Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/080301

On Nonlinear Multi-Term Fractional Integro-Differential Equations with Anti-Periodic Boundary Conditions

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Received: 16 Sep. 2020, Revised: 18 Oct. 2020, Accepted: 21 Oct. 2020
Published online: 1 Jul. 2022

Abstract: We present the existence criteria for solutions of an anti-periodic boundary value problem involving a nonlinear multi-term Caputo-type fractional integro-differential equation with a Riemann-Liouville fractional integral operator. We also extend our study to the case of three different types of nonlinearities. Examples are constructed for demonstrating the application of the obtained results.

Keywords: Caputo derivative, Riemann-Liouville fractional integral, anti-periodic boundary conditions, existence, fixed point.

1 Introduction and formulation of the problem

Fractional-order differential and integral operators are found to be of great value as such operators appear extensively in the mathematical modeling of several problems of applied nature. The importance of these operators owes to their nonlocal nature in contrast to their integer-order counterparts. Unlike the classical derivative, the fractional derivatives can be defined in different ways. This fact led to the huge development of fractional calculus and its applications in diverse ares of natural and social sciences. One can find a theoretical description of the subject in the texts [1]-[4]. For application details, we refer the reader to the works presented in [5]-[8]. In particular, there has been shown a great interest in studying a variety of fractional-order boundary value problems, for instance, see [9]-[24]. In the survey paper [25], some interesting results on anti-periodic fractional-order boundary value problems were presented. In [26], fractional Langevin equation equipped with anti-periodic boundary conditions was studied. A new concept of dual anti-periodic boundary conditions was introduced and considered with fractional integro-differential equations in [27].

On the other hand, multi-term fractional differential equations are also important in view of their occurrence in many real world problems. Examples include Bagley–Torvik equation,[28], Basset equation [29], etc. For some interesting results on boundary value problems involving multi-term fractional equations, for instance, see [30]-[34].

In this paper, we discuss the existence of solutions for a nonlinear anti-periodic boundary value problem involving Caputo derivative operators of orders $\nu \in (1,2], \xi \in (1,\nu)$ and a Riemann-Liouville fractional integral of order $p > 0$ given by

$$\lambda_1\, {}^{C}D^{\nu}x(t) + \lambda_2\, {}^{C}D^{\xi}x(t) = f(t,x(t)) + I^{\varpi}g(t,x(t)), t \in [0,\mathscr{T}], \tag{1}$$

$$x(0) = -x(\mathscr{T}),\ x'(0) = -x'(\mathscr{T}), \tag{2}$$

where $\lambda_1, \lambda_2 \in \mathbb{R},\ \lambda_1 \neq 0,\ {}^{C}D^{\nu}$ and ${}^{C}D^{\xi}$ respectively denote the Caputo fractional derivative operators order ν and ξ, I^{ϖ} denotes Riemann-Liouville fractional integral of order $\varpi > 0$, $f,g : [0,\mathscr{T}] \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

The rest of the article is organized as follows. Some preliminary definitions and an auxiliary lemma associated with the linear variant of the problem (1)-(2) are given in Section 2. Section 3 contains the main results, while the case of mixed nonlinearities is discussed in Section 4. Examples illustrating the obtained results are discussed in Section 5. The paper concludes with some interesting observations.

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2 Auxiliary material

Let us first recall basic definitions of fractional calculus [1].

Definition 1. *The Caputo derivative of fractional order $q \in (n-1, n], n \in \mathbb{N}$ for a function $y \in AC^n[a,b]$ is defined by the following integral*

$$^C D^q y(t) = \frac{1}{\Gamma(n-q)} \int_a^t (t-r)^{n-q-1} y^{(n)}(r) dr, \quad t \in [a,b],$$

which exists almost everywhere on $[a,b]$.

Definition 2. *We define the Riemann-Liouville fractional integral of order $q > 0$ for $y \in L_1[a,b]$ as*

$$I^q y(t) = \frac{1}{\Gamma(q)} \int_a^t \frac{y(r)}{(t-r)^{1-q}} dr, \; t \in [a,b],$$

provided the above integral exists almost everywhere on $[a,b]$.

In the following lemma, we solve the linear variant of the problem (1)-(2).

