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The Ornstein-Uhlenbeck Operator for Uncorrelated Random Variables

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Abstract: This work considers Ornstein-Uhlenbeck operator whose role in sensitivity analysis involving Malliavin calculus is of immense importance in different fields including financial mathematics. There is need to consider Ornstein-Uhlenbeck operator for uncorrelated random variables since certain phenomenon involve uncorrelated Gaussian random variables. Thus, we derive the operator for uncorrelated multivariate Gaussian random variables suitable for phenomenon involving multivariate random variables.

Keywords: Ornstein-Uhlenbeck operator, Density function, Gaussian random variables, Covariance matrix

1 Introduction

A lot of research has been done concerning the Ornstein-Uhlenbeck operator, whose applications are in many fields, namely, geometry, functional calculus, financial mathematics, analysis, etc. Chang and Feng [1] studied the Ornstein-Uhlenbeck operator with quadratic potentials. Otten [2] applied the operator as a basis for proving exponential decay of rotating waves in time-dependent reaction diffusion systems. Cai and Zhang [3] discussed coordinated drift estimation of a mixed fractional Ornstein-Uhlenbeck process. Harris [4] applied abstract holomorphic functional calculus theory and showed that the operator has a bounded functional calculus with an optimal angle. He highlighted its applications in the number generator of quantum field theory, the analogue of the Laplacian in the Malliavin calculus, the generator of the transition semigroup linked with mean-reverting stochastic process. Casarino et al. [5] discussed orthogonality of general eigenspaces of an Ornstein-Uhlenbeck operator. Chen and Liu [6] derived certain characteristics of complex-valued Wiener-Itô multiple integrals and complex Ornstein-Uhlenbeck operators and semigroups. Bally et al. [7] gave a numerical algorithm for sensitivity computation in a Lévy market using the Ornstein-Uhlenbeck operator as a differential operator. Bavouzet and Messaoud [8], Bavouzet et al. [9], Bally & Clement [10] and Udoye et al. [11] used the tool of the Ornstein-Uhlenbeck operator on jump-type market model. Metafune et al. [12] obtained the spectrum of a probably degenerate Ornstein-Uhlenbeck operator in $\mathbb{R}^n$ for L^p_μ spaces where μ is an invariant measure and $1 \leq p < \infty$.

Some phenomenon involve multivariate Gaussian random variables that are uncorrelated. Thus, applying Ornstein-Uhlenbeck operator in such scenario will not yield adequate model if uncorrelation nature of the random variables are not considered. Hence, we focus on the Ornstein-Uhlenbeck operator for uncorrelated multivariate Gaussian random variables.

The rest of the paper is arranged as follows: Section 2 gives the Mathematical foundation for the work, results are presented in Section 3, then conclusion is drawn.

2 Mathematical Foundation

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. For $p \geq 1$ and $n \geq 1$, $f \in C^p(\mathbb{R}^n)$ where $C^p(\mathbb{R}^n)$ is the space of functions $f : \mathbb{R}^n \to \mathbb{R}$ that are p times continuously differentiable. Let $X_1, \ldots, X_n$ be a sequence of random variables and $S_{(n,p)}$ be the set of simple functionals such that $\widehat{F} = f(X_1, \ldots, X_n) \in S_{(n,p)}$. $P_{(n,p)}$ is the space of simple processes $U_i = u_i(X_1, \ldots, X_n)$ of length n, where $u_i \in C^p(\mathbb{R}^n)$, $i = 1, \ldots, n$.
The *Ornstein-Uhlenbeck operator* $L : S_{(n,2)} \to S_{(n,0)}$ on $\widehat{F}$

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is defined as

$$L\widehat{F} = -\sum_{i=1}^{n}[(\partial_{ii}^{2} f)(X_1,...,X_n) + \phi_i(\partial_i f)(X_1,...,X_n)],$$

where

$$\phi_i(x_i) = \partial_{x_i}\ln[g(\mathbf{x})] = \frac{g_i'(\mathbf{x})}{g(\mathbf{x})}, \qquad g(\mathbf{x}) \neq 0; 1 \leq i \leq n$$

otherwise, $\phi_i(\mathbf{x}) = 0$, where g_i denotes the density function of the random variables $X_i, i = 1,...,n$.

