Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/120236

Goodness-of-Fit Tests for the Topp-Leone Distribution Based on Partial Functional Mean

Kamila Grara[*] *and Ahmad Zghoul*

Department of Mathematics, The University of Jordan, Amman 11942, Jordan

Received: 28 Jun. 2021, Revised: 2 Feb. 2022, Accepted: 17 Mar. 2022
Published online: 1 May 2023

Abstract: In this article, we propose two test statistics based on the partial functional mean to test the conformity of a random sample with the Topp-Leone distribution. Characterization of the distribution based on the partial functional mean has been proven. The tests are formed as the integrated deviation (ID) or integral square deviation(ISD) between the sample and population partial functional means. Compared to the Kolmogorov-Smirnov (KS), Cramer-von Mises (CM), and Anderson-Darling (AD) tests, the proposed tests, say $\hat{D}_{n,1}$and $\hat{D}_{n,2}$, generally perform better in terms of their powers. The dependence of the$\hat{D}_{n,1}$, $\hat{D}_{n,2}$,KS,CM, and AD tests on the skewness of the alternative is clear. In fact, we have found that$\hat{D}_{n,2}$,KS,CM, and AD tests generally have higher powers than $\hat{D}_{n,1}$ when testing against distributions with negative skewness and have lower powers when testing against distributions with positive skewness. We also noticed that $\hat{D}_{n,2}$ outperforms the KS,CM, and AD tests when testing against negatively skewed alternatives and $\hat{D}_{n,1}$outperforms all of these tests when testing against positively skewed alternatives. The percentiles and powers calculations were all based on Monte Carlo simulations.

Keywords: Topp-Leone Distribution; goodness-of-fit; partial functional mean;

1 Introduction

A random variable X is said to have Topp-Leone distribution of shape θ parameter and scale parameter β if its distribution function is given by

$$F(x) = \left[1-\left(1-\frac{x}{\beta}\right)^2\right]^\theta ,0<x<\beta,\theta>0 \tag{1}$$

and, consequently, its density is

$$f(x) = \frac{2\theta}{\beta}\left(1-\frac{x}{\beta}\right)\left[1-\left(1-\frac{x}{\beta}\right)^2\right]^{\theta-1} ,0<x<\beta,\theta>0 \tag{2}$$

This family of J-shaped distributions was first introduced by Topp and Leone (TL) [1]. No further developments in the distribution occurred until (2003) when Nadaraja and Kotz [2] drew attention to its suitability for application in the reliability analysis. Since then, many articles have appeared in the literature addressing different probabilistic and inferential aspects of the distribution; Van Dorp and Kotz [3] utilized the distribution to model income data. Some reliability measures of the distribution and their stochastic orderings were studied by Ghitany et al. [4]. Kotz and Seier [5] studied the kurtosis of the distribution. A two-sided generalization of the distribution was provided by Vicari et al. [6]. Zghoul [7,8] studied order statistics and record values for samples from the TL distribution. Order statistics were also considered by Genç [9], who in [10] estimated $P(X>Y)$ when X (strength) and Y (stress) are independent random variables from (TL) distribution. Bayesian inference was carried-out by Sindhu et al. [11] and Bayoud [12]. New classes of distributions based on TL were introduced by Rezaei et al. [13]. Zghoul [14] introduced plug-in estimators for the

[*] Corresponding author e-mail: KMY9160685@ju.edu.jo

shape parameter of TL. Al-Zahrani [15] used the Anderson-Darling goodness of fit test to model the data with the TL distribution. As far as we know, this is the only published article that deals in part with the goodness of fit (GOF) for the TL distribution.

There are good reasons to consider TL distribution for lifetime analysis. First, the closed form of its cumulative distribution function makes the distribution mathematically attractive. Second, its hazard rate function takes various shapes including the bathtub shape, which is the shape of the hazard functions of many real lifetime data. In addition, the distribution is defined on a bounded domain which is the domain of many reliability applications.

Our interest in this article is to test the null hypothesis that the distribution of a random sample is the TL distribution given in (1). Up to our knowledge, no tests have been proposed for this purpose. However, some GOF tests in the literature, could be applied to any model, particularly the empirical distribution function based tests such as the Kolmogorov Smirnov (KS), Cramer-von Mises (CM), and Anderson Darling (AD) tests. The distributions or asymptotic distributions of GOF tests depend not only on the hypothesized model, but also on the parameters, whether known or unknown. When the parameters are unknown and need to be estimated from the sample, the estimation method also affects the performance of the test. Consequently, the performance of any GOF test would be affected by one or more of these factors. For example, the Shapiro-Wilk test, which works very well as a test for normality, has comparatively poor performance when used to test for exponentiality. Since most test statistics, especially when distribution parameters are unknown, do not have a closed form, quantiles and tests power calculations must be either approximated (possibly based on the asymptotic distribution) or simulated. The latter method is generally used because it works for all sample sizes.

GOF tests are generally based on characterizations that uniquely determine the distribution. For example, the moment generating function, when exists, uniquely determines the distribution. Test statistics can therefore be constructed by measuring the difference between the empirical and theoretical moment generating functions of the assumed model.

The classes of GOF tests include, among others: Chi-square-based tests, empirical distribution-based tests, correlation-based tests, tests based on characteristic and moment- generating functions, and tests based on integral of the distribution functions. Besides Pearson [16] Chi-Square test, known as the first GOF test, examples of articles in this class are Fisher [17] and Rao and Robson [18]. The Kolmogorov-Smirnov (KS), Cramer- von Mises (CM) and Anderson-Darling (AD) tests are all based on the empirical distribution function. The book by D'Agostino and Stephens [19] contains detailed discussions of these tests. Examples of correlation-based tests are Shapiro and Wilk [20], Shapiro and Francia [21], and Coin [22]. Henze and Nikitin [23] and Klar [24] suggested tests based on an integral of the distribution function. Tests based on residual lifetime were proposed by Zghoul and Awad [25]. Examples of tests based on the characteristic function or the moment generating function are Epps et al. [26], Henze [27] and Zghoul [28].

