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Optimality of Bayesian Estimators: A Comparative Study Based on Exponential Progressive Type II Censored Data

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Abstract: The paper investigates estimation problem of the parameter of exponential distribution. The maximum likelihood (ML) estimator and Bayesian estimators are obtained based on eight sampling schemes, six families of prior distributions, and seven classes of loss functions. The estimators are compared based on absolute relative error, standard deviation, mean square error, relative error, and loss, risk, and Pitman closeness. The main objective is to select the loss function that yields an optimal estimator and optimal sampling scheme within a given class of estimators.

Keywords: Censoring schemes, Bayes estimators, Loss functions, Pitman closeness, Exponential distribution.
AMS Subject Classification: 62F10, 62F15, 62N01, 62N02.

1 Introduction

Censoring is very important technique which has wide range of applications in reliability studies. There are several types of censoring schemes. This paper is concerned with progressive type II censoring and some of its special cases such as usual left and right type II censoring schemes. The book of Balakrishnan and Aggarwala (2000) is an essential reference for progressive censoring.

Let us put n units under test and select a pre-specified progressive censoring scheme vector $R = (R_1, \dots, R_m)$ that represents the number of units that will be removed respectively at the m observed failure times. Assume that the life times of these units are independent and identically distributed as a continuous random variable X with support $(a, b) = \{x: f(x) > 0\}$, pdf and cdf are f and F respectively. The boundary points of the support $a = F^{-1}(0)$ and $b = F^{-1}(1)$ may be $-\infty$ and ∞ respectively. Assume the distribution depends on a parameter θ. Let $X_{j:m:n}$ denote the j^{th} order statistic in a sequence of m observations under type–II progressive censoring scheme. Then the joint pdf of $\vec{X} = (X_{1:m:n}, \dots, X_{m:m:n})$ at the observed sample $\vec{x} = (x_{1:m:n}, \dots, x_{m:m:n})$ is given by

$$f_{prog}(\vec{x}; \theta) = c \prod_{i=1}^{m} f(x_{i:m:n};\ \theta)(1 - F(x_{i:m:n}; \theta))^{R_i}, \quad (1)$$

if $F^{-1}(0) = a < x_{1:m:n} < \dots < x_{m:m:n} < b = F^{-1}(1)$, where $c = n(n - R_1 - 1) \dots (n - R_1 - R_2 - \dots - R_{m-1} - m + 1)$, (see e.g. Chapter 1 of Balakrishnan, N. and Aggarwala, R. (2000)).

Concerning Bayes estimation, assume that we have a random sample $\vec{x}$ from a distribution with likelihood $f(\vec{x}\,|\theta)$ where the parameter θ is a random variable with prior $p(\theta)$. Let $g(\theta|\,\vec{x})$, $L(\hat{\theta}, \theta)$, $R(\hat{\theta}, \theta)$ and $E_{\theta|\vec{x}}$ denote the posterior distribution of θ given $\vec{x}$, loss incurred due to estimating θ by $\hat{\theta}$, posterior risk, and expectation operator under posterior distribution respectively. Bayes estimator for θ under $L(\hat{\theta}, \theta)$ is $\hat{\theta}$ that minimizes $R(\hat{\theta}, \theta) = E_{\theta|\vec{x}}\, L(\hat{\theta}, \theta)$ provided that the risk is finite.

We will explore the following seven classes of loss functions together with Bayes estimator $\hat{\theta}$ with respect to each of them and corresponding minimum Bayes risk $R_{min}(\hat{\theta})$ provided that the involved expectations are finite.

