty}\left[ab\rho y^{-b-1}e^{y^{-b}}\left[\rho\left(e^{y^{-b}}-1\right)+1\right]^{-a-1}\right]^{\psi}dy,$$

applying the approach for expanding the density function in Lemma (1.1),

$$v(y) = \int_0^{\infty}(ab\rho)^{\psi}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\varpi_{ij}y^{\psi(-b-1)}e^{-(j-i-\psi)y^{-b}}dy,$$

where $\eta_{ij} = \frac{(-1)^{i+j}\psi^{(i)}\rho^{i}(a+1)^{(i)}}{i}\binom{i}{j}$. Let $u=(j-i-\psi)y^{-b}$. As $y\to 0$, $u\to\infty$ and as $y\to\infty$, $u\to 0$. Also, $y = \left(\frac{u}{j-i-\psi}\right)^{-1/b}$ and $dy = \frac{-du}{b(j-i-\psi)y^{-b-1}}$. Thus,

$$v(y) = (a\rho b)^{\psi}\sum_{i=0}^{\infty}\sum_{i=0}^{j}\eta_{ij}\frac{\Gamma\left(\frac{\psi-1}{b}+\psi\right)}{b(j-i-\psi)^{\frac{\psi-1}{b}+\psi}}.$$

Hence,

$$I_R(\psi) = (1-\psi)^{-1}\log\left\{(a\rho b)^{\psi}\sum_{i=0}^{\infty}\sum_{i=0}^{j}\eta_{ij}\frac{\Gamma\left(\frac{\psi-1}{b}+\psi\right)}{b(j-i-\psi)^{\frac{\psi-1}{b}+\psi}}\right\},$$
$$\psi>0 \text{ and } \psi\neq 0.$$

The proof is therefore complete.

4.7 Stress-Strength Reliability

Stress-strength models are of importance in the field of reliability analysis and engineering applications [6]. The reliability is defined as the probability that the unit's strength Y_1 is greater than the stress Y_2, the unit fails when $Y_1 \leq Y_2$. The estimation of the stress-strength reliability of the component R is $P(Y_2 < Y_1)$ [16].

Proposition 48 *Suppose Y_1 and Y_2 follows the MEC distribution, then the reliability of the component is given by,*

$$f(y) = ab\rho \sum_{i=0}^{\infty}\sum_{j=0}^{i} \Pi_{ij} \frac{1}{(j-i-1)^{1-\frac{1}{b}}},$$

where $\Pi_{ij} = \dfrac{(-1)^{i+j}\rho(2a+1)^{(i)}}{i}\dbinom{i}{j}$.

Proof. The stress-strength reliability measure of a component is defined as,

$$R = \int_0^{\infty} f(y)F(y)dy.$$

Substituting the PDF and CDF of the MEC distribution we have,

$$R = ab\rho \int_0^{\infty} y^{-b-1}e^{y^{-b}}\left[\rho\left(e^{y^{-b}}-1\right)+1\right]^{-(2a+1)} dy$$

using similar concept in lemma (31),

$$R = ab\rho \sum_{i=0}^{\infty}\sum_{j=0}^{i} \varpi_{ij} \int_0^{\infty} y^{-b-1}e^{-(j-i-1)y^{-b}}dy.$$

Let $u = (j-i-1)y^{-b}$. As $y \to 0, u \to \infty$ and as $y \to \infty, u \to 0$. Also, $y = \left(\dfrac{u}{j-i-1}\right)^{-1/b}$ and $dy = \dfrac{-du}{b(j-i-1)y^{-b-1}}$. Hence,

$$R = ab\rho \sum_{i=0}^{\infty}\sum_{j=0}^{i} \varpi_{ij} \frac{1}{(j-i-1)^{1-\frac{1}{b}}}.$$

This completes the proof.

4.8 Order Statistics

Let $Y_{(1)}, Y_{(2)}, \ldots, Y(n)$ denote the order statistics of a random sample $Y_1, Y_2, \ldots, Y_n$ from a continuous population with CDF $F_Y(y)$ and PDF $f_Y(y)$, then the PDF of the p^{th} order statistics Y_p is given by,

$$f_{p:n} = U_{r:n}[F(y)]^{p-1}[1-F(y)]^{n-p} f(y),$$

for $r = 1,2,\ldots,n$, where $U_{r:n} = \frac{n!}{(p-1)!(n-p)!} = [B(p,n-p+1)]^{-1}$ is the beta function.

The PDF of the p^{th} order MEC random variable $Y_{(p)}$ is given by

$$f_{p:n}(y) = U_{r:n}\left[\left[\rho\left(e^{y^{-b}}-1\right)+1\right]^{-a}\right]^{p-1}\left[1-\left[\rho\left(e^{y^{-b}}-1\right)+1\right]^{-a}\right]^{n-p} \times ab\rho y^{-b-1}e^{y^{-b}}\left[\rho\left(e^{y^{-b}}-1\right)+1\right]^{-a-1}. \quad (19)$$

Therefore, the PDF of the n^{th} MEC statistic $Y_{n:p}$ is given by,

$$f_{n:p}(y) = n\left[\left[\rho\left(e^{y^{-b}}-1\right)+1\right]^{-a}\right]^{n-1} ab\rho y^{-b-1}e^{y^{-b}} \left[\rho\left(e^{y^{-b}}-1\right)+1\right]^{-a-1}$$

and the PDF of the first MEC statistic $Y_{1:p}$ is given by,

$$f_{1:p}(y) = n\left[1-\left[\rho\left(e^{y^{-b}}-1\right)+1\right]^{-a}\right]^{n-1} ab\rho y^{-b-1}e^{y^{-b}} \left[\rho\left(e^{y^{-b}}-1\right)+1\right]^{-a-1}.$$

4.9 Stochastic Ordering

Proposition 49 *Suppose Y_1 follows the MEC distribution with parameters a_1, b, ρ and Y_2 also follows the MEC distribution with parameters a_2, b, ρ. Then Y_2 is less than Y_1 in likelihood ratio order ($Y_2 \leq_{lr} Y_1$).*

Proof. The density functions of Y_1 and Y_2 are respectively given by

$$f_{Y_1}(y) = a_1 b\rho y^{-b-1}e^{y^{-b}}\left[1+\rho(e^{y^{-b}}-1)\right]^{-(a_1+1)}, y > 0,$$

and

$$f_{Y_2}(y) = a_2 b\rho y^{-b-1}e^{y^{-b}}\left[1+\rho(e^{y^{-b}}-1)\right]^{-(a_2+1)}, y > 0.$$

The ratio of the two densities is

$$\frac{f_{Y_1}(y)}{f_{Y_2}(y)} = \left(\frac{a_1}{a_2}\right)\left[1+\rho(e^{y^{-b}}-1)\right]^{a_2-a_1}, y > 0.$$

The derivative of the ratio of the densities is

$$\frac{d}{dy}\left(\frac{f_{Y_1}(y)}{f_{Y_2}(y)}\right) = (a_2-a_1)\left(\frac{a_1}{a_2}\right)b\rho y^{-b-1}e^{y^{-b}} \times \left[1+\rho(e^{y^{-b}}-1)\right]^{a_2-a_1-1}.$$

Hence, if $a_1 < a_2$, then

$$\frac{d}{dy}\left(\frac{f_{Y_1}(y)}{f_{Y_2}(y)}\right) > 0.$$

This implies that $Y_2 \leq_{lr} Y_1$ and this completes the proof.

5 Parameter Estimation

In this section we obtain estimates of parameters for the MEC distribution by employing three estimation approaches namely the maximum likelihood estimation (MLE) method, method of Least squares (OLS) and Weighted least squares (WLS). We have considered three types of estimation methods such as the MLE , OLS and WLS.