Lemma 1. *For a given $\rho \in C([0,\mathscr{T}],\mathbb{R})$ and $\lambda_1 \neq 0$, the unique solution of the linear multi-term fractional differential equation*

$$\lambda_1 \, {}^C D^\nu x(t) + \lambda_2 \, {}^C D^\xi x(t) = \rho(t), \tag{3}$$

supplemented with the boundary conditions (2) is given by

$$x(t) = \frac{1}{\lambda_1} I^\nu \rho(t) - \frac{\lambda_2}{\lambda_1} I^{\nu-\xi} x(t) - \frac{1}{2\lambda_1} I^\nu \rho(\mathscr{T}) + \frac{\lambda_2}{2\lambda_1} I^{\nu-\xi} x(\mathscr{T}) + \frac{1}{4\lambda_1}(\mathscr{T}-2t)\left(I^{\nu-1}\rho(\mathscr{T}) - \lambda_2 I^{\nu-\xi-1} x(\mathscr{T})\right). \tag{4}$$

Proof. Applying the fractional integral operator I^ν to both sides of the fractional differential equation (3), we obtain

$$x(t) = \frac{1}{\lambda_1} I^\nu \rho(t) - \frac{\lambda_2}{\lambda_1} I^{\nu-\xi} x(t) - c_0 - c_1 t, \tag{5}$$

where c_0 and c_1 are unknown arbitrary constants. Differentiating (5) with respect to t leads to the following expression:

$$x'(t) = \frac{1}{\lambda_1} I^{\nu-1} \rho(t) - \frac{\lambda_2}{\lambda_1} I^{\nu-\xi-1} x(t) - c_1. \tag{6}$$

Inserting (5) and (6) in the conditions (2) and subsequently solving the resulting system for c_0 and c_1, we find that

$$c_0 = \frac{1}{2\lambda_1} I^\nu \rho(\mathscr{T}) - \frac{\mathscr{T}}{4\lambda_1} I^{\nu-1} \rho(\mathscr{T}) + \frac{\mathscr{T}\lambda_2}{4\lambda_1} I^{\nu-\xi-1} x(\mathscr{T}) - \frac{\lambda_2}{2\lambda_1} I^{\nu-\xi} x(\mathscr{T}),$$

$$c_1 = \frac{1}{2\lambda_1} I^{\nu-1} \rho(\mathscr{T}) - \frac{\lambda_2}{2\lambda_1} I^{\nu-\xi-1} x(\mathscr{T}).$$

Substituting the values of c_0 and c_1 into (3) yields the solution (4). The converse of this lemma follows by direct computation. This finishes the proof. □

3 Existence results

Let $\mathscr{C} = C([0,\mathscr{T}],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,\mathscr{T}] \to \mathbb{R}$ endowed with the norm denoted by $\|.\|$.

By Lemma 1, we can convert the problem (1)-(2) into a fixed point problem $\mathscr{U}x = x$, where $\mathscr{U} : \mathscr{C} \to \mathscr{C}$ is defined by

$$\begin{aligned}
(\mathscr{U}x)(t) =& \frac{1}{\lambda_1}\left[\int_0^t \frac{(t-s)^{\nu-1}}{\Gamma(\nu)} f(s,x(s))ds + \int_0^t \frac{(t-s)^{\nu+\varpi-1}}{\Gamma(\nu+\varpi)} g(s,x(s))ds\right] \\
&- \frac{\lambda_2}{\lambda_1}\int_0^t \frac{(t-s)^{\nu-\xi-1}}{\Gamma(\nu-\xi)} x(s)ds - \frac{1}{2\lambda_1}\left[\int_0^{\mathscr{T}} \frac{(\mathscr{T}-s)^{\nu-1}}{\Gamma(\nu)} f(s,x(s))ds\right. \\
&\left. + \int_0^{\mathscr{T}} \frac{(\mathscr{T}-s)^{\nu+\varpi-1}}{\Gamma(\nu+\varpi)} g(s,x(s))ds\right] + \frac{\lambda_2}{2\lambda_1}\int_0^{\mathscr{T}} \frac{(\mathscr{T}-s)^{\nu-\xi-1}}{\Gamma(\nu-\xi)} x(s)ds \\
&+ \frac{(\mathscr{T}-2t)}{4\lambda_1}\left[\int_0^{\mathscr{T}} \frac{(\mathscr{T}-s)^{\nu-2}}{\Gamma(\nu-1)} f(s,x(s))ds + \int_0^{\mathscr{T}} \frac{(\mathscr{T}-s)^{\nu+\varpi-2}}{\Gamma(\nu+\varpi-1)} g(s,x(s))ds\right] \\
&- \frac{(\mathscr{T}-2t)\lambda_2}{4\lambda_1}\int_0^{\mathscr{T}} \frac{(\mathscr{T}-s)^{\nu-\xi-2}}{\Gamma(\nu-\xi-1)} x(s)ds, \; t \in [0,\mathscr{T}].
\end{aligned} \tag{7}$$

Observe that the existence of a fixed point of the operator $\mathscr{U}$ implies the existence of a solution for the problem (1)-(2). Now we present our first main result which deals with the existence of a unique solution for the problem at hand.

