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Integrated Quadrably Reduced Additive Weighted Mixture Distributions

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Abstract: We propose new families of probability distributions derived from mixtures of weighted probability distributions. These distributions have not been addressed in the statistics literature. Many new continuous and discrete probability distributions could be generated from these families. We simplify a mixture of probability density (mass) functions into one unique closed probability density (mass) function. Some of the distributions in the families could be used in modeling survival analysis problems in biostatistics and reliability techniques in engineering. These distributions can easily capture different features of research data sets, such as bimodality, symmetry and asymmetry. A new parameter that vertically translates the sum of the weights is introduced. We call this parameter the translation parameter since it controls the number of the modes for the given distribution. Closed form of a generalized exponential distribution that could be modelled on bimodal data is derived from the mixture and employed by fitting real and simulated data sets. The survival and hazard functions of the exponential distribution are also investigated.

Keywords: Generalized Exponential Distribution, Translation Parameter, Weighted Distributions, Rational Hazard Functions, Bimodal Distributions, Vertically Translated Weights.

1 Introduction

Many attempts have been made to find new families of probability distributions by extending well-known families of distributions to provide more flexibility in modeling various data sets, see for example, [1, 2]. Most of the commonly used distributions are derived from the family of the generalized gamma. Another promising family that has generated many continuous and discrete distributions is the family of the weighted distributions. Weighted distributions were first studied by [3] to investigate the effect of methods of ascertainment upon estimation of frequencies. In extending the basic ideas, [3] and [4, 5] investigated different sampling schemes that can be modeled by weighted distributions. [6] and his co-authors published a series of papers about weighted distributions, see for example [6–8]. Moreover, [9–11] conducted studies about the properties and the characterizations of the weighted distributions in the context of stochastic ordering. Weighted distributions are used extensively in engineering, in general, and reliability in particular. Applications of these distributions in reliability were investigated by many researchers, see for example, [12–15] among others. Mixtures of these distributions have not been addressed in the statistics literature. Distributions derived from these mixtures could be used to generate large families of probability distributions capable of capturing different features of both continuous and count data.

2 Integrated quadrably Reduced Additive Weighted Mixture Distributions and their Properties

Let X be a random variable with a probability density (mass) function $\{\psi(x,\varphi),\varphi\in\Omega\}$ with respect to some σ-finite measure. Let $X^{\omega_1}, X^{\omega_2}, X^{\omega_3},\ldots,X^{\omega_p}$ be weighted versions of X with weight functions $\omega_1(x),\omega_2(x),\ldots,\omega_p(x)$ and probability density (mass) functions $\psi(x,\varphi)^{\omega_1},\psi(x,\varphi)^{\omega_2},\ldots,\psi(x,\varphi)^{\omega_p}$ respectively. The density functions can be

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obtained as $\psi(x,\varphi)^{\omega_j} = [\omega_j(x)/\mathbb{E}(\omega_j(x))]\psi(x,\varphi)$ for $j = 1,...,p$. $\mathbb{E}(\omega_j(x))'s$ are the expected values of the weight functions. All the weights are assumed to be positive $0 < \omega_j(x) < \infty$. Let

$$\phi(x,\varphi) = \sum_{j=1}^{p} \alpha_j \psi(x,\varphi)^{\omega_j} = \sum_{j=1}^{p} \alpha_j [\omega_j(x)/\mathbb{E}(\omega_j(x))]\psi(x,\varphi) \qquad j = 1,...,p \tag{1}$$

be the finite mixture of the probability density (mass) functions $\psi(x,\varphi)^{\omega_1},...,\psi(x,\varphi)^{\omega_p}$ where $\sum_{j=0}^{p} \alpha_j = 1$ and $\psi(x,\varphi)$ is the baseline density function. The resulted density function in (1) contains parsimonious and flexible family of probability distributions.