5.1 Maximum Likelihood Estimation

The method of maximum likelihood estimation in the context of the MEC is described in this subsection. We consider $y_1, y_2, y_3, \ldots, y_n$ as random sample from the MEC distribution. Then the likelihood function of the MEC is given by,

$$L(y;a,b,\rho) = a^n b^n \rho^n \prod_{i=1}^{n} y_i^{-b-1} e^{y_i^{-b}} \left[\rho\left(e^{y_i^{-b}}-1\right)+1\right]^{-a-1}.$$

The log likelihood equation is given by,

$$\begin{aligned}\ln L(y;a,b,\rho) =& n\ln a + n\ln b + n\ln\rho + (-a-1)\\ &\times \ln\sum_{i=1}^{n}\left[\rho\left(e^{y_i^{-b}}-1\right)+1\right]\\ &+(-b-1)\ln\sum_{i=1}^{n} y_i + \sum_{i=1}^{n} y_i^{-b}.\end{aligned} \tag{20}$$

Differentiating equation (20) with respect to a, b and ρ in turn and equating to zero, we obtain the equations,

$$\frac{\partial \ln L(y;a,b,\rho)}{\partial a} = \frac{n}{a} - \ln\sum_{i=1}^{n}\left[\rho\left(e^{y_i^{-b}}-1\right)+1\right] = 0, \tag{21}$$

$$\begin{aligned}\frac{\partial \ln L(y;a,b,\rho)}{\partial b} =& \frac{n}{b} + (-a-1)\\ &\times \sum_{i=1}^{n} \frac{-e^{y_i^{-b}} y^{-b} \ln y_i \rho\left(e^{y_i^{-b}}-1\right)}{\left(e^{y_i^{-b}}-1\right)+1}\\ &-\sum_{i=1}^{n}\ln y_i - \sum_{i=1}^{n} y_i^{-b}\ln(-y_i) = 0,\end{aligned} \tag{22}$$

$$\begin{aligned}\frac{\partial \ln L(y;a,b,\rho)}{\partial \rho} =& \frac{n}{\rho} + (-a-1)\\ &\times \sum_{i=1}^{n} \frac{e^{y_i^{b}}}{\rho\left(e^{y_i^{-b}}-1\right)+1} = 0.\end{aligned} \tag{23}$$

The estimators of the MEC distributions can be obtained by solving equation (21), (22) and (23) in relation to a, b and ρ simultaneously using a numerical procedure.

5.2 Ordinary and Weighted Least Squares

The OLS and WLS estimators are used generally for estimation of parameters in linear models. These estimates were used by [17] to estimate parameters of a beta distribution. Suppose $F(Y_{(j)})$ denotes the MEC distribution function of the ordered random variables $Y_{(1)} < Y_{(2)} < \cdots < Y_{(n)}$ where $\{Y_1, Y_2, \ldots, Y_n\}$ is a random sample of size n from a distribution function $F(y;a,b,\rho)$. Therefore, the least-square estimators of a, b and ρ, say $\hat{a}_{OLS}$ $\hat{b}_{OLS}$ and $\hat{\rho}_{OLS}$ of the MEC distribution can be obtained by minimizing

$$OLS(a,b,\rho) = \sum_{i=1}^{n}\left[F\left(y_{i:n}|a,b,\rho\right) - \frac{i}{n+1}\right]^2, \tag{24}$$

with respect to a, b, and ρ where $F\left(y_{i:n}|a,b,\rho\right)$ is the CDF of the MEC distribution. Partially differentiating equation (24) and equating to zero with respect to a, b, and ρ yields,

$$\begin{aligned}\frac{\partial OLS(a,b,\rho)}{\partial a} =& 2\sum_{i=1}^{n}\left[\left(\rho\left(e^{y_{i:n}^{-b}}-1\right)+1\right)^{-a} - \frac{i}{n+1}\right]\\ &\Omega_1\left(y_{i:n},|a,b,\rho\right) = 0,\end{aligned} \tag{25}$$

$$\begin{aligned}\frac{\partial OLS(a,b,\rho)}{\partial b} =& 2\sum_{i=1}^{n}\left[\left(\rho\left(e^{y_{i:n}^{-b}}-1\right)+1\right)^{-a} - \frac{i}{n+1}\right]\\ &\Omega_2\left(y_{i:n},|a,b,\rho\right) = 0,\end{aligned} \tag{26}$$

$$\begin{aligned}\frac{\partial OLS(a,b,\rho)}{\partial \rho} =& 2\sum_{i=1}^{n}\left[\left(\rho\left(e^{y_{i:n}^{-b}}-1\right)+1\right)^{-a} - \frac{i}{n+1}\right]\\ &\Omega_3\left(y_{i:n},|a,b,\rho\right) = 0,\end{aligned} \tag{27}$$

where

$$\begin{aligned}\Omega_1\left(y_{i:n},|a,b,\rho\right) =& -\ln\left(\rho\left(e^{y_{i:n}^{-b}}-1\right)+1\right)\\ &\times\left(\rho\left(e^{y_{i:n}^{-b}}-1\right)+1\right)^{-a},\end{aligned} \tag{28}$$

$$\begin{aligned}\Omega_2\left(y_{i:n},|a,b,\rho\right) =& \rho a e^{y_{i:n}^{-b}} y^{-b}\ln\left(y_{i:n}\right)\\ &\times\left(\rho\left(e^{y_{i:n}^{-b}}-1\right)+1\right)^{-a-1},\end{aligned} \tag{29}$$

$$\begin{aligned}\Omega_3\left(y_{i:n},|a,b,\rho\right) =& -\rho - a\left(e^{y_{i:n}^{-b}}-1\right)\\ &\times\left(\rho\left(e^{y_{i:n}^{-b}}-1\right)+1\right)^{-a-1}.\end{aligned} \tag{30}$$

Solving equations (25-27) simultaneously we derive the ordinary least squares estimators of the MEC distribution. The WLS estimators of the MEC distribution, can be obtained by minimizing

$$\begin{aligned}WLS(a,b,\rho) =& \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\\ &\left[F\left(y_{i:n}|a,b,\rho\right) - \frac{i}{n+1}\right]^2\end{aligned} \tag{31}$$

The estimators can also be obtained by solving equations (29-31),

$$\frac{\partial WLS(a,b,\rho)}{\partial a} = 2\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \times \left[\left(\rho \left(e^{y_{i:n}^{-b}} - 1 \right) + 1 \right)^{-a} - \frac{i}{n+1} \right] \times \Omega_1(y_{i:n}, |a,b,\rho) = 0 \quad (32)$$

$$\frac{\partial WLS(a,b,\rho)}{\partial b} = 2\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \times \left[\left(\rho \left(e^{y_{i:n}^{-b}} - 1 \right) + 1 \right)^{-a} - \frac{i}{n+1} \right] \times \Omega_2(y_{i:n}, |a,b,\rho) = 0 \quad (33)$$

$$\frac{\partial WLS(a,b,\rho)}{\partial \rho} = 2\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \times \left[\left(\rho \left(e^{y_{i:n}^{-b}} - 1 \right) + 1 \right)^{-a} - \frac{i}{n+1} \right] \times \Omega_3(y_{i:n}, |a,b,\rho) = 0. \quad (34)$$