In this article, we will present tests based on so-called partial functional moments. In Section 2, we will give a characterization of the TL distribution and propose test statistics based on this characterization. Investigation of the proposed test distributions and simulations of some of their quantiles will be presented in Section 3. Also in this section, quantiles of other tests will be simulated for power comparison purposes. In Section 4, we compare the performance of the proposed tests with that of the KS, CM and AD tests in terms of powers. The results of this article are summarized in Section 5.

2 Characterization of the TL distribution and composition of tests

We will denote the random variable X which has a Topp-Leone distribution with shape parameter θ and scale parameter β by $X \sim TL(\theta,\beta)$. If $\beta = 1$, we will simply write $X \sim TL(\theta)$.
Definition : Let X be a random variable with probability density function $f(x)$,$x \in (a,b)$, where a could be $-\infty$ and b could be ∞ . Then for any function $\psi(x)$ and real constants s and t with $a \le s < t \le b$, we call $\mu(s,t) = \int_s^t \psi(x)f(x)dx$ partial functional mean of $\psi(x)$ provided that $\int_s^t |\psi(x)|\, f(x)dx < \infty$.

In particular, if X is nonnegative random variable, $\psi(x) = x, t = \infty$, then $\mu(s) \equiv \int_s^\infty \psi(x)f(x)dx$ is the mean residual lifetime of X.

We now prove a characterization of the TL distribution based on the partial functional mean.

Let $\psi(x) = 1-(1-x)^2$, then the density of $TL(\theta)$ is

$$f(x,\theta) = \theta\psi^{'}(x)\,[\psi(x)]^{\theta-1}\,, 0 < x < 1, \theta > 0\,.$$

Theorem 1: If $g(x)$ is nonnegative function and θ is a positive real number, then

$$\int_s^1 \psi(x)g(x)dx = \mu_\theta(s), s > 0\,, \tag{3}$$

Where

$$\mu_\theta(s) = \frac{\theta}{\theta+1}\left[1-(\psi(s))^{\theta+1}\right],$$

iff $g(x)$ is the $TL(\theta)$ density.

Proof: Assume first that $g(x)$is the $TL(\theta)$ density, then

$$\begin{aligned}
\int_s^1 \psi(x)g(x)dx &= \int_s^1 \psi(x)\left[\theta\psi^{'}(x)(\psi(x))^{\theta-1}\right]dx \\
&= \int_s^1 \theta\psi^{'}(x)\psi(x)^{\theta}dx \\
&= \frac{\theta}{\theta+1}(\psi(x))^{\theta+1}\mid_s^1 \\
&= \frac{\theta}{\theta+1}\left[1-(\psi(s))^{\theta+1}\right] = \mu_\theta(s).
\end{aligned}$$

To show the sufficient condition, differentiate both sides of (3) with respect to s, to get

$$-\psi(s)g(s) = -\theta\psi^{'}(s)(\psi(s))^{\theta},$$

which gives $g(s) = \theta\psi^{'}(s)(\psi(s))^{\theta-1}, 0 < s < 1$; the $TL(\theta)$ density.

Assume $Y_1,...,Y_n$ be a random sample from $TL(\theta,\beta)$ and $X_1,...,X_n$ is the transformed sample with $X_i = \frac{Y_i}{\beta}, i = 1,...,n$. For now, assume that θ and β are known. To construct test statistics based on this sample using the characterization in Theorem 1, we introduce measures of discrepancy between $\mu_\theta(s)$ and the empirical counterpart of the left hand side of (3), which is given by $\frac{1}{n}\sum_{j=1}^{n}\psi(X_j)I_{(s,1)}(X_j)$, where $I_A(x)$ is the usual indicator function.

Let

$$D_n(X,\theta) = \frac{1}{n}\sum_{j=1}^{n}\psi(X_j)I_{(s,1)}(X_j) - \mu_\theta(s), \tag{4}$$

then, we will consider the following measures:

$$D_{n,1} = \int_s^1 D_n(X,\theta)\,w(s)\,ds \tag{5}$$

$$D_{n,2} = \int_s^1 D_n^2(X,\theta)\,w(s)\,ds \tag{6}$$

$$D_{n,3} = \int_s^1 |D_n(X,\theta)|\,w(s)\,ds \tag{7}$$

where $w(s)$ is a suitably chosen weight function. Here, $D_{n,1}$ represents total weighted deviation of the sample partial functional mean from the population partial functional mean, $D_{n,2}$ is the total weighted square deviation, and $D_{n,3}$ is the total weighted absolute deviation.

Because whether $D_n(X,\theta)$ is positive or negative depends on the sample, it seems difficult to handle $D_{n,3}$ analytically, so it will not be considered in this article. Integrating the r.h.s. of (5), we obtain

$$\begin{aligned}
D_{n,1} &= \frac{1}{n}\sum_{j=1}^{n}\int_0^{X_j}\psi(X_j)ds - \int_0^1 \mu_\theta(s)ds \\
&= \frac{1}{n}\sum_{j=1}^{n}X_j\psi(X_j) - \frac{\theta}{\theta+1}\left(1-\frac{1}{2}B\left(\theta+2,\frac{1}{2}\right)\right)
\end{aligned} \tag{8}$$

where $B(a,b)$ is the usual beta fuction given by

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}.$$

In the next theorem, we derive the mean and variance of $D_{n,1}$.

Theorem 2. Let $X_1,...,X_n$ be a random sample from $TL(\theta)$, then $E(D_{n,1}) = 0$ and

$$Var(D_{n,1}) = \sigma_\theta^2 = \frac{1}{n}\left[\frac{\theta}{\theta+2}\left(\frac{\theta+4}{\theta+3} - B(\theta+3,\frac{1}{2}\right) - \left(\frac{\theta}{\theta+1}\right)^2\left(1-\frac{1}{2}B\left(\theta+2,\frac{1}{2}\right)\right)^2\right] \tag{9}$$

Proof: $D_{n,1}$ is a random variable centered at its mean, so it is readily seen that $E(D_{n,1}) = 0$.

To compute the variance of $D_{n,1}$, we first notice that for $k > -1$,

$$\begin{aligned}\int_0^1 (\psi(s))^k ds &= \int_0^1 \left[1-(1-s)^2\right]^k ds \\ &= \frac{1}{2}\int_0^1 u^k (1-u)^{-1/2} du \\ &= \frac{1}{2}B\left(k+1,\frac{1}{2}\right),\end{aligned} \tag{10}$$

where u is set to $\left[1-(1-s)^2\right]$.

