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Strong Convergence of Four Multivalued Nonexpansive Mappings

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Abstract: In this paper, a new three-step iterative scheme is introduced to approximate a common fixed point of four multivalued nonexpansive mappings in a uniformly convex Banach space and establish strong convergence theorems for the proposed process. Our results extend some existing results.

Keywords: Common fixed points, Three-step iterative scheme, Multi-valued mappings, Nonexpansive mapping, Banach spaces
Mathematics Subject Classification $47H10$, $54H25$

1 Introduction

Many authors have studied the fixed-point theorems for finite family of single valued mappings (see [3,8,10,13]). The study of fixed points for multivalued contractions and nonexpansive mappings using the Hausdorff metric was initiated by Markin [5] and Nadler [6]. Later, an interesting and rich fixed point theory for such maps was developed which has applications in control theory, convex optimization, differential inclusion and economics. Iterative methods for approximating fixed points of multivalued mappings in Banach spaces have been studied by Sastry and Babu [11] proved the convergence of Mann and Ishikawa iteration process for multivalued mapping T with a fixed point p converge to a fixed point q of T under certain conditions. They also claimed that the fixed point q may be different from p. Panyanak [9] extended result of Sastry and Babu to uniformly convex Banach spaces but the domain of T remains compact. Song and Wang [12] modified the iteration scheme used in [9] and improved the results presented therein. They further revised the gap and also gave the affirmative answer to Panyanaks open question.

Recently, M. Abbas et al.[1] introduced a new one-step iterative process to compute common fixed points of two multivalued nonexpansive mappings.

Very recently, M. Eslamian, A. Abkar [2] introduced a new one-step iterative process to approximate common fixed points of a finite family of generalized nonexpansive multivalued mappings and prove some weak and strong convergence theorems for such mappings in uniformly convex Banach spaces. They employed the following iterative process: Let E be a Banach space, K be a nonempty convex subset of E and $T_i : K \to CB(K)$ $(i = 1,2,...,m)$ be finitely many given mappings. Then, for $x_0 \in K$, we consider the following iterative process:

$$x_{n+1} = a_{n,0}x_n + a_{n,1}z_{n,1} + a_{n,2}z_{n,2} + ... + a_{n,m}z_{n,m}, \quad n \in \mathbb{N}, \tag{1.1}$$

where $z_{n,i} \in T_i(x_n)$ and $\{a_{n,k}\}$ are sequences of numbers in $[0,1]$ such that for every natural number $n \in \mathbb{N}$ and $\sum_{k=0}^{m} a_{n,k} = 1$.

In this paper, we introduce a new three-step iterative process to approximate the common fixed points of four finite families of multivalued nonexpansive mappings in a uniformly convex real Banach space and establish strong convergence theorems for the proposed process. Our results extend and improve the recent results.

2 Preliminaries

Let E be Banach space with dim $E \geq 2$, the modulus of convexity of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) = \inf\{1 - \frac{1}{2}\|x+y\| : \|x\| = 1, \|y\| = 1, \|x-y\| = \varepsilon\}.$$

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E is uniformly convex if and only if with $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$.

A subset K is called proximinal if for each $x \in E$, there exists an element $k \in K$ such that

$$d(x,k) = \inf\{\|x-y\| : y \in K\} = d(x,K).$$

It is known that a weakly compact convex subsets of a Banach space and closed convex subsets of a uniformly convex Banach space are proximinal. We shall denote the family of nonempty bounded proximinal subsets of K by $P(K)$ and the family of nonempty compact subsets of K by $C(K)$. Consistent with [6], let $CB(E)$ be the class of all nonempty bounded and closed subsets of E. Let H be a Hausdorff metric induced by the metric d of E, given by

$$H(A,B) = \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\},$$

for every $A,B \in CB(E)$. It is obvious that $P(K) \in CB(E)$.

A multivalued mapping $T : K \to P(K)$ is said to be a contraction if there exists a constant $k \in [0,1)$ such that for any $x,y \in K$,

$$H(Tx,Ty) \le k\|x-y\|,$$

and T is said to be nonexpansive if

$$H(Tx,Ty) \le \|x-y\|,$$

for all $x,y \in K$. A point $x \in K$ is called a fixed point of T if $x \in Tx$. Throughout the paper $\mathbb{N}$ denotes the set of all natural numbers.

Let us recall the following definitions.