6 Simulation Study

In this section we present results of some numerical experiments to compare the performance of the different estimators. The main goal of the simulation is to compare the efficiency of the different estimation methods for the parameters of the MEC distribution. The experiment was performed with three different set of parameter values $(a,b,\rho) = (0.8,0.3,0.5), (0.4,8.2,6.5)$ and $(2.5,1.8,4.5)$. The simulation steps are:

i. Specify the value of the parameters a,b,ρ and sample size n.
ii. Generate random observations of size $n = 30,60,100,150,200,300,500$ from the MEC distribution using the quantile function.
iii. Compute MLE, OLS and WLS of parameters a,b and ρ according to section (5).
iv Replicate steps $ii-iii$ for $N = 10,000$ times.
v. To examine the performance of the estimators, we calculated the averages bias (AB) and root mean square error (RMSE) using the formulas,

$$AB = \frac{1}{N}\sum_{i=0}^{N}\left(\hat{\theta}_i - \theta\right) \qquad RMSE = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(\hat{\theta}_i - \theta\right)^2}.$$

The summary of the results using the Monte-Carlo simulation study is reported in Figures 3-20.
The results in Figures 5-10 shows the MLE, OLS and WLS estimators of parameter a. It can be observed that for the different estimators the obtained ABs tend to zero for large n and also, the values of RMSEs tend to zero. However, for some parameter values the ABs of the WLS and OLS estimator exhibits intermittent behaviors.
In addition, it can be observed that as the sample size grows lager the MLE decays faster compared to the other methods of estimation. It is important to note that the MLE, OLS and WLS estimators for the parameter a have smaller RMSE and AB compared to that of the other parameters. In the case of parameter a, the MLE shows better performance.

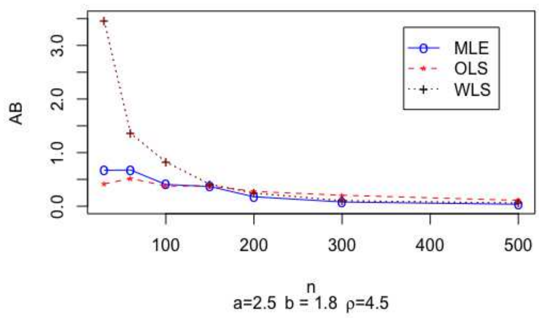


Fig. 5: Plots of the AB of the different estimators of a **for MLE, OLS, and WLS with different sample sizes** $30,60,100,150,200,300$ **and** 500

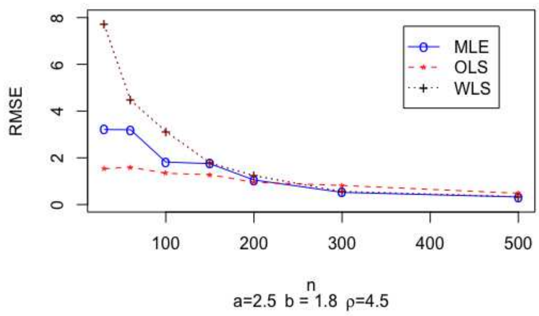


Fig. 6: Plots of the RMSE of the different estimators of a **for MLE, OLS, and WLS with different sample sizes** $30,60,100,150,200,300$ **and** 500

Figure 11-16 displays the numerical efficiency of the MLE, OLS and WLS estimators of parameter b. It can be observed that for the different estimators the obtained ABs and RMSEs decreases to zero as the sample increases. For parameter values $a = 2.5, b = 1.8, \rho = 4.5$ and $a = 0.4, b = 8.2, \rho = 6.5$ we observe that the RMSE of the WLS estimator shows exhibits and irregular behaviour. Also, for the different estimators, it can be noted that both the MLE and WLS estimators decays towards zero as sample size increases for both AB and

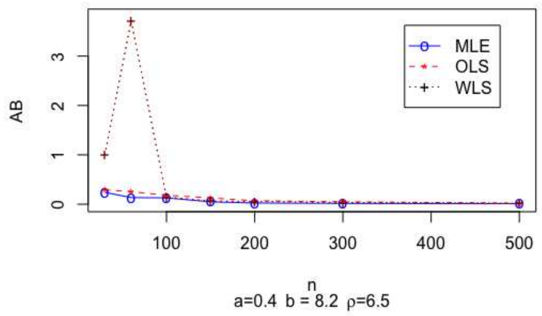


Fig. 7: Plots of the AB of the different estimators of a **for MLE, OLS, and WLS with different sample sizes** $30, 60, 100, 150, 200, 300$ **and** 500

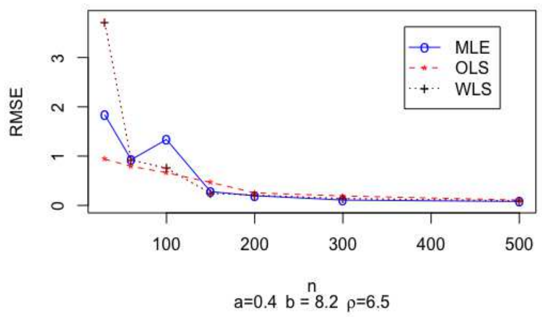


Fig. 8: Plots of the RMSE of the different estimators of a **for MLE, OLS, and WLS with different sample sizes** $30, 60, 100, 150, 200, 300$ **and** 500

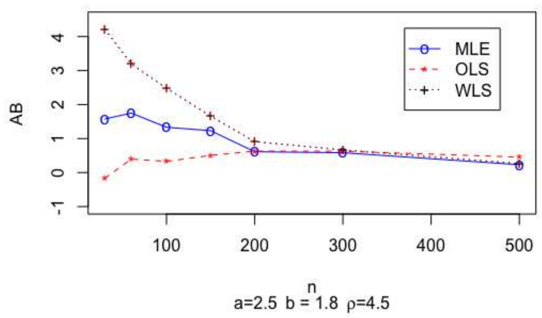


Fig. 9: Plots of the AB of the different estimators of a **for MLE, OLS, and WLS with different sample sizes** $30, 60, 100, 150, 200, 300$ **and** 500

RMSE. However, the MLE decays faster compared to the WLS and OLS estimators.

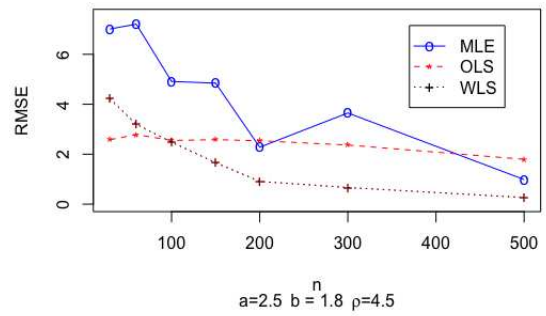


Fig. 10: Plots of the RMSE of the different estimators of a **for MLE, OLS, and WLS with different sample sizes** $30, 60, 100, 150, 200, 300$ **and** 500

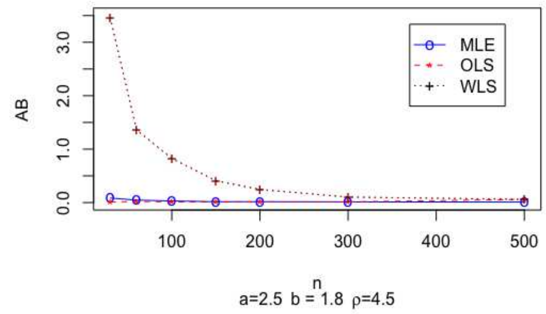


Fig. 11: Plots of the AB of the different estimators of b **for MLE, OLS, and WLS with different sample sizes** $30, 60, 100, 150, 200, 300$ **and** 500

Figure 17-22 displays the numerical efficiency of the MLE, OLS and WLS estimators of parameter ρ. It can be observed that for the different estimators, the obtained ABs and RMSEs decreases to zero as the sample increases. The MLE and the OLS estimators for parameter values $a = 0.4, b - 8.2, \rho = 6.5$ and $a = 2.5, b = 1.8, \rho = 4.5$ have, respectively, the largest AB and RMSE among all the considered estimators for parameter ρ. However, the AB and RMSE of the MLE estimator decays faster towards zero compared the other estimators as the sample size increases. Combining all results with the good properties of the MLE method, such as consistency, asymptotic efficiency, normality, and in-variance, we conclude that the MLE estimator is a highly competitive method compared to the OLS and WLS method for estimating the MEC distribution.