Applying the variance on (8), we obtain

$$\begin{aligned}Var(D_{n,1}) &= \frac{1}{n}Var(X\psi(X)) \\ &= \frac{1}{n}\left[E(X\psi(X))^2 - E^2(X\psi(X))\right]\end{aligned} \tag{11}$$

We have

$$\begin{aligned}E(X\psi(X)) &= \int_0^1 x\psi(x)f(x)dx \\ &= \theta\int_0^1 x\psi'(x)\psi(x)^\theta dx\end{aligned}$$

Integrating by parts and applying the result in (10), we get

$$E(X\psi(X)) = \frac{\theta}{\theta+1}\left[1-\frac{1}{2}Beta(\frac{1}{2},\theta+2)\right] \tag{12}$$

The second moment of $X\psi(X)$ is

$$\begin{aligned}E(X\psi(X))^2 &= \int_0^1 x^2\psi^2(x)f(x)dx \\ &= \theta\int_0^1 x^2\psi'(x)\psi(x)^{\theta+1}dx\end{aligned}$$

Integrating by parts, one has

$$E(X\psi(X))^2 = \frac{\theta}{\theta+2}\left[1-2\int_0^1 x\psi(x)^{\theta+2}dx\right] \tag{13}$$

By the definition of $\psi(x)$, we have $x = 1-\sqrt{1-\psi(x)}$, hence

$$\begin{aligned}\int_0^1 x\psi(x)^{\theta+2}dx &= \int_0^1\left(1-\sqrt{1-\psi(x)}\right)\psi(x)^{\theta+2}dx \\ &= \int_0^1 \psi(x)^{\theta+2}dx - \int_0^1\sqrt{1-\psi(x)}\psi(x)^{\theta+2}dx \\ &= \frac{1}{2}Beta(\theta+3,\frac{1}{2}) - \frac{1}{2(\theta+3)}\end{aligned} \tag{14}$$

Substituting (14) into (13), we obtain

$$E(X\psi(X))^2 = \frac{\theta}{\theta+2}\left[\frac{\theta+4}{\theta+3} - Beta(\theta+3,\frac{1}{2})\right] \tag{15}$$

Plugging (12) and (15) in (11), equation (9) is verified.

In the above discussion, we have assumed that both parameters θ and β are known, which is not the case in most applications. Let $\widehat{\theta}$ and $\widehat{\beta}$ be respective estimators of θ and β, and update $D_{n,1}$ accordingly by first divide each variable in the random sample $Y_1,...,Y_n$, assumed to be selected from $TL(\theta,\beta)$, by $\widehat{\beta}$, then replace θ in (8) by $\widehat{\theta}$. The updated test statistic $\widehat{D}_{n,1}$ is now

$$\widehat{D}_{n,1} = \frac{1}{n}\sum_{j=1}^{n} X_j\psi(X_j) - \frac{\widehat{\theta}}{\widehat{\theta}+1}\left[1-\frac{1}{2}Beta(\widehat{\theta}+2,\frac{1}{2})\right], \tag{16}$$

where $X_j = Y_j/\widehat{\beta}$,$j = 1,...,n$, which is a simple computational Form.

3 Distribution of the proposed tests

It seems that the exact distribution of (16) is difficult to accomplish. However, asymptotic distribution can be derived. If $\widehat{\theta}$ and $\widehat{\beta}$ are the maximum likelihood estimators of θ and β, then $\widehat{\theta} \xrightarrow{P} \theta$ and $\widehat{\beta} \xrightarrow{P} \beta$, as $n \to \infty$, where $\xrightarrow{P}$ denotes convergence in probability. Then, by the continuous map theorem,
$\frac{Y_j}{\widehat{\beta}} \xrightarrow{P} \frac{Y_j}{\beta}$ and $\frac{\widehat{\theta}}{\widehat{\theta}+1}\left[1-\frac{1}{2}Beta(\widehat{\theta}+2,\frac{1}{2})\right] \xrightarrow{P} \frac{\theta}{\theta+1}\left[1-\frac{1}{2}Beta(\theta+2,\frac{1}{2})\right]$.

Thus, by Slutsky's Theorem, $\widehat{D}_{n,1} \xrightarrow{D} D_{n,1}$, as $n \to \infty$, where $\xrightarrow{D}$ refers to convergence in distribution.

The statistic $D_{n,1}$ is a sum of independent and identically random variables with finite variance, so by the central limit theorem we have $\sqrt{n}D_{n,1} \xrightarrow{D} N\left(0,\sigma_\theta^2\right)$, where σ_θ^2 is as given in (9). Thus,$\sqrt{n}\widehat{D}_{n,1}$ is approximately $N\left(0,\sigma_\theta^2\right)$. The accuracy of this approximation is not guaranteed especially for small values of n. Consequently, we will use simulations to calculate some quantiles for $\widehat{D}_{n,1}$ and then to compute approximated powers. Simulated needed quantiles of $\widehat{D}_{n,1}$ and $\widehat{D}_{n,2}$ (to be introduced soon) are given in Table 1.

We now consider $D_{n,2} = \int_s^1 D_n^2(X,\theta)\,w(s)\,ds$. Here, we also assume that $w(s) = 1$. To derive a computational form, we evaluate the following integral

$$\begin{aligned}
D_{n,2} &= \int_0^1 \left(\frac{1}{n}\sum_{j=1}^{n}\psi(X_j)I_{(s,1)}(X_j) - \mu_\theta(s)\right)^2 ds \\
&= \frac{1}{n^2}\sum_{i,j=1}^{n}\psi(X_i)\psi(X_j)\,(X_i\Lambda X_j) - \frac{2\theta}{n(\theta+1)}\sum_{j=1}^{n}\psi(X_j)\int_0^{X_j}\left(1-\psi(s)^{\theta+1}\right)ds + \int_0^1 \mu_\theta^2(s)ds \\
&= \frac{1}{n^2}\sum_{i,j=1}^{n}\psi(X_i)\psi(X_j)\,(X_i\Lambda X_j) - \frac{\theta}{n(\theta+1)}\sum_{j=1}^{n}\psi(X_j)B(\theta+2,\frac{1}{2};X_j) + M(\theta)
\end{aligned}$$

where $X_i\Lambda X_j = Min(X_i,X_j)$, and

$$\begin{aligned}
M(\theta) &= \left(\frac{\theta}{\theta+1}\right)^2 \int_0^1 \left(1-\psi(s)^{\theta+1}\right)^2 ds \\
&= \left(\frac{\theta}{\theta+1}\right)^2 \int_0^1 \left(1-2\psi(s)^{\theta+1}+\psi(s)^{2\theta+2}\right) ds \\
&= \left(\frac{\theta}{\theta+1}\right)^2 \left(1-B(\theta+2,\frac{1}{2})+\frac{1}{2}B(2\theta+3,\frac{1}{2})\right).
\end{aligned}$$