Definition 2.1[4] *A mapping $T : K \to K$ where K a subset of E, is said to satisfy condition (A) if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ such that either $d(x,Tx) \ge f(d(x,F)$ for all $x \in K$, where $d(x,F) = \inf\{\|x-p\| : p \in F\}$.*

The following is the multivalued version of condition $(\grave{A})$;

Definition 2.2*The four multivalued nonexpansive mappings $A,T,S,R : K \to CB(K)$, where K a subset of E, are said to satisfy condition $(\grave{A})$ if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ such that either $d(x,Ax) \ge f(d(x,F)$ or $d(x,Tx) \ge f(d(x,F)$ or $d(x,Sx) \ge f(d(x,F)$ or $d(x,Rx) \ge f(d(x,F)$ for all $x \in K$, where $F = F(A)\bigcap F(T)\bigcap F(S)\bigcap F(R)$, the set of all common fixed points of the mappings A,T,S and R.*

Definition 2.3*The mapping $T : E \to CB(E)$, is called hemicompact if, for any sequence $\{x_n\}$ in E such that $d(x_n,Tx_n) \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_r}\}$ of $\{x_n\}$ such that $x_{n_r} \to p \in E$. We note that if E is compact, then every finite family of multivalued mapping $T : E \to CB(E)$ is hemicompact.*

Next we state the following useful lemma.

Lemma 2.1[7] *Let X be a uniformly convex Banach space and let $B_r(0) = \{x \in X : \|x\| \le r\}, r > 0$. Then there exist a continuous, strictly increasing, and convex function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ such that*

$$\|\alpha x+\beta y+\gamma z+\eta w\|^2 \le \alpha\|x\|^2+\beta\|y\|^2+\gamma\|z\|^2+\eta\|w\|^2-\alpha\beta\varphi(\|x-y\|)$$

for all $x,y,z,w \in B_r(0)$, and $\alpha,\beta,\gamma,\eta \in [0,1]$ with $\alpha+\beta+\gamma+\eta = 1$.

3 Main Results

We now introduce the following iteration scheme. Let E be Banach space, K be a nonempty subset of E and let $A,T,S,R : K \to CB(K)$ be four multivalued nonexpansive mappings with common fixed point P. Our process reads as follows:

$$\begin{aligned} x_0 &\in K,\\ x_{n+1} &= (1-\alpha_n-\beta_n-\gamma_n)a_n+\alpha_n u_n+\beta_n v_n+\gamma_n w_n,\\ y_n &= (1-\beta_n-\gamma_n)a_n+\beta_n v_n+\gamma_n w_n, \qquad (3.1)\\ z_n &= (1-\gamma_n)a_n+\gamma_n w_n, \end{aligned}$$

where $a_n \in Ax_n$, $u_n \in Ty_n$, $v_n \in Sz_n$, $w_n \in Rx_n$ and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequence of numbers in $[0,1]$ satisfying $\alpha_n+\beta_n+\gamma_n < 1$.

Remark 3.1 *1.If $A = S = R = T$. The iterative scheme (3.1) reduce to*

$$\begin{aligned} x_0 &\in K,\\ x_{n+1} &= (1-\alpha_n-\beta_n-\gamma_n)a_n+\alpha_n u_n+\beta_n v_n+\gamma_n w_n,\\ y_n &= (1-\beta_n-\gamma_n)a_n+\beta_n v_n+\gamma_n w_n, \qquad (3.2)\\ z_n &= (1-\gamma_n)a_n+\gamma_n w_n, \end{aligned}$$

where $a_n,w_n \in Tx_n$, $u_n \in Ty_n$ and $v_n \in Tz_n$.

2.If $A = I$. The iterative scheme (3.1) reduce to.

$$\begin{aligned} x_0 &\in K,\\ x_{n+1} &= (1-\alpha_n-\beta_n-\gamma_n)x_n+\alpha_n u_n+\beta_n v_n+\gamma_n w_n,\\ y_n &= (1-\beta_n-\gamma_n)x_n+\beta_n v_n+\gamma_n w_n, \qquad (3.3)\\ z_n &= (1-\gamma_n)x_n+\gamma_n w_n, \end{aligned}$$

where $u_n \in Ty_n$, $v_n \in Sz_n$, $w_n \in Rx_n$.

3.If $\gamma_n \equiv 0$. The iterative scheme (3.1) reduce to

$$\begin{aligned} x_0 &\in K,\\ x_{n+1} &= (1-\alpha_n-\beta_n)a_n+\alpha_n u_n+\beta_n v_n,\\ y_n &= (1-\beta_n)a_n+\beta_n v_n, \qquad (3.4) \end{aligned}$$

where $a_n \in Ax_n$, $u_n \in Ty_n$, $v_n \in Sx_n$.