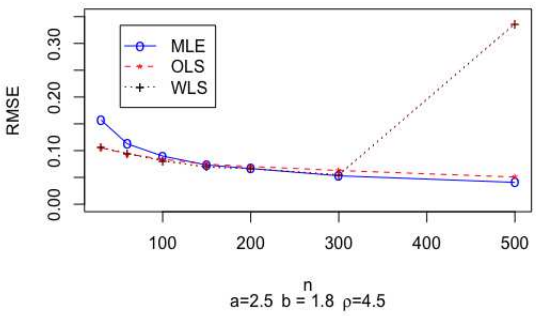


Fig. 12: Plots of the RMSE of the different estimators of b **for MLE, OLS, and WLS with different sample sizes** $30, 60, 100, 150, 200, 300$ **and** 500

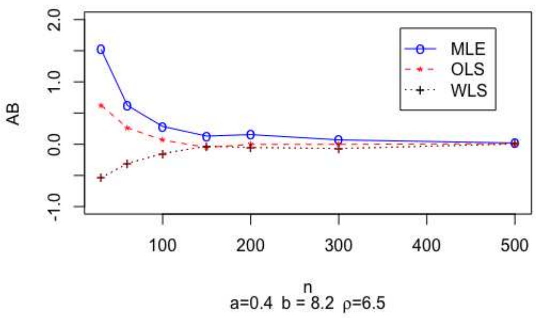


Fig. 13: Plots of the AB of the different estimators of b **for MLE, OLS, and WLS with different sample sizes** $30, 60, 100, 150, 200, 300$ **and** 500

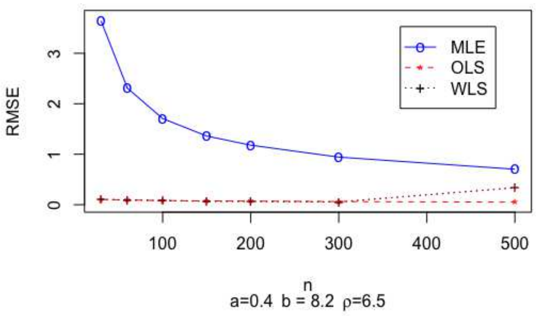


Fig. 14: Plots of the RMSE of the different estimators of b **for MLE, OLS, and WLS with different sample sizes** $30, 60, 100, 150, 200, 300$ **and** 500

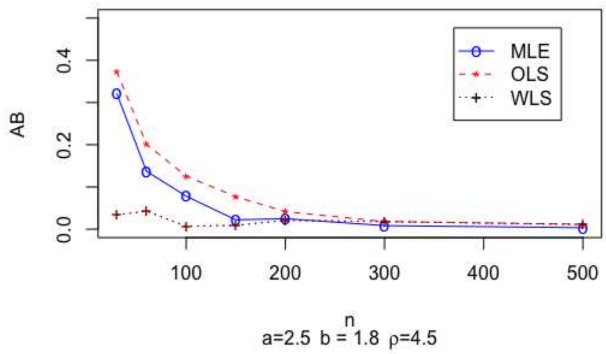


Fig. 15: Plots of the AB of the different estimators of b **for MLE, OLS, and WLS with different sample sizes** $30, 60, 100, 150, 200, 300$ **and** 500

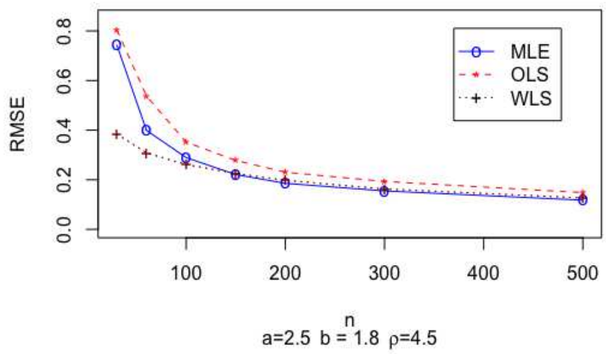


Fig. 16: Plots of the RMSE of the different estimators of b **for MLE, OLS, and WLS with different sample sizes** $30, 60, 100, 150, 200, 300$ **and** 500

7 Applications

To illustrate the application of the MEC model, annual maximum rainfall data in Ghana for three different locations (Greater-Accra, Eastern and Central Regions) were modeled using the MEC distribution. The performance of the MEC distribution is compared with other competing distributions such as Log-logistic (LL) [18], Inverse Weibull (IW) [19], Cauchy [20], Burr (3P) [1], Nakagami (NAK) [21], Gumbel (GUM) [22], Extended Burr XII (EB XII) [23] and Generalised Inverse Weibull (GIW) [24] for all maximum annual rainfall data from all three locations in Ghana.

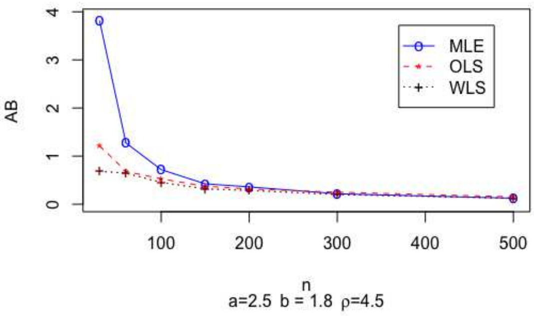


Fig. 17: Plots of the AB of the different estimators of ρ for MLE, OLS, and WLS with different sample sizes $30, 60, 100, 150, 200, 300$ **and** 500

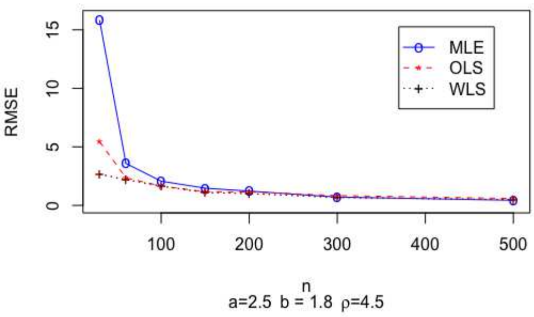


Fig. 18: Plots of the RMSE of the different estimators of ρ for MLE, OLS, and WLS with different sample sizes $30, 60, 100, 150, 200, 300$ **and** 500

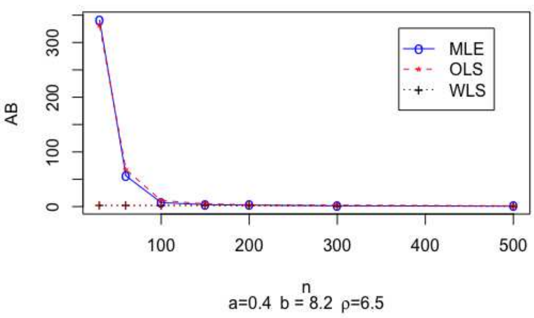


Fig. 19: Plots of the AB of the different estimators of ρ for MLE, OLS, and WLS with different sample sizes $30, 60, 100, 150, 200, 300$ **and** 500