Analogous to the statistic, if we replace θ and β with $\widehat{\theta}$ and $\widehat{\beta}$, respectively, then the test statistic has the following form:

$$\widehat{D}_{n,2} = \frac{1}{n^2}\sum_{i,j=1}^{n} \psi(X_i)\psi(X_j)I_{X_i \wedge X_j > s} - \frac{\widehat{\theta}}{n\left(\widehat{\theta}+1\right)}\sum_{j=1}^{n}\psi(X_j)B(\widehat{\theta}+2,\frac{1}{2};X_j) + M\left(\widehat{\theta}\right).$$

Simulations are carried out to compute the 2.5^{th} and 97.5^{th} percentiles of $\widehat{D}_{n,1}$ and the 95^{th} percentile of $\widehat{D}_{n,2}$ for a range of θ values and samples of sizes $10, 20$, and 50. These percentiles are displayed in Table1.

Table 1. Needed percentile of the two-sided test statistic $\widehat{D}_{n,1}$ and the one-sided test statistic $\widehat{D}_{n,2}$ for samples of sizes $10, 20$, and 50, and for $\theta = 0.5(0.5)5$.

		$\widehat{D}_{n,1}$		$\widehat{D}_{n,2}$		$\widehat{D}_{n,1}$		$\widehat{D}_{n,2}$
θ	n	2.5%	97.5%	95%	θ	2.5%	97.5%	95%
0.5	10	−0.156	0.208	0.171	3	−0.123	0.165	0.941
	20	−0.179	0.233	0.116		−0.144	0.195	0.780
	50	−0.209	0.255	0.084		−0.172	0.201	0.667
1.0	10	−0.158	0.225	0.387	3.5	−0.114	0.155	1.016
	20	−0.199	0.245	0.298		−0.125	0.176	0.855
	50	−0.209	0.257	0.220		−0.158	0.187	0.743
1.5	10	−0.153	0.216	0.587	4	−0.105	0.145	1.072
	20	−0.184	0.244	0.444		−0.116	0.152	0.929
	50	−0.214	0.250	0.355		−0.147	0.172	0.800
2.0	10	−0.147	0.198	0.722	4.5	−0.094	0.144	1.133
	20	−0.166	0.210	0.576		−0.111	0.148	0.972
	50	−0.203	0.230	0.470		−0.135	0.162	0.860
2.5	10	−0.129	0.188	0.833	5	−0.090	0.120	1.181
	20	−0.160	0.208	0.693		−0.104	0.145	1.033
	50	−0.186	0.209	0.581		−0.129	0.158	0.922

4 Power Computations

As mentioned earlier, the percentile and power calculations for the proposed tests, as well as for KS, CM and AD tests are based on Monte Carlo simulations. The first step is to estimate the distribution parameters β and θ . Assume that $y_1, ..., y_n$ are the observed values of a random sample of size n drawn from $TL(\theta, \beta)$ distribution with the density given in (2), without loss of generality, we assume that $y_1 < ... < y_n$ then the likelihood function is

$$\begin{aligned} L(\theta,\beta\,|y_1,...,y_n) &= 2^n\theta^n\beta^{-n}I_{(0,\beta)}y_n\prod_{j=1}^{n}\left(1-\frac{y_j}{\beta}\right)\left[1-\left(1-\frac{y_j}{\beta}\right)^2\right]^{\theta-1}, \\ &= 2^n\theta^n\beta^{-2n\theta}\prod_{j=1}^{n}(\beta-y_j)\prod_{j=1}^{n}[(2\beta-y_j)y_j]^{\theta-1}I_{(0,\beta)}y_n. \end{aligned}$$

Thus, the log-likelihood function is

$$l(\theta,\beta) = n\log 2 + n\log\theta - 2n\theta\log\beta + \sum_{j=1}^{n}\log(\beta-y_j) + (\theta-1)\sum_{j=1}^{n}\log(2\beta-y_j)y_j\,. \tag{17}$$

We observe that for fixed $\theta > 0$, $\lim_{\beta\downarrow y_n} l(\theta,\beta) = \lim_{\beta\uparrow\infty} l(\theta,\beta) = -\infty$. The continuity and concavity of $l(\theta,\beta)$ imply that, for fixed θ, $l(\theta,\beta)$ attains its maximum when β is in (y_n,∞).

Differentiating (17) with respect to β and equating the result to 0, we get

$$\frac{\partial l(\theta,\beta)}{\partial\beta} = -\frac{2n\theta}{\beta}+\sum_{j=1}^{n}\frac{1}{(\beta-y_j)}+2(\theta-1)\sum_{j=1}^{n}\frac{1}{(2\beta-y_j)} \tag{18}$$
$$= \sum_{j=1}^{n}\frac{y_j}{(\beta-y_j)(2\beta-y_j)}-2\theta\sum_{j=1}^{n}\frac{(\beta-y_j)}{\beta(2\beta-y_j)}=0.$$

Now differentiate (17) with respect to θ , then equate the derivative to 0 to get

$$\frac{\partial l(\theta,\beta)}{\partial\theta}=\frac{n}{\theta}-2n\log\beta+\sum_{j=1}^{n}\log[(2\beta-y_j)y_j]=0 \tag{19}$$

Solving (19) for θ , we obtain

$$\theta=-\frac{n}{\sum_{j=1}^{n}\log[(2\beta-y_j)y_j/\beta^2]} \tag{20}$$

Implementing (20) in(18), we have

$$\sum_{j=1}^{n}\frac{y_j}{(\beta-y_j)(2\beta-y_j)}+\frac{2n}{\sum_{j=1}^{n}\log[(2\beta-y_j)y_j/\beta^2]}\sum_{j=1}^{n}\frac{(\beta-y_j)}{\beta(2\beta-y_j)}=0 \tag{21}$$

The MLE for β is computed numerically from equation (21), and then the MLE of θ is obtained from equation(20).