3 Results

To derive the Ornstein-Uhlenbeck operator for uncorrelated random variables, let the uncorrelated Gaussian random set be given by $Z_1,...,Z_n$. Then, $\mathrm{Var}(Z_i) = \mathbb{E}[(Z_i - \mathbb{E}(Z_i))^2]$, $i = 1,...,n$ is the *variance of* Z_i. The *probability density function* of a multivariate Gaussian random vector $\mathbf{Z} \sim \mathscr{N}(\bar{\mu}, \Sigma)$ is given by

$$g(\mathbf{z}) = \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} \exp\left\{ -\frac{1}{2}(\mathbf{z}-\bar{\mu})^T \Sigma^{-1}(\mathbf{z}-\bar{\mu}) \right\} \tag{1}$$

where $\bar{\mu} \in \mathbb{R}^n$ is a vector denoting $\mathbb{E}[\mathbf{Z}]$ (expectation of Z_i, $i = 1,...,n$), $\Sigma \in \mathbb{R}^{n\times n}$ denotes an $n \times n$ covariance matrix, $\mathbf{Z} = [Z_1, Z_2, ..., Z_n]^T \in \mathbb{R}^n$ is a Gaussian random vector, ^{T} and $\det(\Sigma)$ denote the transpose and the determinant of the covariance matrix, respectively.

Theorem 3.1. Let ϖ_{ii} be the diagonal entries of an inverse covariance matrix Σ^{-1}. The density function g of n-dimensional uncorrelated Gaussian random variables $Z_1,...,Z_n$ satisfies the following:

1. $\ln g(\mathbf{z}) = \mathscr{K} - \frac{1}{2}\left[\sum_{i=1}^{n}((z_i-\mu_i)^2\varpi_{ii}\right]$ where

$$\mathscr{K} = -\frac{n}{2}\ln(2\pi) - \frac{1}{2}\ln(\det(\Sigma)) \tag{2}$$

is a constant.

2. $\partial_{z_i}\ln g(z_1,z_2,...,z_n) = -[(z_i-\mu_i)\varpi_{ii}]$.

Proof.

1. From equation (1),

$$\begin{aligned}\ln g(\mathbf{z}) &= -\frac{n}{2}\ln(2\pi) - \frac{1}{2}\ln(\det(\Sigma)) \\ &\quad -\frac{1}{2}[(\mathbf{z}-\bar{\mu})^T\Sigma^{-1}(\mathbf{z}-\bar{\mu})] \\ &= \mathscr{K} - \frac{1}{2}[(\mathbf{z}-\bar{\mu})^T\Sigma^{-1}(\mathbf{z}-\bar{\mu})];\end{aligned}$$

where $\mathscr{K}$ is as given in equation (2).

2. Let Σ and Σ^{-1} be n-dimensional covariance matrix and inverse covariance matrix, respectively. Since the random variables are uncorrelated,

$$\mathbb{E}[(Z_i-\mu_i)(Z_j-\mu_j)] = 0 \,\forall\, i \neq j.$$

Hence,
$\Sigma =$

$$\begin{bmatrix} \mathbb{E}(Z_1-\mu_1)^2 & \cdots & \mathbb{E}[(Z_1-\mu_1)(Z_n-\mu_n)] \\ \mathbb{E}[(Z_2-\mu_2)(Z_1-\mu_1)] & \cdots & \mathbb{E}[(Z_2-\mu_2)(Z_n-\mu_n)] \\ \mathbb{E}[(Z_3-\mu_3)(Z_1-\mu_1)] & \cdots & \mathbb{E}[(Z_3-\mu_3)(Z_n-\mu_n)] \\ \vdots & \vdots & \vdots \\ \mathbb{E}[(Z_n-\mu_n)(Z_1-\mu_1)] & \cdots & \mathbb{E}(Z_n-\mu_n)^2 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbb{E}(Z_1-\mu_1)^2 & 0 & \cdots & 0 \\ 0 & \mathbb{E}(Z_2-\mu_2)^2 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathbb{E}(Z_n-\mu_n)^2 \end{bmatrix}$$

$$(\mathbf{z}-\bar{\mu}) = \begin{bmatrix} Z_1-\mathbb{E}[Z_1] \\ Z_2-\mathbb{E}[Z_2] \\ Z_3-\mathbb{E}[Z_3] \\ \vdots \\ Z_n-\mathbb{E}[Z_n] \end{bmatrix},$$

$$\Sigma^{-1} = \begin{bmatrix} \varpi_{11} & \varpi_{12} & \cdots & \varpi_{1n} \\ \varpi_{21} & \varpi_{22} & \cdots & \varpi_{2n} \\ \varpi_{31} & \varpi_{32} & \cdots & \varpi_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ \varpi_{n1} & \varpi_{n2} & \cdots & \varpi_{nn} \end{bmatrix} = \begin{bmatrix} \varpi_{11} & 0 & \cdots & 0 \\ 0 & \varpi_{22} & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \varpi_{nn} \end{bmatrix}.$$

Let $i = 1,...,n$; $n \in N$.