In comparing the distributions, the best model is chosen as the one having the highest log-likelihood (L)

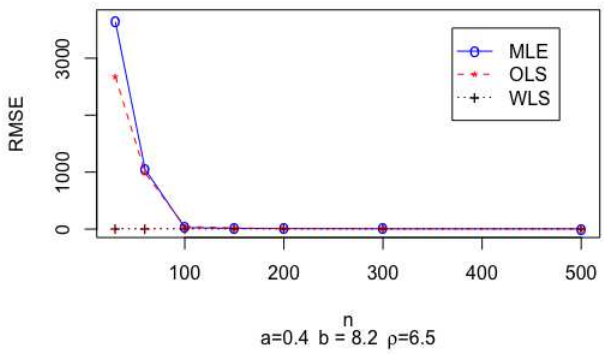


Fig. 20: Plots of the RMSE of the different estimators of ρ for MLE, OLS, and WLS with different sample sizes $30, 60, 100, 150, 200, 300$ **and** 500

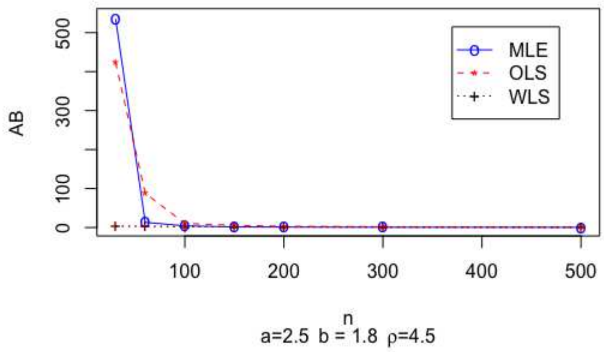


Fig. 21: Plots of the AB of the different estimators of ρ for MLE, OLS, and WLS with different sample sizes $30, 60, 100, 150, 200, 300$ **and** 500

and lowest AIC, BIC, CAIC and HQIC. The Anderson-Darling (A), Cramérvon Mises (W) and Kolmogorov-Smirnov (K-S) statistics for goodness-of-fit were also computed and used in the selection of the best distribution for the rainfall data from the different locations. The maximum likelihood estimates(MLEs) for the parameters of the fitted models are derived through maximizing the log-likelihood function using the *bbmle* package in R software [25].

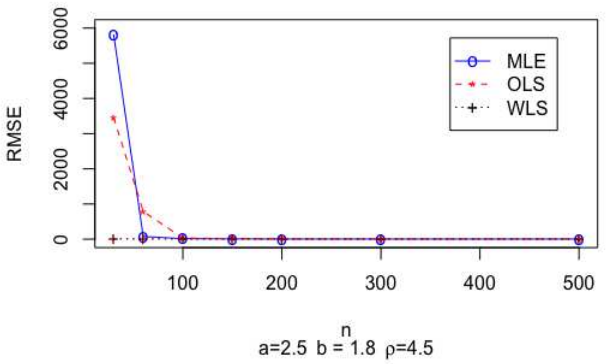


Fig. 22: Plots of the RMSE of the different estimators of ρ for MLE, OLS, and WLS with different sample sizes $30, 60, 100, 150, 200, 300$ **and** 500

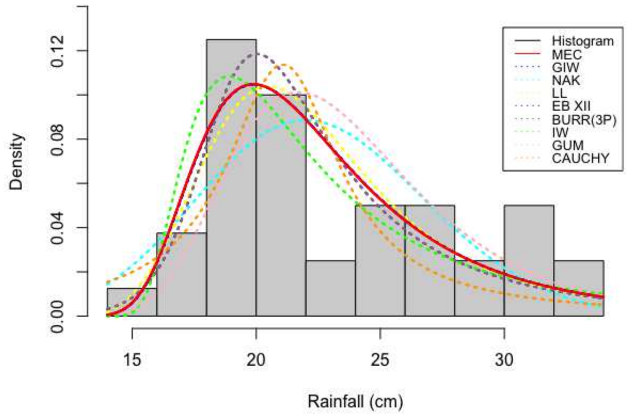


Fig. 23: The estimated PDFs for annual monthly rainfall maxima from Koforidua, Ghana.

7.1 Maximum Annual Rainfall Data

In this section, the annual maximum of the average monthly rainfall data from three locations in Ghana are considered for the period 1980-2019. Monthly average rainfall in centimeter(cm) data used in the study were retrieved from (https://www.globalclimatemonitor.org/) using geographic coordinates in simple decimal standard format of the locations.

7.1.1 Maximum Annual Rainfall Data-Koforidua

The first data represents the maximum annual rainfall of Koforidua in the Eastern region of Ghana. Table 1 displays the maximum likelihood estimates of the parameters and standard errors for the fitted distributions. Table 2 presents the log-likelihood, information criteria and goodness-of-fit statistics for the maximum annual rainfall for Koforidua data. It was obvious that, for the nine mentioned models, the MEC has the highest value of the L and lowest values of AIC, CAIC, BIC and HQIC. On the other hand, it can be observed that the MEC distribution has the smallest W, A and K-S value and the largest corresponding p-values. In light of this evidence it is clear that the MEC distribution provides a better fit to the rainfall data of Koforidua than the other eight candidate models.

In Figure 24, we show the estimated PDFs and estimated CDFs of all fitted distributions using the estimators obtained in Table 6. From Figure 6, it is noticed that the MEC distribution fits the maximum rainfall data for Koforidua better than the other eight models.

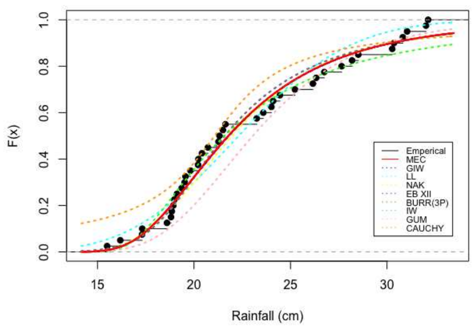


Fig. 24: The estimated CDFs for annual monthly rainfall maxima from Koforidua, Ghana.

7.1.2 Maximum Annual Rainfall Data-Accra

The second data represents the maximum annual rainfall of Accra in the Greater-Accra region of Ghana. Table 3 presents the parameter estimates and the corresponding standard errors. Table 4 shows the goodness of fit, log-likelihood and information criteria values for all nine fitted models.

The parameters of the fitted models along with their AIC, BIC, CAIC, HQIC, A, W and K-S statistics with p-value for the all nine data sets are presented respectively. From the findings, it is evident that, according to the lowest values of the AIC, BIC, CAIC, HQIC, A and W and highest p-values of the K-S, A and W statistic the proposed MEC distribution, could be chosen as a better model than proposed models namely

Table 1: MLEs of parameters and standard errors for maximum annual rainfall for Koforidua