The following is an example of four multivalued nonexpansive mappings with a common fixed point.

Example 1 Let $X = [0,1]$. Define $A,T,S,R : X \to CB(X)$ as follows:

$$Ax = [0, \frac{2x-1}{x^2}],$$

$$Tx = [0, \frac{x^2-\tan(\frac{5\Pi}{4})x+1}{x}],$$

$$Sx=[0,\frac{1}{x}],$$

and

$$Rx=[0,\frac{3\tan(\frac{\Pi}{4})x-x}{x+1}].$$

Then clearly A, T, S and R are four multivalued nonexpansive mappings and have a common fixed point at 1.

In this section, we prove that the iterative process defined by (3.1) converges strongly to a common fixed point.

At first, we shall prove the following lemmas.

Lemma 3.1*Let E be a uniformly convex Banach space and K a nonempty closed convex subset of E. Let $A,T,S,R:K\to CB(K)$ be four multivalued nonexpansive mappings. Let $\{x_n\}$ be the sequence as defined in (3.1). If $F\neq\emptyset$ and $Ap=Tp=Sp=Rp=\{p\}$ for any $p\in F$ then $\lim_{n\to\infty}\|x_n-p\|$ exists for all $p\in F$.*

*Proof.*Assume that $F\neq\emptyset$. Let $p\in F$. Then from (3.1) we have,

$$\begin{aligned}
&\|x_{n+1}-p\|\\
&=\|(1-\alpha_n-\beta_n-\gamma_n)a_n+\alpha_n u_n+\beta_n v_n+\gamma_n w_n-p\|\\
&=\|(1-\alpha_n-\beta_n-\gamma_n)(a_n-p)+\alpha_n(u_n-p)+\beta_n(v_n-p)+\gamma_n(w_n-p)\|\\
&\leq(1-\alpha_n-\beta_n-\gamma_n)\|a_n-p\|+\alpha_n\|u_n-p\|+\beta_n\|v_n-p\|+\gamma_n\|w_n-p\|\\
&\leq(1-\alpha_n-\beta_n-\gamma_n)d(a_n,Ap)+\alpha_n d(u_n,Tp)+\beta_n d(v_n,Sp)+\gamma_n d(w_n,Rp)\\
&\leq(1-\alpha_n-\beta_n-\gamma_n)H(Ax_n,Ap)+\alpha_n H(Ty_n,Tp)+\beta_n H(Sz_n,Sp)+\gamma_n H(Rx_n,Rp)\\
&\leq(1-\alpha_n-\beta_n-\gamma_n)\|x_n-p\|+\alpha_n\|y_n-p\|+\beta_n\|z_n-p\|+\gamma_n\|x_n-p\|\\
&=(1-\alpha_n-\beta_n)\|x_n-p\|+\alpha_n\|y_n-p\|+\beta_n\|z_n-p\|, \qquad (3.5)
\end{aligned}$$

and

$$\begin{aligned}
\|y_n-p\|&=\|(1-\beta_n-\gamma_n)a_n+\beta_n v_n+\gamma_n w_n-p\|\\
&=\|(1-\beta_n-\gamma_n)(a_n-p)+\beta_n(v_n-p)+\gamma_n(w_n-p)\|\\
&\leq(1-\beta_n-\gamma_n)\|a_n-p\|+\beta_n\|v_n-p\|+\gamma_n\|w_n-p\|\\
&\leq(1-\beta_n-\gamma_n)d(a_n,Ap)+\beta_n d(v_n,Sp)+\gamma_n d(w_n,Rp)\\
&\leq(1-\beta_n-\gamma_n)H(Ax_n,Ap)+\beta_n H(Sz_n,Sp)+\gamma_n H(Rx_n,Rp)\\
&\leq(1-\beta_n-\gamma_n)\|x_n-p\|+\beta_n\|z_n-p\|+\gamma_n\|x_n-p\|\\
&=(1-\beta_n)\|x_n-p\|+\beta_n\|z_n-p\|, \qquad (3.6)
\end{aligned}$$

and

$$\begin{aligned}
\|z_n-p\|&=\|(1-\gamma_n)a_n+\gamma_n w_n-p\|\\
&=\|(1-\gamma_n)(a_n-p)+\gamma_n(w_n-p)\|\\
&\leq(1-\gamma_n)\|a_n-p\|+\gamma_n\|w_n-p\|\\
&\leq(1-\gamma_n)d(a_n,Ap)+\gamma_n d(w_n,Rp)\\
&\leq(1-\gamma_n)H(Ax_n,Ap)+\gamma_n H(Rx_n,Rp)\\
&\leq(1-\gamma_n)\|x_n-p\|+\gamma_n\|x_n-p\|\\
&=\|x_n-p\|. \qquad (3.7)
\end{aligned}$$