The simulation proceeds as follows: We generate a random sample of specified size from $TL(\theta,\beta)$ for a given value of the shape parameter θ and, without loss of generality, for $\beta=1$. Based on this sample we compute the MLEs for β and θ , then we compute the values of the underlined tests. To find a simulated percentile of a given test, we repeat this process 10,000 times, then sort the computed test values in increasing order. The $100(1-\alpha)^{th}$ percentile,$0<\alpha<1$, is the $10000\times(1-\alpha)^{th}$ ordered value. Our power computations will be for the nominal value $\alpha=0.05$, so we need to compute the 95^{th} percentile for the one-sided test statistics KS,CM,AD and $\widehat{D}_{n,2}$ and the 2.5^{th} and 97.5^{th} percentiles for the two-sided test statistic $\widehat{D}_{n,1}$. The 95^{th} percentiles for KS,CM and AD tests are depicted in Table 2.

Table 2. $\alpha=0.05$ criticl values of KS, CM, and AD statistics for samples of sizes n=10,20,and 50, and θ= 0.5(0.5)5.

θ	n	KS	CM	AD	θ	n	KS	CM	AD
	10	0.302	0.174	0.953		10	0.297	0.163	0.897
0.5	20	0.223	0.186	1.014	3	20	0.220	0.173	0.963
	50	0.146	0.198	1.075		50	0.143	0.184	1.017
	10	0.302	0.168	0.921		10	0.294	0.158	0.878
1	20	0.221	0.178	1.001	3.5	20	0.220	0.178	0.960
	50	0.143	0.187	1.040		50	0.142	0.182	1.000
	10	0.298	0.164	0.906		10	0.297	0.163	0.899
1.5	20	0.219	0.175	0.966	4	20	0.218	0.173	0.948
	50	0.143	0.186	1.029		50	0.143	0.179	0.988
	10	0.296	0.162	0.895		10	0.295	0.160	0.882
2	20	0.218	0.172	0.962	4.5	20	0.218	0.171	0.941
	50	0.144	0.182	1.010		50	0.142	0.181	1.010
	10	0.299	0.165	0.903		10	0.299	0.165	0.908
2.5	20	0.219	0.174	0.966	5	20	0.218	0.172	0.946
	50	0.143	0.185	1.022		50	0.144	0.187	1.025

To compute the approximate power of a test against some alternative, we first simulate a random sample of a given size from the alternative distribution, then compute the test value. This process is repeated $10,000$ times, the percentage of times the test value exceeds the 95^{th} percentile for a right-tailed test, or falls outside the range of the 2.5^{th} and 97.5^{th} percentiles for a two-tailed test is an approximate power.

The alternatives were selected from the neighbouring beta distribution, $\beta(a,b)$ with $a=0.5,1,2,3,$and 4, and $b=0.5,1.5,2.5,3.5,$ and 5; a total of 25 alternatives covering a wide spectrum of distribution shapes. The power values for the underlined tests for a range of θ values between 0.5 and 5 in increments of 0.5 and for samples of sizes $10,20,$ and 50, when testing $TL(\theta,\beta)$ against the abovementioned alternatives are calculated. This produced a huge array of 750 rows, so only a representative portion of those are shown in Tables A2-A4 in the appendix. We also calculated the power of the underlined tests when the $TL(\theta,\beta)$ is tested against itself. The results are displayed in Table A1. It can be seen from Table A1 that the tests recover the nominal value $\alpha = 0.05$ except in a few cases where it is slightly lower than $\alpha(0.04)$ or slightly higher than $\alpha(0.06)$. In Tables A2-A4 it is noted that the tests under consideration do not restore the nominal value for certain values of θ and / or n, which implies the bias of these tests.

5 Discussion and Conclusions

Following the procedures described in the previous section, we calculated the power of the proposed tests as well as the KS, CM and AD tests. Calculations were made for 10 values of the Topp-Leone shape parameter θ , each tested against the aforementioned 25 beta distributions of the Beta family, which covered a wide range of alternative shapes. As mentioned above, the simulated powers were calculated based on samples of sizes $10, 20$ and 50.

Although the average calculated power is a rough measure of test performance, nevertheless it gives an idea of how tests generally work. Table 3 shows the average of 250 (10 values of θ each tested against 25 alternatives) simulated power values for each of the five underlined tests. The table shows that the power of all the tests considered increases with the size of the sample revealing the consistency of the tests. Table 3 also shows that $\widehat{D}_{n,2}$ on average, outperforms the other four tests for small $(n = 10)$ and moderate $(n = 20)$ sample sizes, while $\widehat{D}_{n,1}$ average power outperforms the other tests for $n = 50$.

Table 3. The power means of KS, CM, AD, $\widehat{D}_{n,1}$ and $\widehat{D}_{n,2}$ tests for samples of sizes 10, 20, and 50, and θ ranges from 0.5 to 5 with increments of 0.5 when testing the null hypothesis against 25 different distributions from the Beta family.

n	KS	CM	AD	$\widehat{D}_{n,1}$	$\widehat{D}_{n,2}$
10	0.126	0.139	0.149	0.179	0.246
20	0.193	0.211	0.218	0.246	0.295
50	0.293	0.318	0.325	0.385	0.346

Another issue to consider is how changes in the shape parameter θ would affect the performance of each test. For $n = 20$ and for each value of $\theta = 0.5(0.5)5$, we calculated the average power resulting from testing H_0 against the 25 alternatives. These averages for the tests considered are given in Table 4**.** It is surprising that the KS, CM and AD tests show no significant sensitivity to variations in θ, while in general $\widehat{D}_{n,1}$ indicates an increase in power and $\widehat{D}_{n,2}$ indicates a decrease in power with θ. The same is true for $n = 10$ and 50

Table 4**.** also tells that, on the average, KS, CM and AD have almost the same power, $\widehat{D}_{n,2}$ outperforms all tests for values of θ less than 2.5, and $\widehat{D}_{n,1}$ outperforms all other tests for θ of value 3.0 or more.