Distribution	Parameters	Estimates	Standard Errors
MEC	$\hat{a}$	4.2719×10^{3}	4.2427×10^{-6}
	$\hat{b}$	5.7431	5.3614×10^{2}
	$\hat{p}$	7.9190×10^{3}	2.2882×10^{6}
LL	$\hat{a}$	3.3763	1.2553
	$\hat{b}$	8.8843	3.0661
	$\hat{p}$	13.0823	2.7824
IW	$\hat{a}$	11.4855	1.8517
	$\hat{b}$	2.3485	0.6112
	$\hat{p}$	0.0063	0.0115
CAUCHY	a	2.7973	0.6339
	$\hat{b}$	21.0703	0.7601
! BURR XII(3P)	$\hat{a}$	15.7030	6.4657
	$\hat{b}$	0.3139	0.1855
	$\hat{p}$	18.8988	1.1413
NAK	$\hat{a}$	525.3651	31.8421
	$\hat{b}$	6.3686	1.4068
GUM	$\hat{a}$	22.0768	0.9913
	$\hat{b}$	5.3473	1.1447
EB XII	$\hat{a}$	20.3492	1.0707
	$\hat{b}$	15.6814	6.4590
	$\hat{p}$	-3.1802	1.8810
GIW	$\hat{a}$	6.9339	0.8697
	$\hat{b}$	5.7430	0.6993
	$\hat{p}$	500.5086	0.0021

Table 2: Comparison criterion of maximum annual rainfall for Koforidua

Distribution	L	AIC	CAIC	BIC	HQIC	W	A	K-S
MEC	**-116.9126**	**239.8252**	**240.0752**	**244.8918**	**13.9890**	**0.0653 (0.7836)**	**0.4658 (0.7809)**	**0.0834 (0.9220)**
LL	-117.3418	240.6836	240.9336	245.7502	15.7045	0.0808 (0.6897)	0.5226 (0.7228)	0.0867 (0.8988)
IW	-119.3193	244.6386	244.8886	249.7052	15.8882	0.0741 (0.7297)	0.6737 (0.5802)	0.1211 (0.5599)
CAUCHY	-127.2068	258.4136	258.6636	261.7913	14.1557	0.3413 (0.1035)	2.2725 (0.0657)	0.1697 (0.1776)
BURR XII(3P)	-117.5054	241.0109	241.2609	246.0775	13.9990	0.08182 (0.684)	0.5510 (0.6944)	0.0910 (0.7919)
NAK	-117.9936	239.9872	240.2372	243.3650	18.7457	0.1327 (0.4486)	1.0388 (0.3371)	0.1109 (0.6679)
GUM	-120.1498	244.2996	253.5742	247.6773	14.1430	0.4612 (0.0495)	2.4763 (0.0513)	0.2128 (0.0455)
EB XII	-117.5054	241.0108	241.2608	246.0775	14.0001	0.0818 (0.6840)	0.5509 (0.6945)	0.0990 (0.7920)
GIW	-116.9129	239.8258	240.0758	244.8924	14.1890	0.0753 (0.7835)	0.5659 (0.7808)	0.0934 (0.9219)

Note: P-value in bracket.

LL, IW, CAUCHY, BURR XII(3P), NAK, GUM, EB XII and GIW. In regards to finding the best model, Figure 26

Table 3: MLEs of parameters and standard errors for maximum annual rainfall for Accra

Distribution	Parameter	Estimates	Standard Errors
MEC	$\hat{a}$	430.0393	300.4484
	$\hat{b}$	4.1352	0.4983
	$\hat{\rho}$	422.8530	301.5543
LL	$\hat{a}$	3.7939	2.3594
	$\hat{b}$	13.0129	7.6657
	$\hat{\rho}$	8.0263	7.1783
IW	$\hat{a}$	8.3647	0.0628
	$\hat{b}$	2.2029	0.2553
	$\hat{\rho}$	0.0064	0.0035
CAUCHY	$\hat{a}$	4.0669	0.7870
	$\hat{b}$	20.8291	1.2012
BURR XII(3P)	$\hat{a}$	96.8084	0.0001
	$\hat{b}$	0.0237	0.0038
	$\hat{\rho}$	13.7815	0.1725
NAK	$\hat{a}$	358.7447	36.0451
	$\hat{a}$	2.4417	0.7047
GUM	$\hat{a}$	20.5562	1.5824
	$\hat{b}$	8.3028	3.5855
EB XII	$\hat{a}$	14.4737	0.55482
	$\hat{b}$	71.74213	63.3608
	$\hat{\rho}$	-30.9269	27.7249
GIW	$\hat{a}$	9.5659	0.7313
	$\hat{b}$	4.1314	0.5018
	$\hat{\rho}$	15.9339	0.1063

displays the estimated PDFs and estimated CDFs of all fitted distributions using the estimates obtained in Table 4. From Figure 26 , it is noticed that the MEC distribution fits the maximum rainfall data for Accra better than the other eight models.

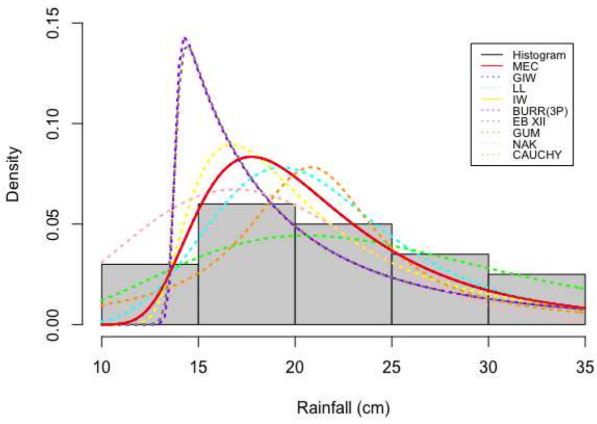


Fig. 25: The estimated PDFs for annual monthly rainfall maxima from Accra, Ghana.

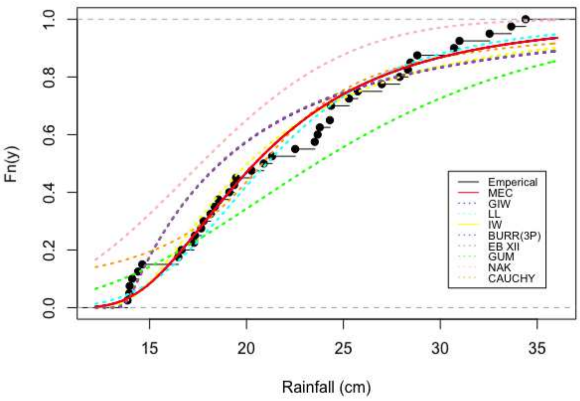


Fig. 26: The estimated CDFs for annual monthly rainfall maxima from Accra, Ghana.

7.1.3 Maximum Annual Rainfall Data-Cape Coast

The third data represents the maximum annual rainfall of Cape Coast in the Central region of Ghana. Table 5 displays the maximum likelihood estimates of the parameters, standard errors and p-values for the compared distributions.

Table 6 presents the log-likelihood, information criteria and goodness-of-fit statistics for the maximum annual rainfall for Cape Coast data.

It was obvious that, for the nine mentioned models, the MEC has the highest value of the L and lowest value of the AIC, CAIC, BIC and HQIC. On the other hand, it can be observed that the MEC distribution has the smallest W, A and K-S value and the largest corresponding p-values. This confirms that the MEC distribution appears to be a very competitive model to the maximum rainfall data of Koforidua than the other eight models.

In regards to finding the best model fit, findings are further validated graphically by providing plots of fitted densities with histogram and fitted CDFs with data in Figure 28 for the all nine models. These plots also indicate that the proposed distributions provide good fit to the data set considered here.