The result is trivial for $i = 1$.

Assume that the result is true for $i = 1,2,...,k$. We show that it is true for $i = 1,...,k,k+1$.

$$\ln g(z_1,z_2,z_3,...,z_{k+1}) = \mathscr{K} - \frac{1}{2}\left[\sum_{i=1}^{k+1}(z_i-\mu_i)^2\varpi_{ii}\right]. \tag{3}$$

$$\begin{aligned}\frac{\partial \ln g(z_1,...,z_{k+1})}{\partial z_1} &= -\frac{1}{2}\frac{\partial}{\partial z_1}((z_1-\mu_1)^2\varpi_{11} \\ &+(z_2-\mu_2)^2\varpi_{22} + \cdots + (z_{k+1}-\mu_{k+1})^2\varpi_{k+1,k+1}) \\ &= -(z_1-\mu_1)\varpi_{11}.\end{aligned}$$

$$\begin{aligned}\frac{\partial \ln g(z_1,...,z_{k+1})}{\partial z_2} &= -\frac{1}{2}(\frac{\partial}{\partial z_2}((z_1-\mu_1)^2\varpi_{11} \\ &+(z_2-\mu_2)^2\varpi_{22} + \cdots + (z_{k+1}-\mu_{k+1})^2\varpi_{k+1,k+1}) \\ &= -(z_2-\mu_2)\varpi_{22}\end{aligned}$$

$$\vdots$$

$$\frac{\partial \ln g(z_1,\ldots,z_{k+1})}{\partial z_k} = -\frac{1}{2}\left[\frac{\partial}{\partial z_k}((z_1-\mu_1)^2\varpi_{11}\right.$$

$$\left.+(z_2-\mu_2)^2\varpi_{22}+\cdots+(z_{k+1}-\mu_{k+1})^2\varpi_{k+1,k+1}\right]$$

$$= -(z_k-\mu_k)\varpi_{kk}.$$

Since it is true for $i=1,\ldots,k$, respectively; there is need to show that it is true for $i=k+1$.
From equation (3),

$$\frac{\partial \ln g(z_1,\ldots,z_k,z_{k+1})}{\partial z_{k+1}} = -\frac{1}{2}\frac{\partial}{\partial z_{k+1}}((z_1-\mu_1)^2\varpi_{11}$$

$$+(z_2-\mu_2)^2\varpi_{22}+\cdots+(z_{k+1}-\mu_{k+1})^2\varpi_{k+1,k+1})$$

$$= -\frac{1}{2}[2(z_{k+1}-\mu_{k+1})\varpi_{k+1,k+1}]$$

$$= -(z_{k+1}-\mu_{k+1})\varpi_{k+1,k+1}.$$

Therefore, the result is true for all i. □

Corrolary 3.2.

1. If $Z_1,Z_2,\ldots,Z_n$ is a sequence of independent Gaussian random variables, then $\varpi_{ij}=0$ for $i\neq j$.
   Thus, $\ln g(z_1,\ldots,z_n) = \mathscr{K} - \frac{1}{2}\sum_{i=1}^{n}(z_i-\mu_i)^2\varpi_{ii}$ and $\partial_{z_i}\ln g(\mathbf{z}) = -(z_i-\mu_i)\varpi_{ii}$ where $\mathscr{K}$ is given by equation (2).
2. If $Z_1,Z_2,\ldots,Z_n$ is a sequence of independent and standardized Gaussian random variables, then the diagonal of the covariance matrix $\varpi_{ii}=1$ and $\varpi_{ij}=0$ for $i\neq j$. Thus,

$$\ln g(\mathbf{z}) = \mathscr{K} - \frac{1}{2}\sum_{i=1}^{n} z_i^2 \quad \text{and}$$

$$\partial_{z_i}\ln g(z_1,z_2,\ldots,z_n) = -z_i.$$

Theorem 3.3. Let $(\Omega,\mathscr{F},\mathbb{P})$ be a probability space. Let $\widehat{F}=f(Z_1,\ldots,Z_n)$ be a functional where $f:\mathbb{R}^n\to\mathbb{R}$. Assume that $Z_1,\ldots,Z_n$ is a sequence of uncorrelated Gaussian random variables with absolute continuous law $g_i(y)dy$ where g_i are piecewise continuous for each $i=1,\ldots,n$. Then,

