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A New Approach for Solving Fredholm Integro-Differential Equations

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Abstract: In the present article, we extend the optimal auxiliary function method (OAFM) to integro-differential equations. The method has especially contained the auxiliary function and convergence control parameters that accelerate the method's convergence. The numerical results, obtained by the OAFM are listed with those obtained by the ADM, CAS Wavelet method, and ESA method. Further, the obtained results have been compared with the exact solution through different graphs and tables, which shows that the proposed method is very effective and easy to implement for different fractional-order PDEs.

Keywords: Optimal Axillary Function Method (OAFM), Fredholm Integro-Differential Equations.

1 Introduction

Integral equations play a key role in the field of Science and technology. Different problems related to these types of problems are given in the series of problems [1-4]. Due to the rare exact solution to these types of problems, researchers used different approximate analytical methods. Some well-known methods are homotopy perturbation method (HPM) [5], Homotopy analysis method (HAM) [6], Differential transformation method (DTM) [7], Genocchi polynomials (GP) [8] reproducing kernel Hilbert space method (RKHSM) [9], Chebyshev wavelets method (CWM) [10] fixed point method (FPM) [11], Sinc-collocation method (SCM) [12], Haar functions method (HFM) [13], Adomian decomposition method (ADM) [14] and relaxed Monte Carlo method (RMCM) [15]. These methods have their advantages and disadvantages. Like perturbation methods have to need small or large parameters and numerical methods have discretization issues. There is a proper method to choose an artificial parameter assumption in the equation.

In the same field of research, we introduce another approach, which does need any small or large parameter assumption and not any discretization. The approach is known, the optimal axillary function method (OAFM). This approach was presented by Marinca et al. and used to find the series solution of the thin-film flow of a fourth-grade fluid down vertical cylinder [16]. Later on, the method has been extended by Laiq Zada et al. to the partial differential equation and used for the Korteweg-Devries equations arising in shallow water waves [18]. The beauty of the method is that there is no need for artificial small or large parameter assumptions like other analytical methods. The OAFM method gives a series solution after only one iteration.

The remaining part of the present article is organized as follows. Section 1 was the introduction. The second section is about OAFM. In section three, about the implementation of OAFM with examples. The last section is devoted for the conclusion.

2 OAFM for the Integro-Differential Equations

The extension of the optimal axillary function method (OAFM) to general integro-differential equations can be discussed in the following steps. Let us take the following general Integro- differential equations as,

$$F'(\tau) = g(\tau) + \int_a^b K(\eta,\tau)F(\tau)\,d\tau = 0, F(0) = F_0. \quad (1)$$

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Here, $\frac{d}{d\tau}$ can be replaced by L. Therefore, Eq. (1) can be written in the following form.

$$L(F(\tau)) = g(\tau) + \int_a^b K(\eta,\tau)F(\tau)\,d\tau = 0. \tag{2}$$

Step1: For finding the series solution for Eq. (2), we take the solution in the form of two components as follow,

$$\tilde{F}(\tau) = F_0(\tau) + F_1(\tau, C_i), i = 1,2,3,\ldots\ldots p \tag{4}$$

Step 2: By substituting Eq. (4) into Eq. (2), we get the zero-order and first-order solution by OAFM method.

$$L(F_0(\tau)) + L(F_1(\tau, C_i)) + g(\tau) + \int_a^b K(\eta,\tau)F_0(\tau)\,d\tau = 0. \tag{5}$$

Step 3: For finding initial approximation $F_0(\tau)$ and first $F_1(\tau)$ can be obtained from the following linear equations:

$$L[F_0(\tau) + g(\tau)] = 0, F_0(0) = 0, \tag{6}$$

$$L\left(F_1(\tau, C_i)\right) + \int_a^b \mathrm{K}(\eta,\tau)\left(F_0(\tau) + F_1(\tau, C_i)\right)\mathrm{d}\tau = 0, \quad F_1(\tau) = 0. \tag{7}$$