Brown (1968) introduced squared log-error loss as

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$$L_1(\hat{\theta},\theta) = \left(\mathrm{Log}(\hat{\theta}) - \log(\theta)\right)^2; \theta > 0, \hat{\theta} > 0.$$

The corresponding Bayes estimator and its minimum risk are

$$\hat{\theta} = \exp\left(E_{\theta|\vec{x}}(\log(\theta))\right), \quad (2)$$

$$R_{min}(\hat{\theta}) = \mathrm{Var}_{\theta|\vec{x}}(\log(\theta)).$$

Calabria and Pulcini (1994) introduced generalized entropy (modified LINEX) loss as

$$L_2(\hat{\theta},\theta;\eta) = \left(\frac{\hat{\theta}}{\theta}\right)^{\eta} - \eta \log\left(\frac{\hat{\theta}}{\theta}\right) - 1;\ \eta \neq 0,\ \theta > 0,\ \hat{\theta} > 0. \quad (3)$$

The corresponding Bayes estimator and its minimum risk are

$$\hat{\Theta} = \left(E_{\theta|\vec{x}}(\theta^{-\eta})\right)^{-\frac{1}{\eta}}, \quad (4)$$

$$R_{min}(\hat{\theta}) = \log\left(E_{\theta|\vec{x}}(\theta^{-\eta})\right) + E_{\theta|\vec{x}}(\log(\theta^{\eta})). \quad (5)$$

Norstrom (1996) defined precautionary loss function as

$$L_3(\hat{\theta},\theta;\eta,\gamma) = \frac{1}{\hat{\theta}^{\gamma}}\left(\hat{\theta}^{\eta} - \theta^{\eta}\right)^2;\quad 0 < \gamma < 2\eta, \theta > 0.$$

The corresponding Bayes estimator and its minimum risk are

$$\hat{\theta} = \left(\frac{(\eta-\gamma)E_{\theta|\vec{x}}(\theta^{\eta}) + \sqrt{(\eta-\gamma)^2\left(E_{\theta|\vec{x}}(\theta^{\eta})\right)^2 + \gamma(2\eta-\gamma)E_{\theta|\vec{x}}(\theta^{2\eta})}}{2\eta-\gamma}\right)^{\frac{1}{\eta}}, \quad (6)$$

$$R_{min}(\hat{\theta}) = \hat{\theta}^{-\eta}\left\{\hat{\theta}^2 - 2\hat{\theta}E_{\theta|\vec{x}}(\theta) + E_{\theta|\vec{x}}(\theta^2)\right\}. \quad (7)$$

El - Sayyad (1967) suggested generalized squared relative error loss as

$$L_4(\hat{\theta},\theta;\eta,\gamma) = \theta^{\gamma}\left(\hat{\theta}^{\eta} - \theta^{\eta}\right)^2, \gamma \neq 0, \eta \neq 0, \theta > 0.$$

The corresponding Bayes estimator and its minimum risk are

$$\hat{\theta} = \left(\frac{E_{\theta|\vec{x}}(\theta^{\eta+\gamma})}{E_{\theta|\vec{x}}(\theta^{\gamma})}\right)^{1/\eta}, \quad (8)$$

$$R_{min}(\hat{\theta}) = E_{\theta|\vec{x}}(\theta^{2\eta+\gamma}) - \frac{\left(E_{\theta|\vec{x}}(\theta^{\eta+\gamma})\right)^2}{E_{\theta|\vec{x}}(\theta^{\gamma})}. \quad (9)$$

Varian(1975) proposed generalized absolute error loss (quantile loss, Lin-Lin Loss) as

$$L_5(\hat{\theta},\theta;p) = p\left(\theta - \hat{\theta}\right)I_{(\hat{\theta},\infty)}(\theta) + (1-p)\left(\hat{\theta} - \theta\right)I_{(-\infty,\hat{\theta})}(\theta);\ 0 < p < 1.$$

The corresponding Bayes estimator and its minimum risk are

$$\hat{\theta} = G^{-1}_{\theta|\vec{x}}(p), \quad (10)$$

$$R_{min}(\hat{\theta}) = p\,E_{\theta|\vec{x}}(\theta) - \int_{-\infty}^{G^{-1}_{\theta|\vec{x}}(p)} \theta\, g(\theta|\vec{x})\, d\theta; \quad (11)$$

where G is the cdf of posterior distribution.