Substituting (3.7) into (3.6) we obtain,

$$\begin{aligned}
\|y_n-p\|&\leq(1-\beta_n)\|x_n-p\|+\beta_n\|x_n-p\|\\
&=\|x_n-p\|. \qquad (3.8)
\end{aligned}$$

Substituting (3.7) and (3.8) into (3.5) we have

$$\begin{aligned}
\|x_{n+1}-p\|&\leq(1-\alpha_n-\beta_n)\|x_n-p\|+\alpha_n\|x_n-p\|+\beta_n\|x_n-p\|\\
&=\|x_n-p\|. \qquad (3.9)
\end{aligned}$$

Thus $\lim_{n\to\infty}\|x_n-p\|$ exists for each $p\in F$, hence $\{x_n\}$ is bounded.

Lemma 3.2*Let E be a uniformly convex Banach space and K be nonempty closed convex subset of E. Let $A,T,S,R:K\to CB(K)$ be four multivalued nonexpansive mappings and $\{x_n\}$ be the sequence as defined in (3.1). If $F\neq\emptyset$ and $Ap=Tp=Sp=Rp=\{p\}$ for any $p\in F$ and*

$$\|x_n-u_n\|\leq\|a_n-u_n\|, \qquad (3.10)$$

then $\lim_{n\to\infty}d(x_n,Ax_n)=\lim_{n\to\infty}d(x_n,Ty_n)=\lim_{n\to\infty}d(x_n,Sz_n)=\lim_{n\to\infty}d(x_n,Rx_n)=0$.

*Proof.*Let $p\in F$. By Lemma (3.1), $\lim_{n\to\infty}\|x_n-p\|$ exists, $\{x_n\}$ is bounded and so $\{y_n\}$ and $\{z_n\}$ are bounded. Therefore, there exists $r>0$ such that $x_n-p,\ y_n-p,\ z_n-p\in B_r(0)$ for all $n\geq 0$. Applying lemma (2.1) and using (3.1) we have

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
&=\|(1-\alpha_n-\beta_n-\gamma_n)a_n+\alpha_n u_n+\beta_n v_n+\gamma_n w_n-p\|^2\\
&=\|(1-\alpha_n-\beta_n-\gamma_n)(a_n-p)+\alpha_n(u_n-p)+\beta_n(v_n-p)+\gamma_n(w_n-p)\|^2\\
&\leq(1-\alpha_n-\beta_n-\gamma_n)\|a_n-p\|^2+\alpha_n\|u_n-p\|^2+\beta_n\|v_n-p\|^2+\gamma_n\|w_n-p\|^2\\
&-\alpha_n(1-\alpha_n-\beta_n-\gamma_n)\varphi(\|a_n-u_n\|)\\
&\leq(1-\alpha_n-\beta_n-\gamma_n)d(a_n,Ap)^2+\alpha_n d(u_n,Tp)^2+\beta_n d(v_n,Sp)^2+\gamma_n d(w_n,Rp)^2\\
&-\alpha_n(1-\alpha_n-\beta_n-\gamma_n)\varphi(\|a_n-u_n\|)\\
&\leq(1-\alpha_n-\beta_n-\gamma_n)H(Ax_n,Ap)^2+\alpha_n H(Ty_n,Tp)^2+\beta_n H(Sz_n,Sp)^2\\
&+\gamma_n H(Rx_n,Rp)^2-\alpha_n(1-\alpha_n-\beta_n-\gamma_n)\varphi(\|a_n-u_n\|)\\
&\leq(1-\alpha_n-\beta_n-\gamma_n)\|x_n-p\|^2+\alpha_n\|y_n-p\|^2+\beta_n\|z_n-p\|^2\\
&+\gamma_n\|x_n-p\|^2-\alpha_n(1-\alpha_n-\beta_n-\gamma_n)\varphi(\|a_n-u_n\|)\\
&\leq(1-\alpha_n-\beta_n)\|x_n-p\|^2+\alpha_n\|y_n-p\|^2+\beta_n\|z_n-p\|^2\\
&-\alpha_n(1-\alpha_n-\beta_n-\gamma_n)\varphi(\|a_n-u_n\|), \qquad (3.11)
\end{aligned}$$