Step 4: Hence, the nonlinear term in Eq. (7), takes the following form,

$$\int_a^b \mathrm{K}(\eta,\tau)\left(F_0(\tau)+F_1(\tau,C_i)\right)\mathrm{d}\tau = \int_a^b \mathrm{K}(\eta,\tau)\left(F_0(\tau)\right)\mathrm{d}\tau + \sum_{k=1}^{\infty}\frac{F_1^k}{k!}\left(\int_a^b \mathrm{K}(\eta,\tau)\left(F_0(\tau)\right)\mathrm{d}\tau\right) \tag{8}$$

Step 5: We see the difficulties in Eq (8), for this purpose we consider another expression for Eq. (8) which can be easily solved. i.e.

$$L(F_1(\tau, C_i)) + A_1(F_0(\tau)) \int_a^b K(\eta,\tau)(F_0(\tau))\,d\tau + A_2\left(F_0(\eta,\tau), C_j\right) = 0, F_1(\tau) = 0, \tag{9}$$

Remark 1: Here A_1 and A_2 are two arbitrary auxiliary functions depending on the initial approximation $F_0(\tau)$ and a number of the unknown parameters C_i and C_j, $i = 1,2,3..., j = s+1, s+2, ..p$.

Remark 2: Here A_1 and A_2 are not unique. They are like $F_0(\tau)$ or $N\left[F_0(\eta,\tau)\right]$ or the combination of both $F_0(\eta,\tau)$ and $\int_a^b K(\eta,\tau)F_0(\tau)\,d\tau$.

Remark 3:

- If $F_0(\eta,\tau)$ or $\int_a^b K(\eta,\tau)F_0(\tau)\,d\tau$ a polynomial function then A_1 and A_2 should be the sum of polynomial functions.
- If $F_0(\eta,\tau)$ or $\int_a^b K(\eta,\tau)F_0(\tau)\,d\tau$ an exponential function then A_1 and A_2 should be the sum of exponential functions.
- If $F_0(\eta,\tau)$ or $\int_a^b K(\eta,\tau)F_0(\tau)\,d\tau$ a trigonometric function then A_1 and A_2 should be the sum of trigonometric functions.
- If in the special case $\int_a^b K(\eta,\tau)F_0(\tau)\,d\tau = 0$ then it is clear that $F_0(\eta,\tau)$ is an exact solution of Eq. (5)

Step 7: using inverse operator after substitution of Auxiliary function to Eq. (12), one can get the first-order approximate solution of $F_1(\eta,\tau)$.

Step 8: Different methods are used for finding the numerical values for C_i. Some of them Ritz method, collocation method, Galerkin's method, or least square method. Here we use the least method to minimize the errors.

$$J\left(C_i, C_j\right) = \int_0^\tau \int_\Omega R^2\left(\eta, \tau; C_i, C_j\right) d\eta d\tau. \tag{13}$$

Here R denotes the residual,

$$R\left(\tau, C_i, C_j\right) = \tilde{F}'(\tau) + g(\tau) + \int_a^b K(\eta,\tau)\tilde{F}(\tau)\,d\tau, i = 1,2,..s, j = S+1, S+2,..p. \tag{14}$$

3 Implementation of OAFM

In this part of the problem, we test our method for the integrodifferential equations. Numerical results and graphical results can prove the efficiency and accuracy of the proposed method. For the sack of simplicity, we used Mathematica 10.