Theorem 1. *Assume that:*

(H_1) *$f,g:[0,\mathscr{T}]\times\mathbb{R}\to\mathbb{R}$ are continuous functions such that $|f(t,0)|\le M_1<\infty$, $|g(t,0)|\le M_2<\infty$ and $M=max\{M_1,M_2\}$ $\forall t\in[0,\mathscr{T}]$ and*

$$|f(t,x)-f(t,y)|\le L_1|x-y|,\ \ |g(t,x)-g(t,y)|\le L_2|x-y|,\ \ L_1,L_2>0,\forall t\in[0,\mathscr{T}],\ x,y\in\mathbb{R};$$

(H_2) *For $L=max\{L_1,L_2\}$,*

$$(L\Omega_1+\Omega_2)<1,\ where\ \Omega_1=\frac{1}{4|\lambda_1|}\Big[\frac{\mathscr{T}^{\nu}(6+\nu)}{\Gamma(\nu+1)}+\frac{\mathscr{T}^{\nu+p}(6+\nu+\varpi)}{\Gamma(\nu+\varpi+1)}\Big],\ \Omega_2=\frac{|\lambda_2|(6+\nu-\xi)\mathscr{T}^{\nu-\xi}}{4|\lambda_1|\Gamma(\nu-\xi+1)}. \quad (8)$$

Then there exists a unique solutions for the problem (1)-(2) on $[0,\mathscr{T}]$.

Proof. Consider a closed ball $B_r=\{x\in\mathscr{C},\|x\|\le r\}$ and show that $\mathscr{U}B_r\subset B_r$, where the operator $\mathscr{U}:\mathscr{C}\to\mathscr{C}$ is defined by (7) and $r\ge M\Omega_1(1-L\Omega_1-\Omega_2)^{-1}$, Ω_i $(i=1,2)$ are given in (8). For any $x\in B_r$, it follows by the condition (H_1) that

$$|f(t,x)|=|f(t,x(t))-f(t,0)+f(t,0)|\le|f(t,x(t))-f(t,0)|+|f(t,0)|\le L_1\|x\|+M_1\le L_1r+M_1.$$

In a similar manner, one can find that $|g(t,x)|\le L_2r+M_2$. Using the foregoing inequalities and the assumption (H_2), we obtain

$$\begin{aligned}
\|(\mathscr{U}x)\| =& \sup_{t\in[0,T]}|(\mathscr{U}x)(t)|\\
\le& \sup_{t\in[0,\mathscr{T}]}\Big\{\frac{1}{|\lambda_1|}\Big[\int_0^t\frac{(t-s)^{\nu-1}}{\Gamma(\nu)}|f(s,x(s))|ds+\int_0^t\frac{(t-s)^{\nu+\varpi-1}}{\Gamma(\nu+\varpi)}|g(s,x(s))|ds\Big]\\
&+\frac{|\lambda_2|}{|\lambda_1|}\int_0^t\frac{(t-s)^{\nu-\xi-1}}{\Gamma(\nu-\xi)}|x(s)|ds+\frac{1}{2|\lambda_1|}\Big[\int_0^{\mathscr{T}}\frac{(\mathscr{T}-s)^{\nu-1}}{\Gamma(\nu)}|f(s,x(s))|ds\\
&+\int_0^{\mathscr{T}}\frac{(\mathscr{T}-s)^{\nu+\varpi-1}}{\Gamma(\nu+\varpi)}|g(s,x(s))|ds\Big]+\frac{|\lambda_2|}{2|\lambda_1|}\int_0^{\mathscr{T}}\frac{(\mathscr{T}-s)^{\nu-\xi-1}}{\Gamma(\nu-\xi)}|x(s)|ds\\
&+\frac{|\mathscr{T}-2t|}{4|\lambda_1|}\Big[\int_0^{\mathscr{T}}\frac{(\mathscr{T}-s)^{\nu-2}}{\Gamma(\nu-1)}|f(s,x(s))|ds+\int_0^{\mathscr{T}}\frac{(\mathscr{T}-s)^{\nu+\varpi-2}}{\Gamma(\nu+\varpi-1)}|g(s,x(s))|ds\Big]\\
&+\frac{|(\mathscr{T}-2t)\lambda_2|}{4|\lambda_1|}\int_0^{\mathscr{T}}\frac{(\mathscr{T}-s)^{\nu-\xi-2}}{\Gamma(\nu-\xi-1)}|x(s)|ds\Big\}\\
\le& \frac{(L_1r+M_1)(6+\nu)\mathscr{T}^{\nu}}{4|\lambda_1|\Gamma(\nu+1)}+\frac{(L_2r+M_2)(6+\nu+\varpi)\mathscr{T}^{\nu+\varpi}}{4|\lambda_1|\Gamma(\nu+\varpi+1)}+\frac{r|\lambda_2|(6+\nu-\xi)\mathscr{T}^{\nu-\xi}}{4|\lambda_1|\Gamma(\nu-\xi+1)}\\
\le& (Lr+M)\Omega_1+r\Omega_2\le r,
\end{aligned}$$

which shows that $\mathscr{U}x\in B_r$ for any $x\in B_r$. Hence $\mathscr{U}B_r\subset B_r$.