1. the Skorohod integral operator $\delta: P_{(n,1)}\to S_{(n,0)}$ given for simple process $U\in P_{(n,1)}$ satisfies

$$\delta(U) = -\sum_{i=1}^{n}\left[\partial_i U_i - (z_i-\mu_i)\varpi_{ii}U_i\right]$$

where ϖ_{ii} is the diagonal element of the $n\times n$ inverse covariance matrix.
$U_i(\omega)=u_i(Z_1,\ldots,Z_n)(\omega)$; $u_i:\mathbb{R}^n\to\mathbb{R}$, $i\in N,\omega\in\Omega$.
2. the Ornstein-Uhlenbeck operator $L: S_{(n,2)}\to S_{(n,0)}$ satisfies

$$L_i\widehat{F} = -\left[\partial_i^2\widehat{F} - (z_i-\mu_i)\varpi_{ii}\partial_i\widehat{F}\right].$$

Proof.

1. In general

$$\delta_{i,\pi}(u) = -[\partial_i(\pi_i u_i) + (\pi_i u_i)\partial_{z_i}\ln g_i](Z_1,\ldots,Z_n).$$

Since Z_i's are Gaussian random variables, its density function g_i, $i=1,\ldots,n$ is everywhere differentiable on $\mathbb{R}$; its weight function $\pi_i=1$ and its derivative $\pi_i'=0$ (Bavouzet et al. (2009, [9])). Thus, we get

$$\delta(U) = -\sum_{i=1}^{n}[(\pi_i\partial_i u_i + u_i\partial_i\pi_i) + (\pi_i u_i)\partial_{z_i}\ln g_i](Z_1,\ldots,Z_n)$$

$$= -\sum_{i=1}^{n}[\partial_i u_i + u_i\partial_{z_i}\ln g_i](Z_1,\ldots,Z_n).$$

From Theorem 3.1,

$$\delta(U) = -\sum_{i=1}^{n}\left[\partial_i u_i + u_i\partial_{z_i}\ln\left\{\frac{1}{\sqrt{(2\pi)^n\det(\Sigma)}}\right.\right.$$

$$\left.\left.\cdot\exp\left\{-\frac{1}{2}(\mathbf{z}-\bar{\mu})^T\Sigma^{-1}(\mathbf{z}-\bar{\mu})\right\}\right\}\right]$$

$$= -\sum_{i=1}^{n}\left[\partial_i u_i + u_i\partial_{z_i}\left\{\mathscr{K} - \frac{1}{2}\sum_{i=1}^{n}(z_i-\mu_i)^2\varpi_{ii}\right\}\right]$$

$$= -\sum_{i=1}^{n}\left[\partial_i U_i - (z_i-\mu_i)\varpi_{ii}U_i\right].$$

Following the same method, the Ornstein-Uhlenbeck aspect is achieved.
2. The **Ornstein-Uhlenbeck operator** $L=D\delta: S_{(n,2)}\to S_{(n,0)}$ satisfies

$$L\widehat{F} = -\sum_{i=1}^{n}\left[(\partial_i(\partial_i f))(Z_1,\ldots,Z_n)\right.$$

$$\left.-(z_i-\mu_i)\varpi_{ii}(\partial_i f)(Z_1,\ldots,Z_n)\right].$$ □

4 Application

The results are to be employed in sensitivity computations using Malliavin calculus in a financial derivative whose underlying is an interest rate with the dynamics

$$dr(t) = \mu(t)dt + \sum_{i=1}^{n}\sigma^i(t)dZ_i(t),$$

where $\mu(t)$ is the drift of the interest rate, $\sigma^i(t)$ denote the volatility function of the interest rate and Z_i is the ith Gaussian process. The stochastic noise is assumed to be determined by n uncorrelated Gaussian process. The result will also be employed in a phenomenon driven by a multi-dimensional jump-diffusion process for future research.

5 Conclusion

Uncorrelated random variables are very common in real life situation. We have derived the Ornstein-Uhlenbeck operator for such random variables. The extended operator derived above provides a better model when working on phenomenon with uncorrelated random variables, including, in sensitivity analysis.

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Conflict of interest

There is no conflict of interest.

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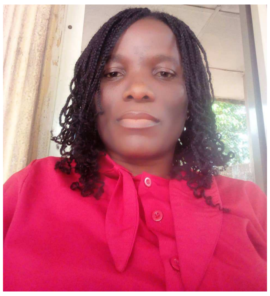

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