3.1 Example 1

Consider the linear Fredholm integral-differential in the following form:

$$F'(\eta) = \eta e^\eta + e^\eta - \eta + \int_0^1 \eta F(\tau)d\tau, F(0) = 0 \tag{15}$$

The exact solution for Eq. (15) is,

$$F(\eta) = \eta e^\eta \tag{17}$$

In Eq. (15), we have linear and nonlinear terms are given below,

$$L(F) = F'(\eta), N(F) = -\int_0^1 \eta F(\tau)d\tau, g(\tau) = \eta e^{\eta} + e^{\eta} - \eta \tag{18}$$

The initial approximate $F_0(\eta,\tau)$ is obtained from eq. (9)

$$F_0'(\eta) - \eta e^{\eta} - e^{\eta} + \eta = 0. F_0(\eta) = 0. \tag{19}$$

Solution for the Eq. (19) is written as follow,

$$F_0(\eta) = \frac{1}{2}\eta(2e^{\eta} - \eta). \tag{20}$$

Using Eq. (20) into Eq. (18), the nonlinear term is,

The first approximation $F_1(\eta,\tau)$ is given by Eq. (12)

$$F_1'(\eta) = -\Delta_1(F_0(\eta))N[F_0(\eta)] - \Delta_2(F_0(\eta), C_j). \tag{22}$$

According to the nonlinear operator, we choose Δ_1 and Δ_2 as

$$\begin{cases} \Delta_1 = C_1\left(e^{\eta} - \frac{\eta}{2}\right) + C_2\left(e^{\eta} - \frac{\eta}{2}\right)^2 + C_3\left(e^{\eta} - \frac{\eta}{2}\right)^3, \\ \Delta_2 = C_4\left(e^{\eta} - \frac{\eta}{2}\right)^4 + C_5\left(e^{\eta} - \frac{\eta}{2}\right)^5 + C_6\left(e^{\eta} - \frac{\eta}{2}\right)^6. \end{cases} \tag{23}$$

Using eq. (20), and (21) into Eq. (22), and apply the inverse operator, we get the first approximation as

$$\begin{aligned} F_1(\eta,\tau) = {} & \frac{1}{2}\left(2e^{\eta} - \eta\right)\eta - 0.1388888888888889(-6C_1 - 13.5c2 - 29.91666666666 \\ & 6664C_3 - 27.7C_4 - 61.273333333333326C_5 - 171.30358333333336C_6 + 6C_1e^{\eta} + \\ & 12C_2e^{\eta} + 27C_3e^{\eta} + 21.599999999999998C_4e^{\eta} + 54C_5e^{\eta} + 162C_6e^{\eta} + 1.5C_2e^{2.\eta} + \\ & 2.25C_3e^{2\eta} + 2.6999999999999997C_4e^{2\eta} + 3.375C_5e^{2\eta} + 5.0625C_6e^{2\eta} + 0.66666666 \\ & 66666666C_3e^{3\eta} + 1.5999999999999999C_4e^{3\eta} + 1.3333333333333333C_5e^{3\eta} + 1.333 \\ & 3333333333333c6e^{3\eta} + 1.7999999999999998C_4e^{4\eta} + 1.125C_5e^{4\eta} + 0.84375C_6e^{4\eta} + \\ & 1.44C_5e^{5\eta} + 0.8639999999999999C_6e^{5\eta} + 1.2C_6e^{6\eta} - 6C_1e^{\eta}\eta - 12C_2e^{\eta}\eta - 27C_3e^{\eta} \\ & \eta - 21.599999999999998C_4e^{\eta}\eta - 54C_5e^{\eta}\eta - 162C_6e^{\eta}\eta - 3C_2e^{2\eta}\eta - 4.5C_3e^{2\eta}\eta - \\ & 5.3999999999999995C_4e^{2\eta}\eta - 6.75C_5e^{2\eta}\eta - 10.125C_6e^{2\eta}\eta - 2C_3e^{3\eta}\eta - 4.8C_4e^{3\eta}\eta \\ & -4C_5e^{3\eta}\eta - 4C_6e^{3\eta}\eta - 4.5C_5e^{4\eta}\eta - 3.375C_6e^{4\eta}\eta - 4.32C_6e^{5\eta}\eta + 6C_2e^{\eta}\eta^2 + 13.5C_3e^{\eta} \\ & \eta^2 + 10.799999999999999C_4e^{\eta}\eta^2 + 27C_5e^{\eta}\eta^2 + 81C_6e^{\eta}\eta^2 + 4.5C_3e^{2\eta}\eta^2 + 5.39999 \\ & 99999999995C_4e^{2\eta}\eta^2 + 6.75C_5e^{2\eta}\eta^2 + 10.125C_6e^{2\eta}\eta^2 + 5.999999999999999C_5e^{3\eta}\eta^2 \\ & +5.999999999999999C_6e^{3\eta}\eta^2 + 6.75C_6e^{4\eta}\eta^2 + C_1\eta^3 - 4.5C_3e^{\eta}\eta^3 - 3.5999999999 \\ & 999996C_4e^{\eta}\eta^3 - 9C_5e^{\eta}\eta^3 - 27C_6e^{\eta}\eta^3 - 4.5C_5e^{2\eta}\eta^3 - 6.75C_6e^{2\eta}\eta^3 - 5.99999999999 \\ & 9999C_6e^{3\eta}\eta^3 - 0.375C_2\eta^4 + 2.25C_5e^{\eta}\eta^4 + 6.75C_6e^{\eta}\eta^4 + 3.375C_6e^{2\eta}\eta^4 + 0.15000000 \\ & 000000002c3\eta^5 + 0.09C_4\eta^5 - 1.3499999999999999c6e^{\eta}\eta^5 - 0.0375C_5\eta^6 + 0.01607 \\ & 142857142857C_6\eta^7) \end{aligned} \tag{24}$$