Next we establish that the operator $\mathscr{U}$ is a contraction. For $x, y \in \mathbb{R}$, we find that

$$
\begin{aligned}
& \|\mathscr{U}x - \mathscr{U}y\| \\
= & \sup_{t\in[0,\mathscr{T}]} |(\mathscr{U}x)(t) - (\mathscr{U}y)(t)| \\
\leq & \sup_{t\in[0,\mathscr{T}]} \Big\{ \frac{1}{|\lambda_1|}\Big[\int_0^t \frac{(t-s)^{\nu-1}}{\Gamma(\nu)}|f(s,x(s)) - f(s,y(s))|ds + \int_0^t \frac{(t-s)^{\nu+\varpi-1}}{\Gamma(\nu+\varpi)}|g(s,x(s)) - g(s,y(s))|ds\Big] \\
+ & \frac{|\lambda_2|}{|\lambda_1|}\int_0^t \frac{(t-s)^{\nu-\xi-1}}{\Gamma(\nu-\xi)}|x(s)-y(s)|ds + \frac{1}{2|\lambda_1|}\Big[\int_0^{\mathscr{T}} \frac{(\mathscr{T}-s)^{\nu-1}}{\Gamma(\nu)}|f(s,x(s)) - f(s,y(s))||ds \\
+ & \int_0^{\mathscr{T}} \frac{(\mathscr{T}-s)^{\nu+\varpi-1}}{\Gamma(\nu+\varpi)}|g(s,x(s)) - g(s,y(s))|ds\Big] + \frac{|\lambda_2|}{2|\lambda_1|}\int_0^{\mathscr{T}} \frac{(\mathscr{T}-s)^{\nu-\xi-1}}{\Gamma(\nu-\xi)}|x(s)-y(s)|ds \\
+ & \frac{|\mathscr{T}-2t|}{4|\lambda_1|}\Big[\int_0^{\mathscr{T}} \frac{(\mathscr{T}-s)^{\nu-2}}{\Gamma(\nu-1)}|f(s,x(s)) - f(s,y(s))|ds + \int_0^{\mathscr{T}} \frac{(\mathscr{T}-s)^{\nu+\varpi-2}}{\Gamma(\nu+\varpi-1)}|g(s,x(s)) - g(s,y(s))|ds\Big] \\
+ & \frac{|(\mathscr{T}-2t)\lambda_2|}{4|\lambda_1|}\int_0^{\mathscr{T}} \frac{(\mathscr{T}-s)^{\nu-\xi-2}}{\Gamma(\nu-\xi-1)}|x(s)-y(s)|ds\Big\} \\
\leq & \Big[\frac{L_1(6+\nu)\mathscr{T}^{\nu}}{4|\lambda_1|\Gamma(\nu+1)} + \frac{L_2(6+\nu+\varpi)\mathscr{T}^{\nu+\varpi}}{4|\lambda_1|\Gamma(\nu+\varpi+1)} + \frac{|\lambda_2|(6+\nu-\xi)\mathscr{T}^{\nu-\xi}}{4|\lambda_1|\Gamma(\nu-\xi+1)}\Big]\|x-y\| \\
\leq & (L\Omega_1 + \Omega_2)\|x-y\|,
\end{aligned}
$$

which shows that $\mathscr{U}$ is a contractive operator as $(L\Omega_1+\Omega_2) < 1$ by (H_2). Thus we deduce by Banach contraction mapping principle that the the operator $\mathscr{U}$ has a unique fixed point, which implies that there exists a unique solution of the (1)-(2) on $[0,\mathscr{T}]$. This completes the proof. □

Now we prove an existence result for the problem (1)-(2) by applying Krasnoselskii's fixed point theorem [35].