$$\begin{aligned}
\|y_n-p\|^2&=\|(1-\beta_n-\gamma_n)a_n+\beta_n v_n+\gamma_n w_n-p\|^2\\
&=\|(1-\beta_n-\gamma_n)(a_n-p)+\beta_n(v_n-p)+\gamma_n(w_n-p)\|^2\\
&\leq(1-\beta_n-\gamma_n)\|a_n-p\|^2+\beta_n\|v_n-p\|^2+\gamma_n\|w_n-p)\|^2\\
&-\beta_n(1-\beta_n-\gamma_n)\varphi(\|a_n-v_n\|)\\
&\leq(1-\beta_n-\gamma_n)d(a_n,Ap)^2+\beta_n d(v_n,Sp)^2+\gamma_n d(w_n,Rp)^2\\
&-\beta_n(1-\beta_n-\gamma_n)\varphi(\|a_n-v_n\|)\\
&\leq(1-\beta_n-\gamma_n)H(Ax_n,Ap)^2+\beta_n H(Sz_n,Sp)^2+\gamma_n d(Rx_n,Rp)^2\\
&-\beta_n(1-\beta_n-\gamma_n)\varphi(\|a_n-v_n\|)\\
&\leq(1-\beta_n-\gamma_n)\|x_n-p\|^2+\beta_n\|z_n-p\|^2+\gamma_n\|x_n-p\|^2\\
&-\beta_n(1-\beta_n-\gamma_n)\varphi(\|a_n-v_n\|)\\
&\leq(1-\beta_n)\|x_n-p\|^2+\beta_n\|z_n-p\|^2-\beta_n(1-\beta_n-\gamma_n)\varphi(\|a_n-v_n\|), \qquad (3.12)
\end{aligned}$$

and

$$\begin{aligned}
\|z_n-p\|^2&=\|(1-\gamma_n)a_n+\gamma_n w_n-p\|^2\\
&=\|(1-\gamma_n)(a_n-p)+\gamma_n(w_n-p)\|^2\\
&\leq(1-\gamma_n)\|a_n-p\|^2+\gamma_n\|w_n-p\|^2-\gamma_n(1-\gamma_n)\varphi(\|a_n-w_n\|)\\
&\leq(1-\gamma_n)d(a_n,Ap)^2+\gamma_n d(w_n,Rp)^2-\gamma_n(1-\gamma_n)\varphi(\|a_n-w_n\|)\\
&\leq(1-\gamma_n)H(Ax_n,Ap)^2+\gamma_n H(Rx_n,Rp)^2-\gamma_n(1-\gamma_n)\varphi(\|a_n-w_n\|)\\
&\leq(1-\gamma_n)\|x_n-p\|^2+\gamma_n\|x_n-p\|^2-\gamma_n(1-\gamma_n)\varphi(\|a_n-w_n\|)\\
&=\|x_n-p\|^2-\gamma_n(1-\gamma_n)\varphi(\|a_n-w_n\|). \qquad (3.13)
\end{aligned}$$

From (3.12) and (3.13) we have

$$\begin{aligned}
\|y_n-p\|^2&\leq(1-\beta_n)\|x_n-p\|^2+\beta_n[\|x_n-p\|^2-\gamma_n(1-\gamma_n)\varphi(\|a_n-w_n\|)]\\
&-\beta_n(1-\beta_n-\gamma_n)\varphi(\|a_n-v_n\|)\\
&\leq\|x_n-p\|^2-\beta_n\gamma_n(1-\gamma_n)\varphi(\|a_n-w_n\|)-\beta_n(1-\beta_n-\gamma_n)\varphi(\|a_n-v_n\|). \qquad (3.14)
\end{aligned}$$