Adding eq. (18) and eq. (22) we get 1st order approximate solution as

$$\tilde{F}(\eta) = F_0(\eta) + F_1(\eta, C_1, C_2, C_3, C_4, C_5, C_6). \tag{25}$$

$$N[F_0(\eta)] = -\int_0^{1\int} \eta F_0(\tau)dt \tag{21}$$

For finding unknown parameters C_i, we used the least square method. The numerical values of C_i are given as

$$C_1 = 3.373733421313527, C_2 = -3.5819082868832943, C_3 = 1.5184569051063876.$$

$$C_4 = -0.24224716321190523, C_5 = 0.2842195008402797, C_6 = -0.04230051739009267$$

Using these values in Eq.(25), we get the first-order approximate solution for prolem 1.

3.2 Example 2

Consider the linear Fredholm integral-differential in the following form:

$$F'(\eta) = 1 - \frac{1}{3}\eta + \int_0^1 \eta F(\tau)d\tau, F(0) = 0 \quad (26)$$

The exact solution for Eq. (26) is,

$$F(\eta) = \eta. \quad (27)$$

In Eq. (26), we have linear and nonlinear terms are given below

$$L(F) = F'(\eta), N(F) = -\int_0^1 \eta F(\tau)d\tau, g(\tau) = -1 + \frac{1}{3}\eta \quad (28)$$

The initial approximate $F_0(\eta,\tau)$ is obtained from eq. (9)

$$F_0'(\eta) - 1 + \frac{1}{3}\eta = 0. F_0(\eta) = \eta. \quad (29)$$

Solution for the Eq. (29) is written as follow,

$$F_0(\eta) = \frac{1}{6}\eta(6-\eta). \quad (30)$$

By substituting Eq. (20) into Eq. (18), the nonlinear term becomes

$$N[F_0(\eta)] = -\int_0^{1\int} \eta F_0(\tau)dt \quad (31)$$

The first approximation $F_1(\eta,\tau)$ is given by Eq. (12)

$$F_1'(\eta) = -\Delta_1(F_0(\eta))N[F_0(\eta)] - \Delta_2(F_0(\eta), C_j). \quad (32)$$

According to the nonlinear operator, we choose Δ_1 and Δ_2 as

$$\begin{cases} \Delta_1 = C_1(\eta)^2 + C_2(2\eta), \\ \Delta_2 = -C_3(2\eta). \end{cases} \quad (33)$$