Theorem 2. *Suppose that $f, g : [0,\mathscr{T}] \times \mathbb{R} \to \mathbb{R}$ are continuous functions, and the following condition is satisfied:*

(H_3) *there exist $\mu_1, \mu_2 \in C([0,\mathscr{T}], \mathbb{R}^+)$ with $\|\mu\| = max\{\|\mu_1\|, \|\mu_2\|\}$ such that $|f(t,x)| \leq \mu_1(t)$ and $|g(t,x)| \leq \mu_2(t)$, for all $(t,x) \in [0,\mathscr{T}] \times \mathbb{R}$.*

Then the problem (1)-(2) has at least one solution on $[0,\mathscr{T}]$ if $\Omega_2 < 1$, where Ω_2 is given in (8).

Proof. Consider $K_\zeta = \{x \in \mathscr{C} : \|x\| \leq \zeta\}$ with $\zeta \geq \|\mu\|\Omega_1(1-\Omega_2)^{-1}$, and define operators $\mathscr{U}_1$ and $\mathscr{U}_2$ on $K_\zeta \to \mathscr{C}$ as follows:

$$
\begin{aligned}
(\mathscr{U}_1 x)(t) = & \frac{1}{\lambda_1}\Big[\int_0^t \frac{(t-s)^{\nu-1}}{\Gamma(\nu)}f(s,x(s))ds + \int_0^t \frac{(t-s)^{\nu+\varpi-1}}{\Gamma(\nu+\varpi)}g(s,x(s))ds\Big] \\
& - \frac{1}{2\lambda_1}\Big[\int_0^{\mathscr{T}} \frac{(\mathscr{T}-s)^{\nu-1}}{\Gamma(\nu)}f(s,x(s))ds + \int_0^{\mathscr{T}} \frac{(\mathscr{T}-s)^{\nu+\varpi-1}}{\Gamma(\nu+\varpi)}g(s,x(s))ds\Big] \\
& + \frac{(\mathscr{T}-2t)}{4\lambda_1}\Big[\int_0^{\mathscr{T}} \frac{(\mathscr{T}-s)^{\nu-2}}{\Gamma(\nu-1)}f(s,x(s))ds + \int_0^{\mathscr{T}} \frac{(\mathscr{T}-s)^{\nu+\varpi-2}}{\Gamma(\nu+\varpi-1)}g(s,x(s))ds\Big], t \in [0,\mathscr{T}], \\
(\mathscr{U}_2 x)(t) = & -\frac{\lambda_2}{\lambda_1}\int_0^t \frac{(t-s)^{\nu-\xi-1}}{\Gamma(\nu-\xi)}x(s)ds + \frac{\lambda_2}{2\lambda_1}\int_0^{\mathscr{T}} \frac{(\mathscr{T}-s)^{\nu-\xi-1}}{\Gamma(\nu-\xi)}x(s)ds \\
& - \frac{(\mathscr{T}-2t)\lambda_2}{4\lambda_1}\int_0^{\mathscr{T}} \frac{(\mathscr{T}-s)^{\nu-\xi-2}}{\Gamma(\nu-\xi-1)}x(s)ds, \ t \in [0,\mathscr{T}].
\end{aligned}
$$

Observe that $\mathscr{U} = \mathscr{U}_1 + \mathscr{U}_2$ on K_ζ. Let us now verify the hypothesis of Krasnoselskii's fixed point theorem [35].

(i) For $x,y\in K_\zeta$, we have

$$\begin{aligned}
&\|\mathscr{U}_1x+\mathscr{U}_2y\| = \sup_{t\in[0,T]} |(\mathscr{U}_1x)(t)+(\mathscr{U}_2y)(t)|\\
\le& \sup_{t\in[0,\mathscr{T}]}\Big\{\frac{1}{|\lambda_1|}\Big[\int_0^t \frac{(t-s)^{\nu-1}}{\Gamma(\nu)}|f(s,x(s))|ds+\int_0^t\frac{(t-s)^{\nu+\varpi-1}}{\Gamma(\nu+\varpi)}|g(s,x(s))|ds\Big]\\
&+\frac{1}{2|\lambda_1|}\Big[\int_0^{\mathscr{T}}\frac{(\mathscr{T}-s)^{\nu-1}}{\Gamma(\nu)}|f(s,x(s))|ds+\int_0^{\mathscr{T}}\frac{(\mathscr{T}-s)^{\nu+\varpi-1}}{\Gamma(\nu+\varpi)}|g(s,x(s))|ds\Big]\\
&+\frac{|\mathscr{T}-2t|}{4|\lambda_1|}\Big[\int_0^{\mathscr{T}}\frac{(\mathscr{T}-s)^{\nu-2}}{\Gamma(\nu-1)}|f(s,x(s))|ds+\int_0^{\mathscr{T}}\frac{(\mathscr{T}-s)^{\nu+\varpi-2}}{\Gamma(\nu+\varpi-1)}|g(s,x(s))|ds\Big]\\
&+\frac{|\lambda_2|}{|\lambda_1|}\int_0^t\frac{(t-s)^{\nu-\xi-1}}{\Gamma(\nu-\xi)}|y(s)|ds+\frac{|\lambda_2|}{2|\lambda_1|}\int_0^{\mathscr{T}}\frac{(\mathscr{T}-s)^{\nu-\xi-1}}{\Gamma(\nu-\xi)}|y(s)|ds\\
&+\frac{|\mathscr{T}-2t|\lambda_2}{4|\lambda_1|}\int_0^{\mathscr{T}}\frac{(\mathscr{T}-s)^{\nu-\xi-2}}{\Gamma(\nu-\xi-1)}|y(s)|ds\Big\}\\
\le& \frac{\|\mu_1\|(6+\nu)\mathscr{T}^{\nu}}{4|\lambda_1|\Gamma(\nu+1)}+\frac{\|\mu_2\|(6+\nu+p)\mathscr{T}^{\nu+\varpi}}{4|\lambda_1|\Gamma(\nu+\varpi+1)}+\frac{\zeta|\lambda_2|(6+\nu-\xi)\mathscr{T}^{\nu-\xi}}{4|\lambda_1|\Gamma(\nu-\xi+1)}\\
\le& \|\mu\|\Omega_1+\zeta\Omega_2\le\zeta,
\end{aligned}$$