Theorem 2.1: The finite mixture in (1) is a probability density (mass) function. **Proof.** Suppose X is a continuous random variable, then
i) it is trivial to show that $\phi(x,\varphi) \geq 0$, secondly
ii) $\int \phi(x,\varphi)dx = \int \sum_{j=1}^{p} \alpha_j [\omega_j(x)/\mathbb{E}(\omega_j(x))]\psi(x,\varphi)dx$
$= \sum_{j=1}^{p} [\alpha_j/\mathbb{E}(\omega_j(x))] \int \omega_j(x)\psi(x,\varphi)dx = \sum_{j=1}^{p} \alpha_j = 1$ Similarly, if X is discrete we have
iii) $\sum_x \phi(x,\varphi) = \sum_x \sum_{j=1}^{p} \alpha_j [\omega_j(x)/\mathbb{E}(\omega_j(x))]\psi(x,\varphi)$
$= \sum_{j=1}^{p} [\alpha_j/\mathbb{E}(\omega_j(x))] \sum_x \omega_j(x)\psi(x,\varphi) = \sum_{j=1}^{p} \alpha_j = 1$

2.1 Reparameterization of the Mixing Parameters

If the mixing proportions $\alpha_j's$ of the mixture in (1) are reparameterized as functions of the weights, a unified probability density function in a closed form is obtained. There are possibly many ways to choose these reparameterizations of the mixing parameters, but the following setup allows components with larger weight expectations to have more effect on the resulted density function, whereas those with lesser weight expectations have minimal impact on the distribution.

$$\alpha_j = \frac{\mathbb{E}(\omega_j(x))}{\sum_{j=1}^{p} \mathbb{E}(\omega_j(x))} \qquad j = 1,...,p \tag{2}$$

Theorem 2.2. The density (mass) function in (1) has a closed form if the mixing parameters are reparameterized as in (2).

Proof. We have $\phi(x,\varphi) = \sum_{j=1}^{p} \alpha_j [\omega_j(x)/\mathbb{E}(\omega_j(x))]\psi(x,\varphi)$
$= \sum_{j=1}^{p} [\frac{\mathbb{E}(\omega_j(x))}{\sum_{j=1}^{p} \mathbb{E}(\omega_j(x))}][\omega_j(x)/\mathbb{E}(\omega_j(x))]\psi(x,\varphi)$
$= \psi(x,\varphi) \sum_{j=1}^{p} [\frac{\omega_j(x)}{\sum_{j=1}^{p} \mathbb{E}(\omega_j(x))}]$

Definition 2.1: A random variable X has the Integrated Quadrably Reduced Additive (IQRA) distribution if its probability density (mass) function $\{\phi(x,\varphi), \varphi \in \Omega\}$ is given by

$$\phi(x,\varphi) = \psi(x,\varphi) \sum_{j=1}^{p} [\frac{\omega_j(x)}{\sum_{j=1}^{p} \mathbb{E}(\omega_j(x))}] \tag{3}$$

If a constant weight δ is chosen for the first component of the mixture, equation (2) can be rewritten as

$$\alpha_1 = \frac{\delta}{\sum_{j=2}^{p} \mathbb{E}(\omega_j(x)) + \delta} \qquad j = 2,...,p$$

and

$$\alpha_j = \frac{\mathbb{E}(\omega_j(x))}{\sum_{j=2}^{p} \mathbb{E}(\omega_j(x)) + \delta} \qquad j = 2,...,p$$

Consequently, the mixture in (1) simplifies is, as follows:

$$\phi(x,\varphi,\delta) = (\frac{\psi(x,\varphi)}{\sum_{j=2}^{p} \mathbb{E}(\omega_j(x)) + \delta})[\delta + \sum_{j=2}^{p} \omega_j(x)] \qquad j = 2,...,p \tag{4}$$

If X is continuous, the moment generating function of the above $IQRA(x,\varphi,\delta)$ distribution is given by

$$\mathbb{M}_{IQRA}(t) = \int (\frac{exp[tx]\psi(x,\varphi)}{\sum_{j=2}^{p}\mathbb{E}(\omega_j(x))+\delta})[\delta+\sum_{j=2}^{p}\omega_j(x)]dx$$

$$= (\frac{1}{\sum_{j=2}^{p}\mathbb{E}(\omega_j(x))+\delta})[\int \delta exp[tx]\psi(x,\varphi)dx + \int exp[tx]\psi(x,\varphi)(\sum_{j=2}^{p}\omega_j(x))dx]$$

$$= (\frac{[\delta\mathbb{M}_X(t)+\sum_{j=2}^{p}\mathbb{E}(exp[tx]\omega_j(x))]}{\sum_{j=2}^{p}\mathbb{E}(\omega_j(x))+\delta})$$

where $\mathbb{M}_X(t)$ is the moment generating function of the baseline distribution.