Using eq. (30), and (31) into Eq. (32), and apply the inverse operator, we get the first approximation as $F_1(\eta) = C_3\eta^2 + 0.19444444444444445C_2\eta^3 + 0.07291666666666667C_1\eta^4$. (34)

Adding eq. (18) and eq. (22) we get 1st order approximate solution as

$$\tilde{F}(\eta,\tau) = C_3\eta^2 + 0.19444444444444445C_2\eta^3 + 0.07291666666666667C_1\eta^4 + \frac{1}{6}\eta(6-\eta). \quad (35)$$

For finding unknown parameters C_i, we used the least squre method. The numerical values of C_i are given as,

$$C_1 = 4.4900966975061266 \times 10^{-14}, C_2 = -3.1540280819972604 \times 10^{-14}, C_3 = 0.16666666666666968.$$

Using these values in Eq. (35), we get the first-order approximate solution for problem 2`.

3.2 Example 3

Consider the linear Fredholm integral-differential in the following form:

$$F'(\eta) = e^{-\eta} + e^{-1} - 1 + \int_0^1 F(\tau)d\tau, F(0) = 1 \quad (36)$$

The exact solution for Eq. (36) is,

$$F(\eta) = e^{-\eta}. \quad (37)$$

In Eq. (36), we have linear and nonlinear terms are given below,

$$L(F) = F'(\eta), N(F) = -\int_0^1 F(\tau)d\tau, g(\tau) = e^{-\eta} + e^{-1} + 1 \quad (38)$$

The initial approximate $F_0(\eta,\tau)$ is obtained from eq. (9)

$$F_0'(\eta) + e^{-\eta} + e^{-1} + 1 = 0. F_0(\eta) = 1. \quad (39)$$

Solution for the Eq. (39) is written as follow,

$$F_0(\eta) = -e^{-(1+\eta)}(-e - \eta e^{\eta} + \eta e^{1+\eta}). \quad (40)$$

Using Eq. (39) into Eq. (42), the nonlinear term tekes the following form,

$$N[F_0(\eta)] = -\int_0^{1\int} F_0(\tau)dt \quad (41)$$

The first approximation $F_1(\eta,\tau)$ is given by Eq. (12

$$F_1'(\eta) = -\Delta_1(F_0(\eta))N[F_0(\eta)] - \Delta_2(F_0(\eta), C_j). \quad (42)$$

According to the nonlinear operator, we choose Δ_1 and Δ_2 as

$$\begin{cases} \Delta_1 = C_1\eta e^{\eta} + C_2\eta e^{2\eta}. \\ \Delta_2 = -C_3. \end{cases} \quad (43)$$

Using eq. (43), and (41) into Eq. (42), and apply the inverse operator, we get the first approximation as

$$F_1(\eta) = e^{-\eta} + C_2\big(0.07901506985356971 + e^{2\eta}(-0.07901506985356971 + 0.15803013970713942\eta)\big) + C_1\big(0.31606027941427883 + e^{\eta}(-0.31606027941427883 +$$

$0.31606027941427883\eta)\big) - 0.6321205588285576\eta + C_3\eta.$ (44)

Adding eq. (18) and eq. (22) we get 1st order approximate solution as $\tilde{F}(\eta,\tau) = C_3\eta^2 + 0.19444444444444445C_2\eta^3 + 0.07291666666666667C_1\eta^4 + \frac{1}{6}\eta(6-\eta).$ (45)

For finding unknown parameters C_i, we used the least squre method. The numerical values of C_i are given as,

$$C_1 = 2.932751768256368\times10^{-15}, C_2 = -1.1755675453600516\times10^{-15}, C_3 = 0.6321205588285577.$$

Using these values in Eq. (45), we get the first-order approximate solution for problem 3.

4 Numerical and Graphical Discussions

Tables (1-2) shows the numerical comparison of OAFM, CAS Wavelet method, and ESA method for problem 1 and 2 respectively. Similarly, table 3, shows the numerical solution of OAFM and Block Pulse Functions and Operational Matrices for problem 3. Furthermore, in figures (1-3), we present the comparison between OAFM and the exact solution for problems 1, 2, and 3 respectively. Similarly figure (3-4) shows the absolute errors obtained by the proposed method.