which implies that $\mathscr{U}_1x+\mathscr{U}_2y\in K_\zeta$.

(ii) In this step we show that $\mathscr{U}_1$ is compact and continuous. Continuity of the operator $\mathscr{U}_1$ follows from that of f and g. Furthermore, $\mathscr{U}_1$ is uniformly bounded on K_ζ as $\|\mathscr{U}_1x\|\le\|\mu\|\Omega_1$. Next we show that the operator $\mathscr{U}_1$ is compact. Define $\sup_{(t,x)\in[0,\mathscr{T}]\times B_\zeta}|f(t,x)|=f_1$ and $\sup_{(t,x)\in[0,\mathscr{T}]\times B_\zeta}|g(t,x)|=g_1$. Then, for $t_1,t_2\in[0,\mathscr{T}], t_1<t_2$, we have

$$\begin{aligned}
&|(\mathscr{U}_1x)(t_2)-(\mathscr{U}_1x)(t_1)|\\
=&\Big|\frac{1}{\lambda_1}\Big[\int_0^{t_1}\frac{((t_2-s)^{\nu-1}-(t_1-s)^{\nu-1})}{\Gamma(\nu)}f(s,x(s))ds+\int_{t_1}^{t_2}\frac{(t_2-s)^{\nu-1}}{\Gamma(\nu)}f(s,x(s))ds\\
&+\int_0^{t_1}\frac{((t_2-s)^{\nu+\varpi-1}-(t_1-s)^{\nu+\varpi-1})}{\Gamma(\nu+\varpi)}g(s,x(s))ds+\int_{t_1}^{t_2}\frac{(t_2-s)^{\nu+\varpi-1}}{\Gamma(\nu+\varpi)}g(s,x(s))ds\\
&+\frac{(t_1-t_2)}{2\lambda_1}\Big[\int_0^{\mathscr{T}}\frac{(\mathscr{T}-s)^{\nu-2}}{\Gamma(\nu-1)}f(s,x(s))ds+\int_0^{\mathscr{T}}\frac{(\mathscr{T}-s)^{\nu+\varpi-2}}{\Gamma(\nu+\varpi-1)}g(s,x(s))ds\Big]\Big|\\
\le&\frac{f_1}{2|\lambda_1|\Gamma(\nu+1)}\Big(2|t_2^{\nu}-t_1^{\nu}|+4(t_2-t_1)^{\nu}+\nu\mathscr{T}^{\nu-1}|t_2-t_1|\Big)\\
&+\frac{g_1}{2|\lambda_1|\Gamma(\nu+\varpi+1)}\Big(2|t_2^{\nu+\varpi}-t_1^{\nu+\varpi}|+4(t_2-t_1)^{\nu+\varpi}+(\nu+\varpi)\mathscr{T}^{\nu+\varpi-1}|t_2-t_1|\Big),
\end{aligned}$$

which tends to zero independent of $x\in K_\zeta$. Therefore $\mathscr{U}_1$ is equicontinuous. So the conclusion of the Arzelá-Ascoli theorem applies and hence $\mathscr{U}_1$ is relatively compact on K_ζ.