7.2 Return level estimation for MEC model

The return level can be defined as the level which is associated to be exceeded one every $1/S(y_T)$ period, where $S(y_T)$ is the exceedance probability which is obtained from the survival function, which is known as a return period. The return level of the MEC distribution is

Table 4: Comparison criterion of maximum annual rainfall for Accra

Distribution	-L	AIC	CAIC	BIC	HQIC	W	A	K-S
MEC	**-127.4893**	**260.9785**	**261.2285**	**266.0452**	**14.1601**	**0.083686 (0.6734)**	**0.65537 (0.5962)**	**0.12868 (0.5218)**
LL	-127.5129	261.0258	261.2758	266.0924	14.1605	0.1394 (0.4251)	1.012 (0.3505)	0.1339 (0.4702)
IW	-129.2508	264.5015	264.7515	269.5682	14.1872	0.1240 (0.4813)	0.96402 (0.3761)	0.1258 (0.5513)
CAUCHY	-137.5277	279.0554	279.3054	282.4332	14.3100	0.1676 (0.341)	1.3042 (0.2309)	0.16815 (0.2081)
BURR XII(3P)	-130.4125	266.8249	267.0749	271.8916	14.2049	0.45627 (0.05101)	2.3099 (0.0628)	0.2055 (0.0683)
NAK	-136.2867	276.5733	276.8233	279.9511	14.2921	1.4978 (0.0001404)	8.8262 (5.67×10^{-05})	0.2766 (0.004387)
GUM	-133.4749	270.9497	271.1997	274.3275	14.2508	0.4504 (0.0529)	2.6581 (0.04125)	0.1843 (0.1320)
EB XII	-130.7633	267.5267	267.7767	272.5933	14.2102	0.4701 (0.0470)	2.3595 (0.05907)	0.2075 (0.06385)
GIW	-127.491	260.9821	261.2321	266.0487	14.3601	0.0940 (0.6715)	0.6575 (0.5943)	0.2288 (0.5202)

Note: P-value in bracket.

Table 5: MLEs of parameters and standard errors for maximum annual rainfall for Cape Coast

Distribution	Parameter	Estimate	Standard Errors
MEC	$\hat{a}$	1638.8000	0.04417
	b	3.18997	0.39446
	$\hat{\rho}$	7.96994	8.99409
LL	$\hat{a}$	2.38744	0.72117
	$\hat{b}$	12.01639	3.30573
	$\hat{\rho}$	10.26471	2.55645
IW	$\hat{a}$	6.796415	3.477481
	$\hat{b}$	2.041814	0.677570
	$\hat{\rho}$	0.006030	0.013904
CAUCHY	$\hat{a}$	5.9170	1.1778
	$\hat{b}$	22.0349	1.6754
BURR XII(3P)	$\hat{a}$	15.09062	10.55420
	$\hat{b}$	0.14399	0.11896
	$\hat{\rho}$	14.88027	1.19011
NAK	$\hat{a}$	579.3763	57.7134
	$\hat{b}$	2.0012	0.4368
GUM	$\hat{a}$	21.8021	2.0737
	$\hat{b}$	11.9467	5.1737
EB XII	$\hat{a}$	16.9582	1.9213
	$\hat{b}$	14.8170	10.5008
	ρ	-6.8057	5.7172
GIW	$\hat{a}$	16.9582	1.9213
	$\hat{b}$	14.8170	10.5008
	$\hat{\rho}$	-6.8057	5.7172

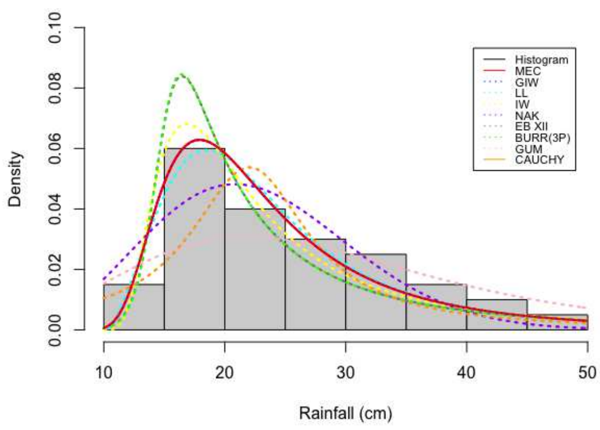


Fig. 27: The estimated PDFs for annual monthly rainfall maxima from Cape Coast, Ghana.

obtained using:

$$y_T = \left[\log\left(\frac{\left(1-\frac{1}{T}\right)^{-\frac{1}{a}}-1}{\rho}+1\right)\right]^{-\frac{1}{b}},$$

where $T \geq 1$ and the return period is:

$$T = \frac{1}{1-\left(\rho\left(e^{y_T^{-b}}-1\right)+1\right)^{-a}}.$$

Table 6: Comparison criterion of maximum annual rainfall for Cape Coast

Distribution	-L	AIC	CAIC	BIC	HQIC	W	A	K-S
MEC	**-141.7993**	**289.5986**	**289.8486**	**294.6652**	**14.3706**	**0.1086 (0.5464)**	**0.7033 (0.5551)**	**0.1118 (0.6581)**
LL	-142.1333	290.2666	290.5166	295.3333	14.3752	0.1123 (0.5298)	0.8420 (0.4510)	0.1457 (0.3315)
IW	-142.8083	291.6166	291.8666	296.6832	14.3846	0.1125 (0.5288)	0.7548 (0.5139)	0.1076 (0.7031)
CAUCHY	-153.6206	311.2412	311.4912	314.6190	14.5291	0.2294 (0.2173)	1.6749 (0.1399)	0.1807 (0.1294)
BURR XII(3P)	-143.3844	292.7689	293.0189	297.8355	14.6338	0.19757 (0.2728)	1.0894 (0.3132)	0.13682 (0.4059)
NAK	-143.8845	291.7691	292.0191	295.1468	14.39942	0.19602 (0.2759)	1.6196 (0.1505)	0.1314 (0.4564)
GUM	-148.5592	301.1184	301.3684	304.4962	14.4627	0.3619 (0.09089)	2.2971 (0.0638)	0.1498 (0.3000)
EB XII	-143.3840	292.7680	293.0180	297.8347	14.39252	0.1966 (0.2747)	1.0854 (0.3150)	0.1367 (0.4068)
GIW	-141.7994	289.5987	289.8488	294.6664	14.39082	0.1200 (0.5400)	0.7090 (0.5441)	0.1129 (0.6471)

Note: P-value in bracket.

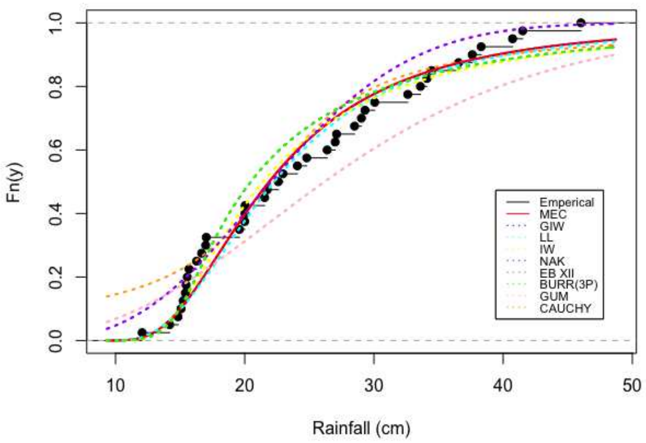


Fig. 28: The estimated CDFs for annual monthly rainfall maxima from Cape Coast, Ghana.

A summary of the return values for all location using the MEC distribution are presented in Table 7. To know the approximate range of maximum annual rainfall for various return periods in each location, the MEC distribution was applied to analyze rainfall locations. Considering a return period of 250 years, the minimum value is reached in Koforidua with an estimated value of 53.5054 *cm*; and the maximum value is obtained in Cape Coast reaching an extreme rainfall value of 110.2881 *cm*.