Table 1: Comparison of absolute errors obtained by a different method and OAFM for problem 1 $\eta \in (0.0,1.0)$.

η	CAS Wavelet [18]	ESA [18]	The method in [20]	OAFM
0.1	1.34917637×10^{-3}	1.00118319×10^{-2}	2.433330×10^{-5}	5.44857×10^{-6}
0.2	6.38548213×10^{-3}	2.78651355×10^{-2}	9.735080×10^{-5}	2.80414×10^{-6}
0.3	7.91370487×10^{-3}	5.08730892×10^{-2}	2.193150×10^{-4}	2.57483×10^{-5}
0.4	2.15586005×10^{-2}	7.55356316×10^{-2}	3.917420×10^{-4}	2.01829×10^{-7}
0.5	4.99358429×10^{-2}	9.71888592×10^{-2}	6.200050×10^{-4}	5.44857×10^{-6}
0.6	2.21728810×10^{-2}	1.09551714×10^{-2}	9.184720×10^{-4}	2.80414×10^{-6}
0.7	1.05645449×10^{-2}	1.04133232×10^{-2}	1.319230×10^{-3}	2.57483×10^{-6}
0.8	1.43233681×10^{-2}	6.94512700×10^{-2}	1.885530×10^{-3}	2.01829×10^{-6}
0.9	2.07747461×10^{-2}	1.00034260×10^{-2}	2.731360×10^{-3}	5.44857×10^{-6}

Table 2: Comparison of absolute errors obtained by a different method and OAFM for problem 2 $\eta \in (0.0,1.0)$.

η	CAS Wavelet [18]	ESA [18]	The method in [20]	OAFM
0.1	2.17942375×10^{-4}	1.66666667×10^{-3}	2.06509×10^{-4}	2.77556×10^{-17}
0.2	6.38548213×10^{-4}	6.09388620×10^{-3}	8.04069×10^{-4}	8.32667×10^{-17}
0.3	7.91370487×10^{-4}	1.32017875×10^{-2}	1.72624×10^{-3}	1.11022×10^{-16}
0.4	2.15586005×10^{-2}	2.29140636×10^{-2}	2.86044×10^{-3}	2.22045×10^{-16}
0.5	4.99358429×10^{-3}	3.51578404×10^{-2}	4.04527×10^{-3}	1.66533×10^{-16}
0.6	2.21728810×10^{-2}	6.69648304×10^{-2}	9.184720×10^{-3}	1.11022×10^{-16}
0.7	1.05645449×10^{-4}	7.12430514×10^{-2}	5.06663×10^{-3}	1.11022×10^{-16}
0.8	1.43233681×10^{-3}	8.63983845×10^{-2}	5.65279×10^{-3}	1.11022×10^{-16}
0.9	2.07747461×10^{-2}	1.08103910×10^{-1}	4.10753×10^{-3}	0.0

Table 3: Comparison of absolute errors obtained by a different method and OAFM for problem 2 $\eta \in (0.0,1.0)$.

η	Exact Solution [19]	The method in [19]	OAFM Solution
0.1	0.904837	0.910993	0.904837
0.2	0.818731	0.804005	0.818731
0.3	0.740818	0.755324	0.740818
0.4	0.670320	0.666636	0.670320
0.5	0.606530	0.588375	0.606530
0.6	0.548812	0.552766	0.548812
0.7	0.496585	0.487894	0.496585
0.8	0.449329	0.458378	0.449329
0.9	0.406570	0.404606	0.406570