2.2 Limiting Distributions

Theorem 2.3. If $X^* \sim IQRA(x,\varphi,\delta)$ and $\delta \to \infty$ then $X^* \to X$.
The Proof of theorem 2.3 follows from (3) since if $\delta \to \infty$, $\phi(x,\varphi,\delta) \to \psi(x,\varphi)$
Theorem 2.4. If $X^* \sim IQRA(\varphi,\delta)$ and $\delta \to 0$, then X^* converges to X^{**} with the probability density(mass) function $\phi^*(x,\varphi) = (\frac{\psi(x,\varphi)}{\sum_{i=2}^{p}\mathbb{E}(\omega_j(x))})\sum_{j=2}^{p}\omega_j(x)$.
Proof. Clearly, $\phi(x,\varphi,\delta) \to \phi^*(x,\varphi)$ as $\delta \to 0$

3 Examples of existing probability distributions derived from the IQRA distributions in (4)

3.1 Lindley Distribution

The probability distribution by [16] is one of the oldest probability distributions in statistics. It has many applications in statistics and probability theory. Extensions of its original version are currently available in the statistics literature. Its probability density function is, as follows:

$$\phi(x) = \frac{\theta^2}{1+\theta}(1+x)e^{-\theta x} \qquad x>0, \theta>0 \tag{5}$$

Using the mixture in (1), we can show the distribution in (5) is a mixture of an exponential distribution and its weighted version with weights $\omega_2(x) = 1$, and $\omega_2(x) = x$ with number of component $p = 2$.
Suppose $X \sim Exp(\theta)$, $p = 2$, $\omega_1(x) = 1$ and $\omega_2(x) = x$ then $X^{\omega_1} \sim Exp(\theta)$
and $X^{\omega_2} \sim \Gamma(2,\theta)$. Using equation (2), $\alpha_1 = \frac{\theta}{\theta+1}$ and $\alpha_2 = \frac{1}{\theta+1}$.
Finally, using (4) we have $\phi(x) = \alpha_1\psi(x,\varphi)^{\omega_1} + \alpha_2\psi(x,\varphi)^{\omega_2} = \frac{\theta^2}{1+\theta}(1+x)e^{-\theta x}$

3.2 Bimodal Skew-Symmetric Normal Distribution

The Bimodal Skew-Symmetric Normal Distribution (BSSN) is a probability distribution proposed by [17]. The probability density function of this distribution is given by

$$\phi(x) = \frac{2\kappa^{\frac{3}{2}}[\delta+(x-\beta)^2]e^{-\kappa(x-\mu)^2}}{\sqrt{\pi}(1+2\kappa[(\beta-\mu)^2+\delta])} \qquad -\infty < x < \infty \tag{6}$$

This distribution is also a special case of the distribution (4) and this can be shown as follows: If $X \sim N(\mu, 1/2\kappa)$ and we choose $\omega_1(x) = \delta$, $\omega_2(x) = (x-\beta)^2$,, we have $\mathbb{E}(\omega_2(x)) + \delta) = \frac{1+2\psi[(\beta-\mu)^2+\delta]}{2\psi}$ and $\delta + \sum_{j=2}^{p}\omega_j(x) = \delta + (x-\beta)^2$.
Using (4), we have

$\phi(x,\varphi,\delta) = (\frac{\psi(x,\varphi)}{\sum_{j=2}^{p}\mathbb{E}(\omega_j(x))+\delta})[\delta+\sum_{j=2}^{p}\omega_j(x)] = \frac{2\kappa^{\frac{3}{2}}[\delta+(x-\beta)^2]e^{-\kappa(x-\mu)^2}}{\sqrt{\pi}(1+2\kappa[(\beta-\mu)^2+\delta])}$