(iii) As an immediate consequence of the condition $\Omega_2<1$, the operator $\mathscr{U}_2$ is a contraction.
Thus the hypothesis of Krasnoselskii's fixed point theorem [35] is satisfied and hence its conclusion implies that the problem (1)-(2) has at least one solution on $[0,\mathscr{T}]$. This finishes the proof. □

4 Mixed nonlinearities case

Here we consider the following anti-periodic boundary value problem involving three types of nonlinearities:

$$\begin{cases}\lambda_1\,{}^CD^{\nu}x(t)+\lambda_2\,{}^CD^{\xi}h(t,x(t))=f(t,x(t))+I^{\varpi}g(t,x(t)),\, t\in[0,\mathscr{T}],\\ x(0)=-x(\mathscr{T}),\ x'(0)=-x'(\mathscr{T}),\end{cases}\tag{9}$$

where $h:[0,\mathscr{T}]\times\mathbb{R}\to\mathbb{R}$ is a continuous function, while the rest of the quantities are the same as defined in the problem (1)-(2).

In relation to the problem (9), we define a fixed point operator $\mathscr{V}:\mathscr{C}\to\mathscr{C}$ as

$$\begin{aligned}(\mathscr{V}x)(t)=&\frac{1}{\lambda_1}\Big[\int_0^t\frac{(t-s)^{\nu-1}}{\Gamma(\nu)}f(s,x(s))ds+\int_0^t\frac{(t-s)^{\nu+\varpi-1}}{\Gamma(\nu+\varpi)}g(s,x(s))ds\Big]\\&-\frac{1}{2\lambda_1}\Big[\int_0^{\mathscr{T}}\frac{(\mathscr{T}-s)^{\nu-1}}{\Gamma(\nu)}f(s,x(s))ds+\int_0^{\mathscr{T}}\frac{(\mathscr{T}-s)^{\nu+\varpi-1}}{\Gamma(\nu+\varpi)}g(s,x(s))ds\Big]\\&+\frac{(\mathscr{T}-2t)}{4\lambda_1}\Big[\int_0^{\mathscr{T}}\frac{(\mathscr{T}-s)^{\nu-2}}{\Gamma(\nu-1)}f(s,x(s))ds+\int_0^{\mathscr{T}}\frac{(\mathscr{T}-s)^{\nu+\varpi-2}}{\Gamma(\nu+\varpi-1)}g(s,x(s))ds\Big]\\&-\frac{\lambda_2}{\lambda_1}\int_0^t\frac{(t-s)^{\nu-\xi-1}}{\Gamma(\nu-\xi)}h(s,x(s))ds+\frac{\lambda_2}{2\lambda_1}\int_0^{\mathscr{T}}\frac{(\mathscr{T}-s)^{\nu-\xi-1}}{\Gamma(\nu-\xi)}h(s,x(s))ds\\&-\frac{(\mathscr{T}-2t)\lambda_2}{4\lambda_1}\int_0^{\mathscr{T}}\frac{(\mathscr{T}-s)^{\nu-\xi-2}}{\Gamma(\nu-\xi-1)}x(s)ds,\ t\in[0,\mathscr{T}].\end{aligned}\tag{10}$$

Now we present a result concerning the uniqueness of solutions for the problem (9).

Theorem 3. *Assume that:*

(H_4) $f,g,h:[0,\mathscr{T}]\times\mathbb{R}\to\mathbb{R}$ *are continuous functions and there exist* $L_1,L_2,L_3>0$ *such that*

$$|f(t,x)-f(t,y)|\le L_1|x-y|,\ |g(t,x)-g(t,y)|\le L_2|x-y|,\ |h(t,x)-h(t,y)|\le L_3|x-y|,\ \forall t\in[0,\mathscr{T}],\ x,y\in\mathbb{R},$$

and $|f(t,0)|\le M_1<\infty,\ |g(t,0)|\le M_2<\infty,\ |h(t,0)|\le M_3<\infty,$ *for all* $t\in[0,T]$*;*

(H_5) $L\Lambda<1$, *where* $L=max\{L_1,L_2,L_3\}$,

$$\Lambda=\frac{1}{4|\lambda_1|}\Big[\frac{\mathscr{T}^{\nu}(6+\nu)}{\Gamma(\nu+1)}+\frac{\mathscr{T}^{\nu+p}(6+\nu+\varpi)}{\Gamma(\nu+\varpi+1)}+\frac{|\lambda_2|\mathscr{T}^{\nu-\xi}(6+\nu-\xi)}{\Gamma(\nu-\xi+1)}\Big].\tag{11}$$

Then the problem (9) has a unique solution on $[0,\mathscr{T}]$.

Proof. The proof is similar to that of Theorem 1. So we omit it.

5 Examples

Here we illustrate the results obtained in the previous sections with the aid of examples.