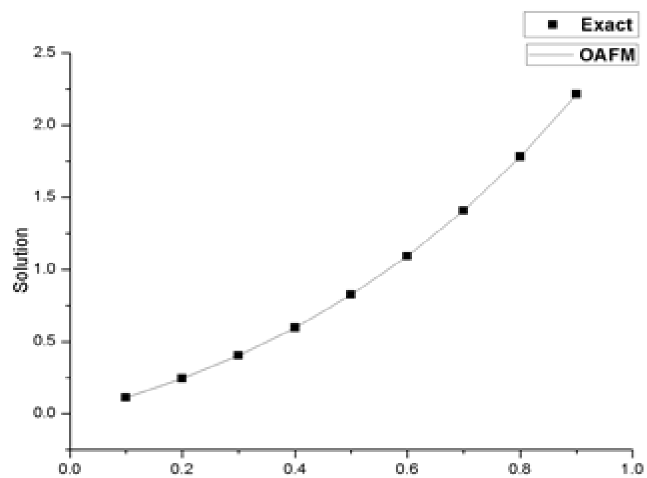

Fig. 1: 2D surfaces show the comparasion of OAFM and exact solution for the numerical example 1 when $\boldsymbol{\eta} \in$ $\mathbf{(0.0, 1.0)}$.

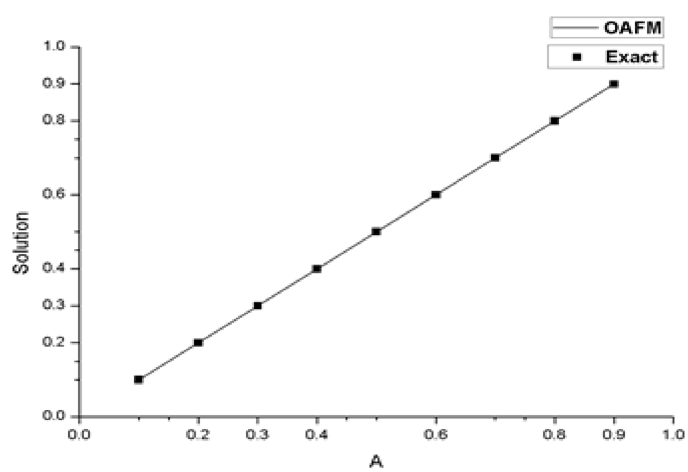

Fig.2: 2D surfaces show the comparison of OAFM and exact solution for the numerical example 2 when $\boldsymbol{\eta} \in$ $\mathbf{(0.0, 1.0)}$.

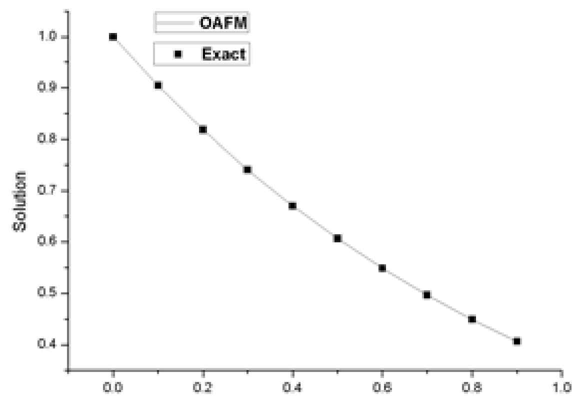

Fig.3: 2D surfaces show the comparison of OAFM and exact solution for the numerical example 3 when $\boldsymbol{\eta} \in (\mathbf{0.0}, \mathbf{1.0})$.