4 New probability distributions derived from the IQRA distributions

4.1 IQRA-Exponential Distribution

IQRA-Exponential represents the finite weighted mixture with a baseline exponential distribution. Here, we consider a case that contains both unimodal and bimodal special cases.
If $X \sim Exp(\beta)$, $p = 2$, $\omega_1(x) = \delta$ and $\omega_2(x) = x^\tau$ then $X^{\omega_1} \sim Exp(\beta)$ and $X^{\omega_2} \sim \Gamma(\tau+1,\beta)$. we have $\mathbb{E}(\omega_1(x)) = \delta$ and $\mathbb{E}(\omega_2(x)) = \beta^\tau\Gamma(\tau+1)$, from equation (1), we get

$$\phi(x,\varphi) = \sum_{j=1}^{2} \alpha_j[\omega_j(x)/\mathbb{E}(\omega_j(x))]\psi(x,\varphi) = \alpha_1\psi(x,\varphi)^{\omega_1} + \alpha_2\psi(x,\varphi)^{\omega_2},$$

where $\alpha_1 = \delta/(\beta^\tau\Gamma(\tau+1)+\delta)$ and $\alpha_2 = (\beta^\tau\Gamma(\tau+1))/(\beta^\tau\Gamma(\tau+1)+\delta)$
thus

$$\phi(x,\delta) = \alpha_1\psi(x,\varphi)^{\omega_1} + \alpha_2\psi(x,\varphi)^{\omega_2} = \frac{e^{-x/\beta}(\delta+x^\tau)}{\beta(\delta+\beta^\tau\Gamma(\tau+1))}, \qquad x>0, \tau>0, \delta\geq 0, \beta>0 \tag{7}$$

The resulted density function in (7) is in closed form and it has only three parameters β, δ and τ, where β is a scale parameter, τ is a shape parameter and δ is the translation parameter. The number of modes for this distribution depends on the value of τ. For example, if $\tau = 1$, it is unimodal. Whereas, if $\tau = 2$, it is bimodal.
The cumulative distribution function of the above distribution is given by

$$\Psi(x) = \int_0^x \frac{e^{-u/\beta}(\delta+u^\tau)}{\beta(\delta+\beta^\tau\Gamma(\tau+1))}du = \frac{\delta(1-e^{-x/\beta}) + \beta^\tau(\Gamma(\tau+1) - \Gamma(\tau+1,x/\beta))}{\delta+\beta^\tau\Gamma(\tau+1)} \tag{8}$$

where $\Gamma(\tau+1,x/\beta)$ is the lower incomplete gamma function. The survival and the hazard functions of the above cdf are, as follows respectively,

$$S(x) = \frac{\delta e^{-x/\beta} + \beta^\tau\Gamma(\tau+1,x/\beta))}{\delta+\beta^\tau\Gamma(\tau+1)}$$

and

$$H(x) = \frac{(\delta+x^\tau)}{\beta\delta+\beta^{\tau+1}e^{x/\beta}\Gamma(\tau+1,x/\beta)}$$

Since the study of Multi-modal data has received much attention in recent years and most of the currently used hazard functions do not explicitly consider this property which restricts their applications to a limited number of cases, investigating alternative hazard functions is needed. For example, the hazard rate for many diseases has been found to be bimodal, but many widely used hazard functions lack the flexibility of capturing this bimodality. The above family of the hazard functions explicitly considers the shape of the distribution and it can capture the nature of many different types of survival data. A major advantage of this family is that it generates realistic hazard functions for unimodal and multimodal distributions based on the value of τ. We consider special cases of the IQRA-Exponential distribution hazard functions, for example, see Figure 2 for the hazards of four different cases of the distribution function in (7).