Example 1. Consider the following problem

$$\begin{cases}10\,{}^{C}D^{1.25}x(t)+3\,{}^{C}D^{1.15}x(t)=f(t,x(t))+I^2g(t,x(t)), & t\in[0,3],\\ x(0)=-x(3),\quad x'(0)=-x'(3),\end{cases}\tag{12}$$

where $\nu=1.25$, $\xi=1.15$, $\varpi=2$, $\lambda_1=10$, $\lambda_2=3$, $\mathscr{T}=3$, and

$$f(t,x(t))=\frac{e^{-2t}}{(t+5)^2}\frac{|x(t)|}{(1+|x(t)|)}+\frac{1}{2},\ g(t,x(t))=\frac{\sin t}{\sqrt{49+t}}tan^{-1}x+\frac{1}{7}.$$

Using the given data, we find that $\Omega_1=1.623420081$, $\Omega_2=0.5367382548$. From the inequalities:

$$|f(t,x)-f(t,y)|\le\frac{e^{-2t}}{(t+5)^2}|x-y|\le\frac{1}{25}|x-y|,$$

$$|g(t,x)-g(t,y)|\le\frac{\sin t}{\sqrt{49+t}}|x-y|\le\frac{1}{7}|x-y|,$$

we have $L_1=\frac{1}{25}$, $L_2=\frac{1}{7}$ and $L=max\{L_1,L_2\}=\frac{1}{7}$. Moreover, $L\Omega_1+\Omega_2\approx0.7686554092$. As all the assumptions of Theorem 1 are satisfied, therefore, the problem (12) has a unique solution on $[0,3]$.

Example 2. Consider the following anti-periodic problem:

$$\begin{cases} 5\,{}^{C}D^{1.75}x(t) - {}^{C}D^{1.60}x(t) = f(t,x(t)) + I^{\frac{1}{2}}g(t,x(t)),\ t \in [0,5], \\ x(0) = -x(5), \quad x'(0) = -x'(5), \end{cases} \tag{13}$$

where $\nu = 1.75$, $\xi = 1.60$, $\varpi = \frac{1}{2}$, $\lambda_1 = 5$, $\mathscr{T} = 5$, and $\lambda_2 = -1$,

$$f(t,x) = \frac{\cos t}{(2t+7)}(\sin x + tan^{-1}x) + \frac{2}{9},\ g(t,x) = \frac{tan^{-1}t}{15}\left(\frac{|x(t)|}{1+x(t)|} + cot^{-1}x\right) + e^{-5t}.$$

Using the given values, it is found that $\Omega_2 \approx 0.4195559892 < 1$, $\|\mu_1\| = (46+9\pi)/126$ and $\|\mu_2\| = (\pi^2+\pi+30)/30$. Obviously the hypothesis of Theorem 2 holds true and consequently its conclusion applies to the problem (13).

Example 3. Consider the following problem

$$\begin{cases} 7\,{}^{C}D^{1.95}x(t) - {}^{C}D^{1.55}h(t,x(t)) = f(t,x(t)) + I^{3}g(t,x(t)),\ t \in [0,2], \\ \\ x(0) = -x(2), \quad x'(0) = -x'(2), \end{cases} \tag{14}$$

where $\nu = 1.95$, $\xi = 1.55$, $\varpi = 3$, $\lambda_1 = 7$, $\lambda_2 = -1$, $\mathscr{T} = 2$, and

$$f(t,x) = \frac{1}{t^2+2}tan^{-1}x + \frac{1}{4},\ g(t,x) = \frac{e^{-2t}|x|}{(4+t)(1+|x|)} + \frac{1}{3},\ h(t,x) = \frac{e^{-3t}(\sin x + x)}{\sqrt{t^4+36}} + \frac{1}{\sqrt{t^2+25}}.$$

With the given data, it is found that $\Lambda = 1.023731638, L_1 = 1/2, L_2 = 1/4, L_3 = 1/3, L = max\{L_1, L_2, L_3\} = 1/2$, $L\Lambda \approx 0.5118658190$. Clearly all the assumptions of Theorem 3 are satisfied. Hence there exists a unique solution for the problem (14) on $[0,2]$.

6 Conclusion

In this work, we have derived the existence and uniqueness results for an anti-periodic boundary value problem involving two linear Caputo type fractional-order terms together non-integral and integral (Riemann-Liouville type) nonlinearities. In the mixed nonolinearities case, our problem consists of one linear Caputo type fractional-order term, together with fractional-order, non-integral and Riemann-Liouville type integral nonlinear terms. Our results are new and contribute to the existing material on fractional-order anti-periodic boundary value problems.

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