Higgins –Tsokos (1980) defined exponential loss by

$$L_6(\hat{\theta},\theta;\eta;\gamma) = \frac{\gamma e^{-\eta(\hat{\theta}-\theta)} + \eta e^{\gamma(\hat{\theta}-\theta)}}{\gamma+\eta} - 1,\ \eta > 0,\ \gamma > 0.$$

The corresponding Bayes estimator and its minimum risk are

$$\hat{\theta} = \frac{1}{\gamma+\eta}\log\left(\frac{E_{\theta|\vec{x}}(e^{\eta\theta})}{E_{\theta|\vec{x}}(e^{-\gamma\theta})}\right), \quad (12)$$

$$R_{min}(\hat{\theta}) = \left(\frac{E_{\theta|\vec{x}}(e^{\eta\theta})}{E_{\theta|\vec{x}}(e^{-\gamma\theta})}\right)^{-\frac{\eta}{\gamma+\eta}} E_{\theta|\vec{x}}\left(e^{\eta\theta}\right) - 1. \quad (13)$$

Varian (1975) defined LINEX (Linear Exponential) loss as

$L_7(\hat{\theta},\theta;\gamma) = e^{-\gamma(\hat{\theta}-\theta)} + \gamma(\hat{\theta}-\theta) - 1, \gamma \neq 0.$

The corresponding Bayes estimator and its minimum risk are

$$\hat{\theta} = \frac{1}{\gamma}\log(E_{\theta|\bar{x}}(e^{\gamma\theta})), \gamma > 0, \quad (14)$$

$$R_{min}(\hat{\theta}) = \log\left(E_{\theta|\bar{x}}(e^{\gamma\theta})\right) - \gamma\, E_{\theta|\bar{x}}(\theta). \quad (15)$$

Remark 1: It should be noted that $L_2(\hat{\theta},\theta;1)$ is called James and Stein (1961) loss function, Robert (1996) used $L_2(\hat{\theta},\theta;-1)$, the loss $L_5\left(\hat{\theta},\theta;\frac{1}{2}\right)$ is the usual absolute loss up to a multiplicative constant, and $L_6(\hat{\theta},\theta;\gamma;\gamma) = \cosh(\gamma(\hat{\theta}-\theta)) - 1;\ \gamma \neq 0$ is called catenary loss function (see e.g. Raeside Owen , 1973) .

Finally, we give the following lemma that reports basic properties of gamma distribution that are needed to obtain the posterior estimators and their corresponding risks due to the above mentioned loss functions.

Lemma 1: Let $W: G\left(k\,,\,\frac{1}{s}\right)$ with probability density functions $g(w)$. Then

$$E(W^r) = \frac{s^{-r}\Gamma(k+r)}{\Gamma(k)}, \quad (16)$$

$$Var(W^r) = s^{-2r}\left(\frac{\Gamma(k+2r)}{\Gamma(k)} - \frac{\Gamma^2(k+r)}{\Gamma^2(k)}\right), \quad (17)$$

$$\frac{E(W^r)}{E(W^v)} = \frac{s^{v-r}\Gamma(k+r)}{\Gamma(k+v)}, \quad (18)$$

$$E(e^{t\,W}) = \frac{1}{(1-\frac{t}{s})^k}, \quad (19)$$

$$E(\log(W)) = \log\left(\frac{1}{s}\right) + \psi(k), \quad (20)$$

$$E\left(\log^2(W)\right) = \log^2\left(\frac{1}{s}\right) + 2\log\left(\frac{1}{s}\right)\psi(k) + \psi^2(k) + \psi'(k), \text{ and} \quad (21)$$

$$Var(\log(W)) = \psi'(k)\ . \quad (22)$$

where $\psi(k) = \frac{\Gamma\prime(k)}{\Gamma(k)}$ is the digamma function.

$$\int_L^U w\, g(w)dw = k\, s \int_L^U g^*(w)dw = k\, s\, (G^*(U) - G^*(L)), \quad (23)$$

where $0 \leq L \leq U < \infty$, $g^*(w)$ and $G^*(w)$ are pdf and cdf of a $G\left(k+1\,,\,\frac{1}{s}\right)$ random variable.