Table 4. The power means computed according to the skewness of the alternatives based on samples of sizes 20 and $\theta = 0.5(0.5)5$ when testing against 25 different distributions from the Beta family.

	Test				
θ	KS	CM	AD	D1	D2
0.5	0.18	0.20	0.21	0.17	0.78
1	0.19	0.21	0.21	0.14	0.58
1.5	0.19	0.21	0.22	0.16	0.46
2	0.20	0.21	0.22	0.20	0.34
2.5	0.19	0.21	0.22	0.21	0.24
3	0.19	0.21	0.22	0.24	0.18
3.5	0.19	0.21	0.22	0.29	0.13
4	0.20	0.21	0.22	0.34	0.10
4.5	0.19	0.21	0.22	0.29	0.13
5	0.20	0.21	0.22	0.34	0.10

We calculated the skewness and kurtosis of each of the alternatives and found that there is a connection between the skewness of the alternative and the powers of the underlined tests. However, kurtosis does not seem to have such a connection. Table 5 shows the mean power of the tests under consideration computed according to the skewness values. We notice that $\widehat{D}_{n,2}$ and AD tests outperform the other tests for skewness less than -1, and that KS, CM and AD, roughly, perform the same for other skewness values. Although $\widehat{D}_{n,2}$ appears to be superior to other tests for negative and small positive skewness alternatives, it is nevertheless inferior to all other tests for alternatives with a skewness of values of about 0.5 or more. In the latter case, $\widehat{D}_{n,1}$ does better than any other underlined test.

Generally speaking, $\widehat{D}_{n,2}$ outperforms all underlined tests when testing against negatively skewed alternatives and $\widehat{D}_{n,1}$ outperforms all considered tests when testing against positively skewed alternatives. It does not seem that the Kurtosis has a clear impact on tests powers.

Table 5. The power means computed according to the skewness of the alternatives based on samples of sizes $10,20$ **and** 50**, and** $\theta = 0.5(0.5)5$ **when testing against** 25 **different distributions from the Beta family.**

Alternativ skeness	KS	CM	AD	$\widehat{D}_{n,1}$	$\widehat{D}_{n,2}$
< -1	0.775	0.822	0.856	0.404	0.862
$-1.0 to -0.5$	0.376	0.417	0.450	0.333	0.564
$-0.5 to 0$	0.158	0.169	0.181	0.194	0.381
$0.0 to 0.5$	0.084	0.093	0.091	0.205	0.201
$0.5 to 1$	0.097	0.109	0.105	0.281	0.067
$1 to 1.5$	0.122	0.140	0.130	0.344	0.018
> 1.5	0.132	0.153	0.144	0.374	0.000

References

[1] Topp, C.W., Leone, F.C. A family of J-shaped frequency functions. J. Am. Stat. Assoc.50:209-219,1955.
[2] Nadarajah, S, Kotz, S. Moments of some J-shaped distributions. J. Appl. Stat. 30:311-317,2007,2003.
[3] Van Dorp, J.R. and Kotz. S. Modeling income distributions using elevated distributions on a bounded domain. Distribution Models Theory. Herrerias- Pleguezuelo, R. , Callejon-Cespedes, J. , and Herrerias-Velasco. J. M. 1-25. World Scientific Publishing Co. Pte. Ltd,2006.
[4] Ghitany,M.E.,Kotz,S.,Xie,M.On some reliability measures and their stochastic orderings for the Topp-Leone distribution.J.Appl.Stat.32:715-722,2005.
[5] Kotz, S. and Seier, E. Kurtosis of the Topp Leone distributions. Interstat, 1- 15.
[6] Vicari, D., van Dorp, J.R. and Kotz, S. . Two-sided generalized Topp and Leone (TS-GTL) distributions. J. Appl. Stat. 35: 1115-1129,2008.
[7] Zghoul, A. A.Order statistics from a family of J-shaped distributions,Metron, LXVIII, (2): 127-136,2010.
[8] Zghoul, A. A. Record values from a family of J-shaped distributions.Statistica 71:355-365,2011.
[9] Genc, A. Moments of order statistics of Topp Leone distribution. Stat. Pap. 51, 1-15,2012.
[10] Genc,A . Estimation of the $P(X > Y)$ with Topp Leone distribution. J. Stat. Comput. Simul.83:326-339,2013.
[11] Sindhu, T.N., Saleem, M. and Aslam, M. Bayesian estimation for Topp Leone distribution under trimmed samples. J. Basic Appl. Sci. Res., 3, 347-360,2013.
[12] Bayoud, H. A. Admissible minimax estimators for the shape parameter of Topp-Leone distribution. Communications in Statistics - Theory and Methods.45: (1) 71-82,2016.
[13] Rezaei, S., B. B. Sadr, M. Alizadeh, and S. Nadarajah . Topp-Leone generated family of distributions: Properties and applications. Communications in Statistics- Theory and Methods 46: (6):2893-2909,2017.
[14] Zghoul, Ahmad A. Efficient Plug-In Estimators of the Topp-Leone Distribution Shape Parameter. Pak. J. Statist. 2020 Vol. 36(4), 305-320,2010.
[15] Al-Zahrani, B. Goodness-of-Fit for the Topp-Leone Distribution with unknown parameters. Applied Mathematical Sciences, Vol. 6, 128: 6355 – 63,2012.
[16] Pearson, K. On the criterion that a given system of deviations from the probable in the case of a correlated system of variables is such that it can be reasonable supposed to have arisen from random sampling. Philosophical Magazine. 50:157,1990.
[17] Fisher, R. A. The conditions under which χ^2 measures the discrepancy between observation and hypothesis. Journal of the Royal Statistical Society 87:442–450,1924.
[18] Rao, K.C. and Robson, D.S. A chi-square statistic for goodness of fit tests within the Exponential family, Commun. Stat. 3:1139–1153,1974.
[19] D'Agostino, R. B., Stephens, M. A.Goodness-of-Fit Techniques. New York: Marcel Dekker, Inc,1986.
[20] Shapiro, S. S., Wilk, M. B. An analysis of variance test for normality (complete samples). Biometrika 52:591–611,1965.
[21] Shapiro, S. S., Francia, R. S. An approximate analysis of variance test for normality. Journal of the American Statistical Association 67: 215–216,1972.
[22] Coin, D. A goodness-of-fit test for normality based on polynomial regression. Computational Statistics & Data Analysis, 52: 2185-98,2006.
[23] Henze, N., Nikitin, Y. A new approach to goodness-of-fit testing based on the integrated empirical process. Journal of Nonparametric Statistics 12:391–416,2000.