Substituting (3.13) and (3.14) into (3.11) we obtain

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
\le\ &(1-\alpha_n-\beta_n)\|x_n-p\|^2+\alpha_n[\|x_n-p\|^2-\beta_n\gamma_n(1-\gamma_n)\varphi(\|a_n-w_n\|)]\\
-\ &\beta_n(1-\beta_n-\gamma_n)\varphi(\|a_n-v_n\|)+\beta_n[\|x_n-p\|^2-\gamma_n(1-\gamma_n)\varphi(\|a_n-w_n\|)]\\
-\ &\alpha_n(1-\alpha_n-\beta_n-\gamma_n)\varphi(\|a_n-u_n\|)\\
\le\ &\|x_n-p\|^2-\beta_n(1-\beta_n-\gamma_n)\varphi(\|a_n-v_n\|)-\beta_n\gamma_n(1-\gamma_n)(\alpha_n+1)\varphi(\|a_n-w_n\|)\\
-\ &\alpha_n(1-\alpha_n-\beta_n-\gamma_n)\varphi(\|a_n-u_n\|) \qquad (3.15)
\end{aligned}$$

From (3.15)we obtain,

$$\begin{aligned}
&\alpha_n(1-\alpha_n-\beta_n-\gamma_n)\varphi(\|a_n-u_n\|)\\
\le\ &\|x_n-p\|^2-\|x_{n+1}-p\|^2-\beta_n\gamma_n(1-\gamma_n)(\alpha_n+1)\varphi(\|a_n-w_n\|)\\
-\ &\beta_n(1-\beta_n-\gamma_n)\varphi(\|a_n-v_n\|) \qquad (3.16)
\end{aligned}$$

Thus,

$$\alpha_n(1-\alpha_n-\beta_n-\gamma_n)\varphi(\|a_n-u_n\|)\le\|x_n-p\|^2-\|x_{n+1}-p\|^2,$$

this implies,

$$\alpha_n(1-\alpha_n-\beta_n-\gamma_n)\varphi(\|a_n-u_n\|)\le\|x_1-p\|^2\le\infty.$$

$$\sum_{n=1}^{\infty}\varphi(\|a_n-u_n\|)\le\|x_1-p\|^2\le\infty.$$

Since φ is continuous at 0 and is strictly increasing, we have

$$\lim_{n\to\infty}\|a_n-u_n\|=0. \qquad (3.17)$$

Using (3.10) and (3.17) it follows then that

$$\begin{aligned}\|a_n-x_n\|&\le\|a_n-u_n\|+\|u_n-x_n\|\\&\le 2\|a_n-u_n\|\to 0\quad n\to\infty.\end{aligned} \qquad (3.18)$$

and using (3.17) and (3.18) we have

$$\begin{aligned}\|u_n-x_n\|&\le\|u_n-a_n\|+\|a_n-x_n\|\\&\to 0\quad n\to\infty.\end{aligned} \qquad (3.19)$$

Similarly from (3.16) we obtain that

$$\lim_{n\to\infty}\|a_n-v_n\|=0. \qquad (3.20)$$

Using (3.18) and (3.20) it follows then that

$$\begin{aligned}\|x_n-v_n\|&\le\|a_n-x_n\|+\|a_n-v_n\|\\&\to 0\quad n\to\infty.\end{aligned} \qquad (3.21)$$

Similarly from (3.16) we obtain that

$$\lim_{n\to\infty}\|a_n-w_n\|=0. \qquad (3.22)$$

Using (3.18) and (3.22) it follows then that

$$\begin{aligned}\|x_n-w_n\|&\le\|x_n-a_n\|+\|a_n-w_n\|\\&\to 0\quad n\to\infty.\end{aligned} \qquad (3.23)$$

Thus from (3.18), (3.19), (3.21) and (3.23), we have $\lim_{n\to\infty}d(x_n,Ax_n)=\lim_{n\to\infty}d(x_n,Ty_n)=\lim_{n\to\infty}d(x_n,Sz_n)=\lim_{n\to\infty}d(x_n,Rx_n)=0.$

This completes the proof.

The following result gives a necessary and sufficient condition for strong convergence of the sequence (3.1) to a common fixed point of four mappings on a real Banach space.

Theorem 3.1*Let E be a uniformly convex Banach space and K, $\{x_n\}$ be as in lemma (3.2). Let $A,T,S,R:K\to CB(K)$, be four multivalued nonexpansive mappings satisfying condition $(\grave{A})$. If $F\neq\emptyset$ and $Ap=Tp=Sp=Rp=\{p\}$ for any $p\in F$, then $\{x_n\}$ converges strongly to a common fixed point of A, T, S and R.*