Proof: Standard integration methods yield all the above results. Thus, details are deleted.

The organization of the paper is, as follows: Section 2 addresses the progressive type II censoring model when the underlying distribution of the units under test is exponential with mean $\frac{1}{\theta}$. Six classes of prior distributions are considered and the corresponding posterior distributions are obtained. Then, the corresponding Bayes estimators and their minimum risks with respect to each of the above suggested seven loss functions are derived. Moreover, the MLE is reported. Section 3 describes an extensive simulation study that aims at achieving the goals of this work. For this purpose, a Mathematica10 code is designed to implement Monte Carlo simulation study that calculates estimators, biases, and values of all optimality criteria together with matrices of relative efficiencies of estimators with respect to each of stated optimality criterion and selects the corresponding optimal estimator with respect to that criterion. Moreover, it classifies the obtained optimal estimators with respect to sampling schemes, prior distributions, and classes of loss functions. Furthermore, it calculates efficiency of each censoring scheme with respect to the complete sampling scheme. To minimize the size of the paper, a representative portion of the output of simulation study is reported in the Appendix. Finally, discussions and conclusions are presented in Sections 4 and 5.

2 Application to Exponential Distribution

Under the setup of progressive type II sampling, if the sampling units follow an exponential distribution with mean $\frac{1}{\theta}$ and pdf $f(x|\theta) = \theta e^{-\theta x};\ x > 0,\ \theta > 0$, then the joint pdf of the progressive type II censoring with the scheme

$$C_{m,n} = \left\{(R_1, R_2, \dots, R_m): R_j \in \{0,1,\dots,n-m\}, \qquad \sum_{i=1}^{m} R_i = n - m\right\}$$

is $\quad f_{prog}(\vec{x}|\theta) = c\ \theta^m \exp\left(-\theta \sum_{i=1}^{m} x_{i:m:n}\ (1 + R_i)\right).$ (24)

It is clear that $\sum_{i=1}^{m} x_{i:m:n}\ (1 + R_i)$ is sufficient statistic for θ and the MLE for θ is

$$\hat{\theta} = m\left(\sum_{j=1}^{m} x_{j:m:n}(1 + R_j)\right)^{-1}. \quad (25)$$

The Fisher information $I(\theta) = \frac{m}{\theta^2}$. (26)

To obtain Bayes estimators, we will consider the following prior distributions for θ:

a) $\theta: G\left(\alpha, \frac{1}{\beta}\right)$; Gamma conjugate prior with $(\alpha, 1/\beta)$ parameters, and pdf

$$p(\theta) = \frac{\beta^{\alpha}\theta^{\alpha-1}\, e^{-\beta\theta}}{\Gamma(\alpha)}, \alpha > 0,\ \beta > 0. \quad (27)$$

b) $\theta: IL(\tau)$; Inverse Levy conjugate prior with parameter τ, and pdf

$$p(\theta) = \sqrt{\frac{\tau}{2\pi\,\theta}}\, e^{-\theta\tau/2} \quad \text{where } \tau > 0, \text{ i.e.} G\left(\frac{1}{2}, \frac{2}{\tau}\right). \quad (28)$$

c) $\theta: MJ(h)$; Shemyakin (2014) modified Jeffreys with parameter h, and pdf

$$p(\theta) \propto I(\theta))^{h} = \frac{1}{\theta^{2h}},\ h > 0, \quad (29)$$

d) $\theta: J$; Jeffreys (1946) prior with pdf $p(\theta) \propto I(\theta))^{\frac{1}{2}} = \frac{1}{\theta}$; (30)

i.e. $\theta: MJ\left(\frac{1}{2}\right)$

e) $\theta: H$; Hartigan (1964) prior with pdf $p(\theta) \propto (I(\theta))^{\frac{2}{3}} = \frac{1}{\theta^{\frac{4}{3}}}$; (31)

i.e. $\theta: MJ\left(\frac{2}{3}\right)$

f) $\theta: U$; Improper uniform prior distribution with pdf $p(\theta) \propto 1$. (32)

The following lemma gives the posterior distribution under each of the above prior distributions.