[24] Klar, B. Goodness-of-fit tests for the exponential and the normal distributions based on the integrated distribution function. Annals of the Institute of Statistics and Mathematics 53(2):338–353,2001.

[25] Zghoul, A. A. , Awad, A. M. Normality test based on a truncated mean characterization. Communications in Statistics - Simulation and Computation. 39:1818-43,2010.

[26] Epps,T.W. & Pulley, L. B. A test for normality based on the empirical characteristic function. Biometrika. 70, (3): 723-26,1983.

[27] Henze, N. An approximation to the limit distribution of the Epps-Pulley test statistic for normality. Metrika, 37: 7–18,1990.

[28] Zghoul, Ahmad A. A goodness of fit test for normality based on the empirical moment generating function. Communications in Statistics - Simulation and Computation. 39:1292-1304,2010.

Appendix

Table A1. Computed powers of the proposed tests and the other considered tests when, for each value of $\theta = 0.5(0.5)5$, the $TL(\theta,\beta)$ is tested against itself at nominal $\alpha = 0.05$ based on samples of sizes $n = 10,20$,and 50.

θ	n	KS	CM	AD	$\widehat{D}_{n,1}$	$\widehat{D}_{n,2}$
0.5	10	0.055	0.053	0.051	0.053	0.051
0.5	20	0.049	0.051	0.052	0.051	0.052
0.5	50	0.051	0.051	0.051	0.051	0.051
1	10	0.054	0.054	0.054	0.054	0.054
1	20	0.051	0.054	0.051	0.054	0.051
1	50	0.048	0.052	0.053	0.052	0.053
1.5	10	0.053	0.055	0.052	0.055	0.052
1.5	20	0.050	0.044	0.044	0.044	0.044
1.5	50	0.053	0.054	0.054	0.054	0.054
2	10	0.052	0.051	0.053	0.051	0.053
2	20	0.052	0.053	0.048	0.053	0.048
2	50	0.053	0.061	0.059	0.061	0.059
2.5	10	0.054	0.048	0.052	0.048	0.052
2.5	20	0.053	0.052	0.054	0.052	0.054
2.5	50	0.051	0.052	0.052	0.052	0.052
3	10	0.052	0.048	0.047	0.048	0.047
3	20	0.049	0.048	0.046	0.048	0.046
3	50	0.050	0.048	0.048	0.048	0.048
3.5	10	0.054	0.054	0.052	0.054	0.052
3.5	20	0.043	0.046	0.049	0.046	0.049
3.5	50	0.051	0.051	0.057	0.051	0.057
4	10	0.043	0.043	0.043	0.043	0.043
4	20	0.054	0.050	0.051	0.050	0.051
4	50	0.049	0.051	0.053	0.051	0.053
4.5	10	0.053	0.056	0.054	0.056	0.054
4.5	20	0.049	0.046	0.049	0.046	0.049
4.5	50	0.052	0.052	0.049	0.052	0.049
5	10	0.041	0.046	0.046	0.046	0.046
5	20	0.052	0.056	0.056	0.056	0.056
5	50	0.042	0.043	0.045	0.043	0.045

Table A2. Computed powers of the considered tests based on samples of size 20 when testing TL with $\theta = 0.5$ against 25 different distributions from the Beta family.

$Alternative$	$Skewness$	$Kurtosis$	KS	CM	AD	$\widehat{D}_{n,1}$	$\widehat{D}_{n,2}$
$Beta(3,0.5)$	-1.57	5.22	0.83	0.89	0.92	0.10	1.00
$Beta(2,0.5)$	-1.25	3.82	0.81	0.88	0.92	0.33	1.00
$Beta(4,0.5)$	-0.69	2.93	0.15	0.18	0.19	0.01	1.00
$Beta(1,0.5)$	-0.64	2.14	0.76	0.84	0.89	0.76	1.00
$Beta(3,1.5)$	-0.51	2.54	0.13	0.15	0.17	0.04	1.00
$Beta(4,2.5)$	-0.31	2.49	0.05	0.05	0.05	0.01	1.00
$Beta(2,1.5)$	-0.22	2.14	0.11	0.13	0.14	0.09	1.00
$Beta(4,3)$	-0.18	2.44	0.04	0.04	0.04	0.01	1.00
$Beta(3,2.5)$	-0.12	2.31	0.04	0.04	0.04	0.01	1.00
$Beta(4,3.5)$	-0.08	2.44	0.04	0.04	0.03	0.01	1.00
$Beta(0.5,0.5)$	0.00	1.50	0.65	0.72	0.82	0.88	0.84
$Beta(4,4)$	0.00	2.45	0.05	0.05	0.04	0.03	1.00
$Beta(3,3.5)$	0.10	2.38	0.04	0.05	0.04	0.04	1.00
$Beta(2,2.5)$	0.16	2.23	0.04	0.04	0.04	0.05	1.00
$Beta(3,5)$	0.31	2.59	0.08	0.09	0.07	0.15	1.00
$Beta(1,1.5)$	0.34	2.05	0.07	0.09	0.10	0.14	0.95
$Beta(2,3.5)$	0.39	2.49	0.06	0.06	0.05	0.11	1.00
$Beta(2,5)$	0.60	2.88	0.10	0.12	0.10	0.28	1.00
$Beta(1,2.5)$	0.73	2.76	0.05	0.05	0.04	0.08	0.72
$Beta(1,3.5)$	0.96	3.41	0.07	0.08	0.06	0.17	0.51
$Beta(0.5,1.5)$	1.00	3.00	0.07	0.07	0.08	0.08	0.08
$Beta(1,5)$	1.18	4.20	0.11	0.13	0.10	0.35	0.29
$Beta(0.5,2.5)$	1.43	4.56	0.05	0.05	0.04	0.05	0.01
$Beta(0.5,3.5)$	1.69	5.82	0.07	0.08	0.06	0.11	0.00
$Beta(0.5,5)$	1.93	7.25	0.10	0.13	0.11	0.25	0.00