*Proof.*Since A, T,S and R, satisfies condition $(\grave{A})$, we have $\lim_{n\to\infty}f(d(x_n,F))=0$. Thus there is a subsequence $\{x_{n_r}\}$ of $\{x_n\}$ and a sequence $\{p_r\}\subset F$ such that

$$\|x_{n_r}-p_r\|<\frac{1}{2^r},$$

for all $r>0$. By lemma (3.1) we obtain that

$$\|x_{n_{r+1}}-p_r\|\le\|x_{n_r}-p_r\|<\frac{1}{2^r}.$$

We now show that $\{p_r\}$ is a Cauchy sequence in K. Observe that

$$\begin{aligned}\|p_{r+1}-p_r\|&\le\|p_{r+1}-x_{n_{r+1}}\|+\|x_{n_{r+1}}-p_r\|\\&<\frac{1}{2^{r+1}}+\frac{1}{2^r}\\&<\frac{1}{2^{r-1}}.\end{aligned}$$

This shows that $\{p_r\}$ is a Cauchy sequence in K and thus converges to $p\in K$. Since

$$\begin{aligned}d(p_r,Ap)&\le H(Ap,Ap_r)\\&\le\|p-p_r\|,\end{aligned}$$

and $p_r\to p$ as $r\to\infty$, it follows that $d(p,Ap)=0$, which implies that $p\in Ap$.

Similarly,

$$\begin{aligned}d(p_r,Tp)&\le H(Tp,Tp_r)\\&\le\|p-p_r\|,\end{aligned}$$

and $p_r\to p$ as $r\to\infty$, it follows that $d(p,Tp)=0$, which implies that $p\in Tp$.

Similarly,

$$\begin{aligned}d(p_r,Sp)&\le H(Sp,Sp_r)\\&\le\|p-p_r\|,\end{aligned}$$

and $p_r\to p$ as $r\to\infty$, it follows that $d(p,Sp)=0$, which implies that $p\in Sp$.

Similarly,

$$\begin{aligned}d(p_r,Rp)&\le H(Rp,Rp_r)\\&\le\|p-p_r\|,\end{aligned}$$

and $p_r\to p$ as $r\to\infty$, it follows that $d(p,Rp)=0$, which implies that $p\in Rp$. Consequently, $p\in F\neq\infty$. $\lim_{n\to\infty}\|x_n-p\|$ exists, we conclude that $\{x_n\}$ converges strongly to a common fixed point p.

Theorem 3.2 *Let E be a real Banach space and K, $\{x_n\}$, A,T,S,R, be as in Lemma(3.2). If $F \neq \emptyset$ and $Ap, Tp = Sp = Rp = \{p\}$ for any $p \in F$, then $\{x_n\}$ converges strongly to a common fixed point of A, T, S and R iff $\liminf_{n\to\infty} d(x_n,F) = 0$.*

Proof. The necessity is obvious. Conversely, suppose that $\liminf_{n\to\infty} d(x_n,F) = 0$. As proved in lemma (3.1),

$$\|x_{n+1} - p\| \leq \|x_n - p\|.$$

This gives

$$d(x_{n+1},F) \leq d(x_n,F),$$

so that $\lim_{n\to\infty} d(x_n,F)$ exists. But, by hypothesis, $\liminf_{n\to\infty} d(x_n,F) = 0$. Therefore we must have $\lim_{n\to\infty} d(x_n,F) = 0$. Next we show that $\{x_n\}$ is a Cauchy sequence in K. Let $\varepsilon > 0$ be arbitrarily chosen. Since $\lim_{n\to\infty} d(x_n,F) = 0$, there exists a constant n_0 such that for all $n \geq n_0$, we have

$$\lim_{n\to\infty} d(x_n,F) < \frac{\varepsilon}{4}.$$

In particular, $\inf\{\|x_{n_0} - p\| : p \in F\} < \frac{\varepsilon}{4}$. There must exist a $p^* \in F$ such that

$$\|x_{n_0} - p^*\| < \frac{\varepsilon}{2}.$$

Now for $m,n \geq n_0$, we have

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - p^*\| + \|x_n - p^*\| \\ &\leq 2\|x_{n_0} - p^*\| \\ &< 2(\frac{\varepsilon}{2}) = \varepsilon. \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in a closed subset K of a Banach space E, and therefore it must converge in K. Let $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = p$ and

$$\begin{aligned} d(x_n,y_n) &\leq d(x_n,p) + d(p,y_n) \\ &\to 0 \ \ as \ \ n \to \infty, \end{aligned}$$

and

$$\begin{aligned} d(x_n,z_n) &\leq d(x_n,p) + d(p,z_n) \\ &\to 0 \ \ as \ \ n \to \infty. \end{aligned}$$