4.1.1 Moment Generating Function and Moments

$$\mathbb{M}_X(t) = \frac{\delta - \tau(\beta/(1-\beta t))^\tau\Gamma(\tau)}{(\beta t-1)(\delta+\beta^\tau\Gamma(\tau+1))} \qquad x>0, \tau\geq 0, \delta\geq 0, \beta>0$$

$$\mathbb{E}(X^n) = \frac{\beta^n(\delta\Gamma(n+1) + \beta^\tau\Gamma(\tau+n+1))}{\delta+\beta^\tau\Gamma(\tau+1)} \qquad x>0, \tau\geq 0, \delta\geq 0, \beta>0$$

4.1.2 A Bimodal IQRA-Exponential with a Z-type Hazard

We consider a bimodal special case with a Z-type hazard function. If we choose $\tau = 2$ and x^2 is horizontally translated by θ in (7), the following probability density function is obtained

$$\psi^*(x) = \frac{e^{-x/\beta}(\delta + (x-\theta)^2)}{\beta(2\beta^2 - 2\beta\theta + \theta^2 + \delta)} \qquad x > 0, \theta \geq 0, \delta \geq 0, \beta > 0 \tag{9}$$

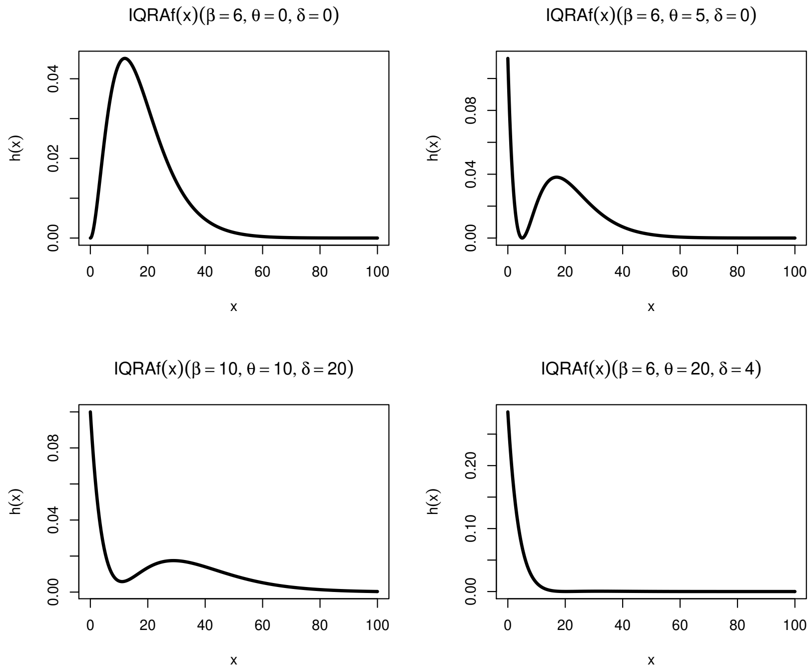

Fig. 1: Different shapes of the bimodal IQRA- Exponential probability density function

The horizontal translation parameter θ controls the location of the concavity of the density function, whereas δ controls the vertex of the concavity. Figure 1 displays the plots of the distribution with different values of θ, δ and λ. The cumulative distribution function is in closed form and is given by

$$\Psi^*(x) = 1 - \frac{e^{-x/\beta}(2\beta^2 + \delta + (x-\delta)^2) + 2\beta(x-\theta)}{(2\beta^2 - 2\beta\theta + \theta^2 + \delta)} \qquad x > 0, \theta \geq 0, \delta \geq 0, \beta > 0 \tag{10}$$

Its survival function is given by

$$S^*(x) = \frac{e^{-x/\beta}(2\beta^2 + \delta + (x-\theta)^2) + 2\beta(x-\theta)}{(2\beta^2 - 2\beta\theta + \theta^2 + \delta)} \qquad x > 0, \theta \geq 0, \delta \geq 0, \beta > 0$$

The rational hazard function of the distribution function in (8) is given by

$$H^*(x) = \frac{(\delta + (x-\theta)^2)}{\beta(2\beta^2 + \delta + (x-\theta)^2 + 2\beta(x-\theta))} \qquad x > 0, \theta \geq 0, \delta \geq 0, \beta > 0$$