A summary of the periods are presented in Table 8. The estimated results indicate an increasing trend in the return periods as the return levels increases. Considering a minimum accumulated rainfall of 29 *cm* annually, the return period for Koforidua, Accra, and Cape Coast is approximately 8, 7, and 4 years, respectively. Also, with a maximum accumulated rainfall of 52 *cm* annually, the return period for Koforidua, Accra and Cape Coast is approximately 212, 69, and 23 annually, respectively. If this type of rainfall accumulation hits Cape Coast in the future, with an accumulated rainfall of up to 52 *cm* a year, Cape Coast will likely experience extreme rainfall with a return period of approximately 23 years.

8 Conclusion

In the present study, we have introduced a new modification of the Chen distribution called the MEC distribution using the Burr-Hatke differential equation. Some mathematical properties along with three different estimation approaches are addressed. Expressions for some of its statistical features including the quantile, generating functions, ordinary and incomplete moments, stochastic ordering, entropy and order statistics are derived. The MEC parameters are estimated by the

Table 7: Return level results of all locations

Location	Return Period (Years)								
	2	**5**	**10**	**20**	**50**	**100**	**150**	**200**	**250**
Koforidua	21.8135	26.5729	30.2822	34.3259	40.3720	45.5910	48.9404	51.4619	53.5054
Accra	20.4340	26.8812	32.2312	38.3604	48.0555	56.8941	62.7819	67.3188	71.0591
Cape Coast	21.8898	31.2411	39.5368	49.5569	66.3876	82.6498	93.9137	102.8133	110.2881

Table 8: Return period results of all locations

Location	Return Levels (cm)							
	29	**32.5**	**34.37**	**39.32**	**43.54**	**46.02**	**49**	**52**
Koforidua	7.9144	14.7491	20.1444	41.0859	76.8862	105.4971	151.0482	212.2815
Accra	6.64604	10.33124	12.8859	21.3793	33.4168	41.8891	54.1483	69.0908
Cape Coast	4.0586	5.5981	6.5882	9.5958	13.4127	15.9034	19.3105	23.2304

maximum likelihood, ordinary least squares and weighted least squares methods. Monte Carlo simulation results are reported for all the estimation methods. The proposed distribution provides a better fit than some other distributions by using three annual extreme rainfall data sets from three locations in Ghana. Finally, the return periods and levels were estimated using the MEC distribution for the three given locations in Ghana.

9 Acknowledgment

The authors would like to thank the anonymous referees for carefully examining the contents and providing valuable remarks that helped enhance this article.

Conflict of Interest The authors declare that they have no conflict of interest.

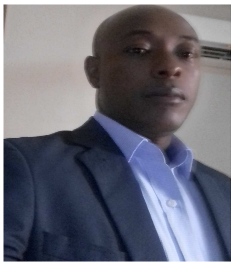
References

[1] I. W. Burr, Cumulative frequency functions, *The Annals of mathematical statistics*, **13**, 215-232 (1942).
[2] K. Pearson, On skew probability curves, *Nature*, **52**, 317, (1895).
[3] N. L. Johnson, Systems of frequency curves generated by methods of translation, *Biometrika*, **36**, 149-176, (1949).
[4] J. W. Tukey, *Exploratory data analysis*, Reading Mass., (1977).
[5] A. Azzalini, A class of distributions which includes the normal ones, *Scandinavian journal of statistics*, 171-178, (1985).
[6] N. Eugene, C. Lee and F. Famoye, Beta-normal distribution and its applications, *Communications in Statistics-Theory and methods*, **31**, 497-512, (2002).
[7] G. S. Mudholkar anfd D. K. Srivastava, Exponentiated Weibull family for analyzing bathtub failure-rate data, *IEEE transactions on reliability*, **42**, 299-302, (1993).
[8] A. Alzaatreh, C. Lee and F. Famoye, A new method for generating families of continuous distributions, *Metron*, **71**, 63-79, (2013).
[9] Z. Chen, A new two-parameter lifetime distribution with bathtub shape or increasing failure rate function, *Statistics & Probability Letters*, **49**, 155-161 (2000).
[10] M. Xie, Y. Tang, and T. N. Goh, A modified Weibull extension with bathtub-shaped failure rate function, *Reliability Engineering & System Safety*, **76**, 279-285 (2002).
[11] V. Pappas, K. Adamidis, and S. Loukas, A family of lifetime distributions, *Journal of Quality and Reliability Engineering*, (2012).
[12] Y. P. Chaubey, and R. Zhang, An extension of Chen's family of survival distributions with bathtub shape or increasing hazard rate function, *Communications in Statistics-Theory and Methods*, **44**, 4049-4064 (2015).
[13] C. Bonferroni, Elemente di statistica generale, (1930).
[14] F. A. Bhatti, A. Ali, G. G. Hamedani, and M. Ahmad, On the generalised log Burr III distribution: development, properties, characterizations and applications, *Pak. J. Statistics*, **35**, 25-51 (2019).
[15] A. Rényi, Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, *Contributions to the Theory of Statistics*, 547-561, (1961).
[16] S. Nasiru, *A New Generalization of Transformed-Transformer Family of Distributions*, PhD thesis, JKUAT, Kenya, (2017).
[17] J. J. Swain, S. Venkatraman and R. James, Least-squares estimation of distribution functions in Johnson's translation system, *Journal of Statistical Computation and Simulation*, **29**, 271-297, (1988).
[18] V. P. Singh, H. Guo and F. X. Yu, Parameter estimation for 3-parameter log-logistic distribution, *Stochastic Hydrology and Hydraulics*, **7**, 163-177, (1993).

[19] A. Z. Keller, A. R. Kamath and U. D. Perera, Reliability analysis of CNC machine tools, *Reliability engineering*, **3**, 449-473, (1982).
[20] R. P. Rider, The distribution of the product of ranges in samples from a rectangular population, *Journal of the American Statistical Association*, **48**, 546-549, (1953).
[21] M. Nakagami, The m-distribution—A general formula of intensity distribution of rapid fading, *Statistical methods in radio wave propagation*, 3-36, (1960).
[22] E. J. Gumbel, Les valeurs extrêmes des distributions statistiques, *Annales de l'institut Henri Poincaré*, **5**, 115-158, (1935).
[23] Q. Shao, H. Wong, J. Xia and W. Ip, Models for extremes using the extended three-parameter Burr XII system with application to flood frequency analysis, *Hydrological Sciences Journal*, **49**, (2004).
[24] F. R. S. Gusmao, R. S. Felipe, E. M. M. Ortega and M. Gauss, The generalized inverse Weibull distribution, *Statistical Papers*, **52**, 591-619, (2011).
[25] B. Bolker, Tools for general maximum likelihood estimation. R development core team, (2014).

Abdulzeid Yen Anafo is a Doctor of Philosophy candidate in Mathematics at the University of Mines and Technology, Tarkwa. He has BSc degree in Mathematical Sciences (Mathematics option) from the University for Development Studies and MSc. degree in Mathematical Sciences from the Africa Institute for Mathematical Sciences Ghana. His research interest are in the areas of distribution theory, statistical analytics and visualisation.

Lewis Brew is a senior lecturer and Head of mathematical sciences department at the University of Mines and Technology, Ghana. He received his PhD in Mathematics from the University of Mines and Technology, Ghana. His research areas are: Algebraic Topology, Statistical Analysis, Time Series Analysis and Linear Algebra.



Suleman Nasiru is an associate professor of statistics at the C. K. Tedam University of Technology and Applied Sciences, Ghana. He received his PhD in Mathematics (Statistics Option) from the Pan African University, Institute of Basic Sciences, Technology and Innovation, Kenya. He has published several research articles in several reputable international journals. His research interest are in the areas of Distribution theory, Extreme Value Analyses and Quality Control.