Now for each we obtain,

$$\begin{aligned} d(p,Ap) &\leq d(p,x_n) + d(x_n,Ax_n) + H(Ax_n,Ap) \\ &\leq d(p,x_n) + d(x_n,a_n) + d(x_n,p) \\ &\to 0 \ \ as \ \ n \to \infty, \end{aligned}$$

gives that $d(p,Ap) = 0$, implies that $p \in Ap$. Consequently, $p \in F \neq \infty$.
Similarly, we obtain

$$\begin{aligned} d(p,Tp) &\leq d(p,y_n) + d(y_n,x_n) + d(x_n,Ty_n) + H(Ty_n,Tp) \\ &\leq d(p,y_n) + d(y_n,x_n) + d(x_n,u_n) + d(y_n,p) \\ &\to 0 \ \ as \ \ n \to \infty, \end{aligned}$$

gives that $d(p,Tp) = 0$, which implies that $p \in T_i p$.
Similarly, we obtain

$$\begin{aligned} d(p,Sp) &\leq d(p,z_n) + d(z_n,x_n) + d(x_n,Sz_n) + H(Sz_n,Sp) \\ &\leq d(p,z_n) + d(z_n,x_n) + d(x_n,v_n) + d(z_n,p) \\ &\to 0 \ \ as \ \ n \to \infty, \end{aligned}$$

gives that $d(p,Sp) = 0$, which implies that $p \in Sp$.
Similarly, we obtain

$$\begin{aligned} d(p,Rp) &\leq d(p,x_n) + d(x_n,Rx_n) + H(Rx_n,Rp) \\ &\leq d(p,x_n) + d(x_n,w_n) + d(x_n,p) \\ &\to 0 \ \ as \ \ n \to \infty, \end{aligned}$$

gives that $d(p,Rp) = 0$, implies that $p \in Rp$. Consequently, $p \in F \neq \infty$.

Theorem 3.3 *Let E be a uniformly convex Banach space and K, $\{x_n\}$ be as in Lemma (3.2). Let $A,T,S,R : K \to CB(K)$, be four multivalued nonexpansive mappings and A,T,S and R are hemicompact and continuous. If $F \neq \emptyset$ and $Ap = Tp = Sp = Rp = \{p\}$ for any $p \in F$, then $\{x_n\}$ converges strongly to a common fixed point of A,T, S and R.*

Proof. Since $\lim_{n\to\infty} d(x_n,Ax_n) = \lim_{n\to\infty} d(x_n,Ty_n) = \lim_{n\to\infty} d(x_n,Sz_n) = 0 = \lim_{n\to\infty} d(x_n,Rx_n)$, and A,T,S and R_i are hemicompact, there is a subsequence $\{x_{n_r}\}$ of $\{x_n\}$ such that $x_{n_r} \to p$ as $r \to \infty$ for some $p \in K$. Since A,T,S and R are continuous, we have

$$d(x_{n_r},Ax_{n_r}) \to d(p,Ap),$$

$$d(x_{n_r},Tx_{n_r}) \to d(p,Tp),$$

$$d(x_{n_r},Sx_{n_r}) \to d(p,Sp),$$

and

$$d(x_{n_r},Rx_{n_r}) \to d(p,Rp).$$

As a result, we have that $d(p,Ap) = d(p,Tp) = d(p,Sp) = d(p,Rp) = 0$ and $p \in F$. Since $\lim_{n\to\infty} \|x_n - p\|$ exists, it follows that $\{x_n\}$ converges strongly to p. This completes the proof.

Corollary 3.1 *Let E be a uniformly convex Banach space and K a nonempty closed convex subset of E. Let T,S,R, be three multivalued nonexpansive mappings and $\{x_n\}$ (3.3) and T,S and R are hemicompact and continuous. If $F \neq \emptyset$ and $Tp = Sp = Rp = \{p\}$ for any $p \in F$, then $\{x_n\}$ converges strongly to a common fixed point of T, S and R.*

Corollary 3.2 *Let E be a uniformly convex Banach space and K a nonempty closed convex subset of E. Let A,T,S, be three multivalued nonexpansive mappings and $\{x_n\}$ (3.4) and A, T and S are hemicompact and continuous. If $F \neq \emptyset$ and $Ap = Tp = Sp = \{p\}$ for any $p \in F$, then $\{x_n\}$ converges strongly to a common fixed point of A_i, T_i and S_i.*

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