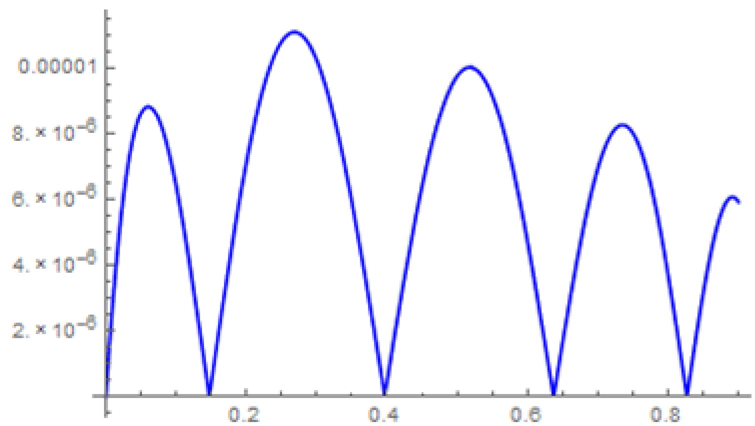

Fig. 4: 2D surface shows the absolute errors, obtained by the OAFM for the numerical example 1 when $\boldsymbol{\eta} \in (\mathbf{0.0}, \mathbf{1.0})$.

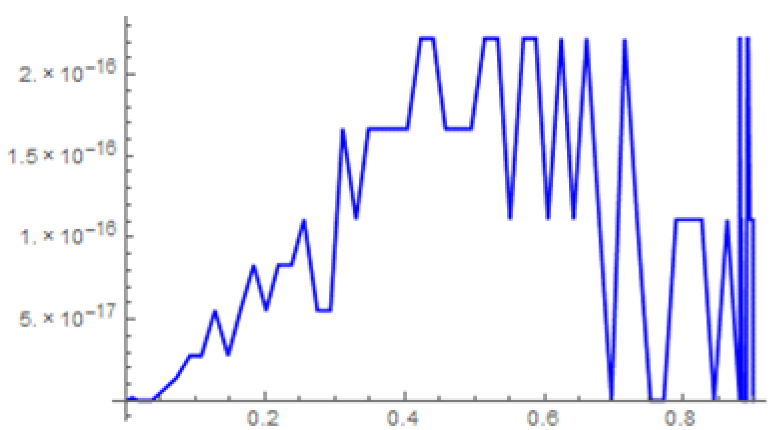

Fig.5: 2D surface shows the absolute errors, obtained by the OAFM for the numerical example 2 when $\boldsymbol{\eta} \in (\mathbf{0.0}, \mathbf{1.0})$.

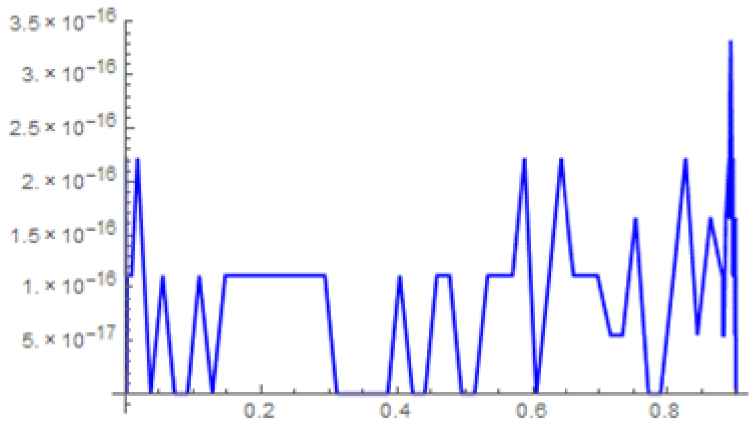

Fig.6: 2D surface shows the absolute errors, obtained by the OAFM for the numerical example 3 when $\boldsymbol{\eta} \in (\mathbf{0.0}, \mathbf{1.0})$

4 Conclusions

The optimal auxiliary function method (OAFM) has been extended to integrodifferential equations and has been used for a family of integrodifferential equation. After obtaining numerical and graphical results, we can say the proposed method has the following advantages.

- The method is very easy to implement and gives an approximate solution after only one iteration. and
- The proposed method contains auxiliary functions and convergence control parameters that control the convergence of the method.
- No need for small or large parameter assumptions into the equation to solve.
- If we want to increase the accuracy of the method, we just increase the number of convergence control parameters.

From the above conclusion, it clear that the method is very effective and can be extended to other nonlinear problems arising in different science and technology.

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