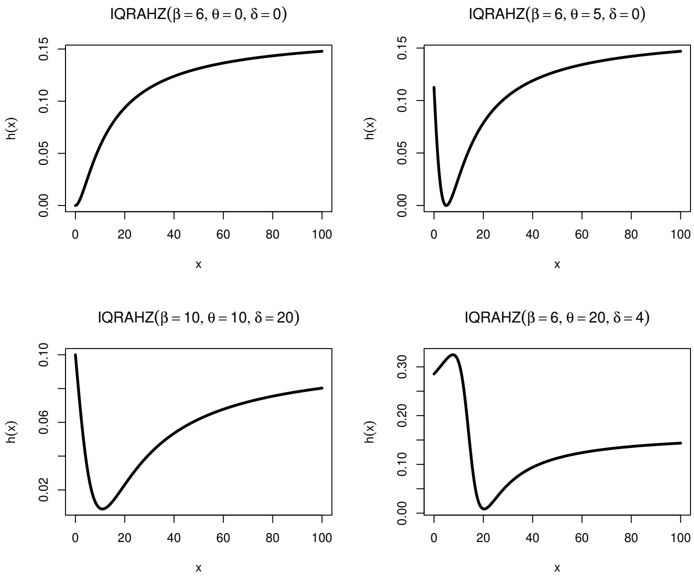

Fig. 2: Some shapes of the bimodal IQRA- Exponential hazard functions

5 Simulation Study

Example 1: **Bimodal IQRA-Exponential and 2 Component Gamma Mixture**

The simulation study illustrates that IQRA-Exponential distribution can be used as an alternative to the mixture of the gamma mixture. A data set of size 100 was simulated from the following two component gamma mixture:

$$f(x;\tau,\beta)=\sum_{j=1}^{p}\alpha_j f_i(x;\tau,\beta) \qquad j=1,2,...,p. \tag{11}$$

Where $f_i(x;\tau,\beta)=\frac{x^{\tau_i-1}e^{-x/\beta_i}}{\beta_i^{\tau_i}\Gamma(\tau_i)}$ and $p=2,\alpha_1=0.8,\tau_1=1,\tau_2=3,\beta_1=2,\beta_2=2.$

Maximum likelihood estimates of the parameters for the IQRA-Exponential and the two component gamma mixture distributions are given in Table 1. The fitted IQRA-Exponential and the gamma mixture models are displayed in Figure 3. Based on the BIC, the IQRA-Exponential model has better captured the simulated data than the gamma mixture. Clearly, the IQRA-Exponential is more parsimonious than the gamma mixture, and the number of estimated independent parameters for the two models is three and five, respectively.

Table 1: Fitted Gamma Mixture and IQRA-Exponential models.

Estimates	α_1	α_2	τ_1	τ_2	β_1	β_2	*BIC*
Gamma Mixture	0.25	0.75	0.52	1.1	2.15	3	410
IQRA-Exp	β	θ	δ	–	–	–	
Estimates	2.51	3.73	57.40	–	–	–	404

Moreover, in Figure 3, it seems that the two models fit the data very well but the fitted IQRA-Exponential model has adapted closely to the shape of the simulated data.

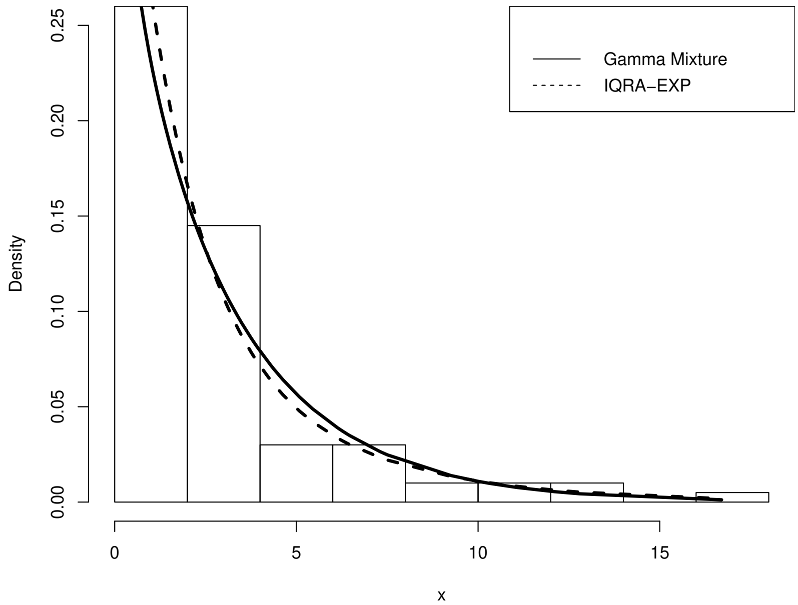

Fig. 3: Fitted Gamma Mixture and IQRA- Exponential Distribution

6 Applications

Example 3: **The Survival Time Data for the Comparison of Two Treatments for Prostatic Cancer**