Table A3. Computed powers of the considered tests based on samples of size 20 **when testing TL with** $\theta = 2.5$ **against** 25 **different distributions from the Beta family.**

Alternative	*Skewness*	*Kurtosis*	*KS*	*CM*	*AD*	$\widehat{D}_{n,1}$	$\widehat{D}_{n,2}$
$Beta(3,0.5)$	-1.57	5.22	0.83	0.90	0.93	0.14	1.00
$Beta(2,0.5)$	-1.25	3.82	0.83	0.89	0.93	0.41	0.91
$Beta(4,0.5)$	-0.69	2.93	0.16	0.19	0.22	0.03	0.98
$Beta(1,0.5)$	-0.64	2.14	0.78	0.85	0.90	0.82	0.13
$Beta(3,1.5)$	-0.51	2.54	0.15	0.17	0.19	0.07	0.70
$Beta(4,2.5)$	-0.31	2.49	0.06	0.06	0.05	0.01	0.71
$Beta(2,1.5)$	-0.22	2.14	0.12	0.14	0.16	0.13	0.10
$Beta(4,3)$	-0.18	2.44	0.05	0.05	0.04	0.01	0.52
$Beta(3,2.5)$	-0.12	2.31	0.05	0.05	0.05	0.03	0.20
$Beta(4,3.5)$	-0.08	2.44	0.05	0.05	0.05	0.03	0.37
$Beta(0.5,0.5)$	0.00	1.50	0.67	0.73	0.84	0.91	0.00
$Beta(4,4)$	0.00	2.45	0.06	0.06	0.05	0.05	0.25
$Beta(3,3.5)$	0.10	2.38	0.05	0.05	0.04	0.07	0.05
$Beta(2,2.5)$	0.16	2.23	0.05	0.05	0.05	0.08	0.01
$Beta(3,5)$	0.31	2.59	0.09	0.10	0.08	0.21	0.01
$Beta(1,1.5)$	0.34	2.05	0.08	0.10	0.11	0.19	0.00
$Beta(2,3.5)$	0.39	2.49	0.07	0.07	0.06	0.15	0.00
$Beta(2,5)$	0.60	2.88	0.11	0.12	0.10	0.33	0.00
$Beta(1,2.5)$	0.73	2.76	0.05	0.05	0.05	0.11	0.00
$Beta(1,3.5)$	0.96	3.41	0.08	0.08	0.07	0.22	0.00
$Beta(0.5,1.5)$	1.00	3.00	0.07	0.07	0.09	0.14	0.00
$Beta(1,5)$	1.18	4.20	0.13	0.15	0.13	0.45	0.00
$Beta(0.5,2.5)$	1.43	4.56	0.06	0.07	0.06	0.08	0.00
$Beta(0.5,3.5)$	1.69	5.82	0.08	0.09	0.07	0.17	0.00
$Beta(0.5,5)$	1.93	7.25	0.12	0.14	0.12	0.34	0.00

Table A4. Computed powers of the considered tests based on samples of size 20 when testing TL with $\theta = 4.0$ against 25 different distributions from the Beta family.

$Alternative$	$Skewness$	$Kurtosis$	KS	CM	AD	$\widehat{D}_{n,1}$	$\widehat{D}_{n,2}$
$Beta(3,0.5)$	-1.57	5.22	0.67	0.73	0.85	0.96	0.00
$Beta(2,0.5)$	-1.25	3.82	0.07	0.07	0.09	0.29	0.00
$Beta(4,0.5)$	-0.69	2.93	0.06	0.07	0.06	0.19	0.00
$Beta(1,0.5)$	-0.64	2.14	0.08	0.09	0.08	0.34	0.00
$Beta(3,1.5)$	-0.51	2.54	0.12	0.14	0.13	0.58	0.00
$Beta(4,2.5)$	-0.31	2.49	0.77	0.84	0.90	0.92	0.02
$Beta(2,1.5)$	-0.22	2.14	0.09	0.10	0.11	0.36	0.00
$Beta(4,3)$	-0.18	2.44	0.06	0.06	0.05	0.24	0.00
$Beta(3,2.5)$	-0.12	2.31	0.08	0.08	0.07	0.36	0.00
$Beta(4,3.5)$	-0.08	2.44	0.13	0.15	0.13	0.65	0.00
$Beta(0.5,0.5)$	0.00	1.50	0.82	0.88	0.93	0.63	0.54
$Beta(4,4)$	0.00	2.45	0.12	0.14	0.17	0.29	0.00
$Beta(3,3.5)$	0.10	2.38	0.05	0.05	0.05	0.17	0.00
$Beta(2,2.5)$	0.16	2.23	0.07	0.07	0.06	0.28	0.00
$Beta(3,5)$	0.31	2.59	0.11	0.13	0.12	0.52	0.00
$Beta(1,1.5)$	0.34	2.05	0.85	0.91	0.94	0.33	0.94
$Beta(2,3.5)$	0.39	2.49	0.14	0.17	0.20	0.18	0.15
$Beta(2,5)$	0.60	2.88	0.06	0.06	0.06	0.10	0.01
$Beta(1,2.5)$	0.73	2.76	0.06	0.06	0.06	0.18	0.00
$Beta(1,3.5)$	0.96	3.41	0.10	0.12	0.10	0.41	0.00
$Beta(0.5,1.5)$	1.00	3.00	0.17	0.20	0.23	0.10	0.61
$Beta(1,5)$	1.18	4.20	0.06	0.06	0.06	0.05	0.14
$Beta(0.5,2.5)$	1.43	4.56	0.05	0.05	0.04	0.07	0.05
$Beta(0.5,3.5)$	1.69	5.82	0.05	0.05	0.04	0.09	0.02
$Beta(0.5,5)$	1.93	7.25	0.06	0.07	0.06	0.14	0.01