The data " The survival time data "comparison of two treatments for prostatic cancer [18] " is fitted to the unimodal ($\tau = 1$ in (7)), bimodal IQRA-Exponential and a two component gamma mixture. Table 2 presents the estimated parameters for the three models and Figure 4 displays a qualitative comparison for the three competing models. Based on the BIC criteria, the best fitted model is the most parsimonious model which has only two paramters, the unimodal IQRA-Exponential.

Table 2: Fitted Gamma Mixture and IQRA-Exponential models.

Est	α_1	α_2	τ_1	τ_2	β_1	β_2	BIC
GM	0.32	0.68	0.4.44	0.98	2.4	29.85	413.76
IQRA-Exp 3 parameter	β	θ	δ	–	–	–	
Est	17.28	11.6	1917	–	–	–	409.60
IQRA-Exp 2 parameter	β	δ	–	–	–		
Est	20	100	–	–	–	–	406.39

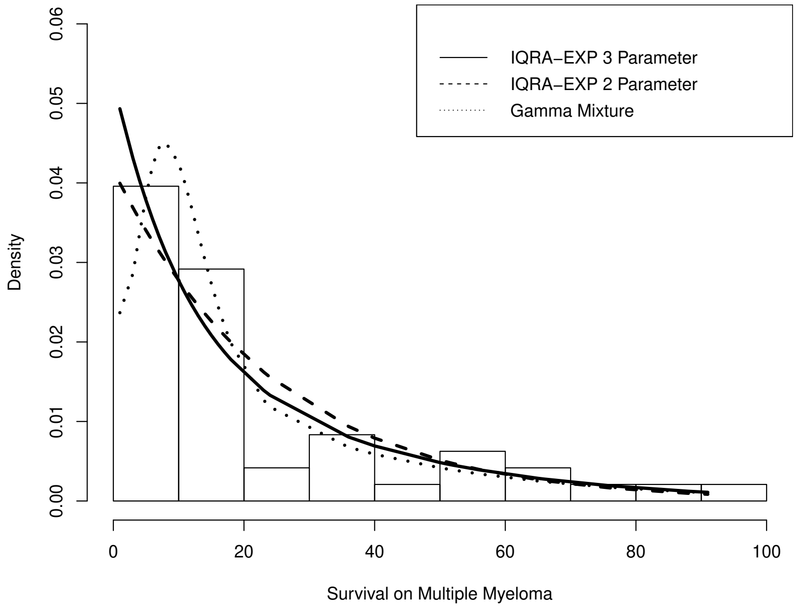

Fig. 4: Fitted Gamma Mixture, Unimodal and Bimodal IQRA-Exponential models

7 Conclusion and Possible Extensions

We have introduced the families of the IQRA distributions derived from the mixture of the weighted distributions to generate various probability distributions and some of their properties are investigated. Mixing parameters are reparameterized and a new translation parameter for the unification of the mixture is proposed. We presented the densities of the probability distributions generated from the families of the IQRA distributions. These distributions contain bimodal and unimodal special cases which are proposed and investigated. The PDF's, hazard functions and moment generating functions are derived. Two examples, including both real and simulated data, are presented. These distributions have adapted closely to the shape of the data simulated from mixtures of two and three component gamma distributions. Moreover, the general form of exponential bimodal distribution is presented and special cases of the distribution are proposed and investigated. It has been shown that this distribution can capture bimodal features of the given data. The results of an example involving real data indicated the simplicity and potential superiority of the parsimonious IQRA-Exponential model compared with the popular gamma mixture distribution. It is hoped that the parsimonious IQRA distributions will attract many researchers who can benefit the flexibility of these distributions . This family can be extended in many different ways based on the baseline distributions and the type of the selected weights.

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