Lemma 2:

a) If $\theta : G\left(\alpha, \frac{1}{\beta}\right)$, then $\theta|\vec{x} : G\left(m + \alpha, \left(\beta + \sum_{j=1}^{m} x_{j:m:n}(1 + R_j)\right)^{-1}\right)$ (33)

b) If $\theta: IL(\tau)$, then $\theta|\vec{x} : G\left(m + \frac{1}{2}, \left(\frac{\tau}{2} + \sum_{j=1}^{m} x_{j:m:n}(1 + R_j)\right)^{-1}\right)$ (34)

c) If $\theta: MJ(h)$, then $\theta|\vec{x} : G\left(m + 1 - 2h, \left(\sum_{j=1}^{m} x_{j:m:n}(1 + R_j)\right)^{-1}\right)$, (35)

where $0 < h < \frac{m+1}{2}$.

d) If $\theta: J$, then $\theta|\vec{x} : G\left(m, \left(\sum_{j=1}^{m} x_{j:m:n}(1 + R_j)\right)^{-1}\right)$ (36)

e) If $\theta: H$, then $\theta|\vec{x} : G\left(m - \frac{1}{3}, \left(\sum_{j=1}^{m} x_{j:m:n}(1 + R_j)\right)^{-1}\right)$ (37)

f) If $\theta: U$, then $\theta|\vec{x} : G\left(m + 1, \left(\sum_{j=1}^{m} x_{j:m:n}(1 + R_j)\right)^{-1}\right)$. (38)

Proof:

a) Using the likelihood in (24) and the prior in (27), we get the posterior

$$g(\theta|\vec{x}) \propto p(\theta)f(\vec{x}|\theta) \propto \theta^{m+\alpha-1} \exp\left(-\theta(\beta + \sum_{j=1}^{m} x_{j:m:n}(1+R_j)\right).$$

Hence, (33) holds.

b) Since, the prior in (28) is equivalent to the prior in (27) when $\beta = \frac{\tau}{2}$ and $\alpha = \frac{1}{2}$, the posterior in (34) follows from that in (33).

c) Using the likelihood in (24) and the prior in (29), we get the posterior

$$g(\theta|\vec{x}) \propto p(\theta)f(\vec{x}|\theta) \propto \theta^{m-2h} \exp\left(-\theta \sum_{j=1}^{m} x_{j:m:n}(1+R_j)\right); \text{ when } 0 < h < \frac{m+1}{2}.$$

Hence, (35) is obtained.

d) Considering (35) when $h = \frac{1}{2}$, and $h = \frac{2}{3}$, we get (36) and (37) respectively.

e) Finally, the posterior (38) is obvious.

Remark 2: It is interesting to note that, for all considered prior distributions, the posterior distribution is Gamma with generic parameters $(k, \frac{1}{s})$. Hence, we introduce the following Lemma 3 that will provide a unified method to evaluate the Bayes estimators and minimum risks with respect to each of the above stated loss functions and stated prior distributions.

Lemma 3: Assume that the posterior distribution of θ given $\vec{x}$ is $G(k, \frac{1}{s})$. With respect to a loss function $L(\hat{\theta}, \theta)$, let $\hat{\theta}$ and $R_{min}(\hat{\theta})$ denote Bayes estimator and its minimum risk, respectively. Then, with respect to

1. $L_1(\hat{\theta}, \theta)$, $\hat{\theta} = \frac{1}{s}\exp(\psi(k))$, and $R_{min}(\hat{\theta}) = \psi'(k)$, (39)

2. $L_2(\hat{\theta}, \theta; \eta)$, $\hat{\theta} = \frac{1}{s}\left(\frac{\Gamma(k)}{\Gamma(k-\eta)}\right)^{\frac{1}{\eta}}$; $k - \eta > 0$, and

$$R_{min}(\hat{\theta}) = \log\left(\frac{\Gamma(k-\eta)}{\Gamma(k)}\right) + \eta\, \psi(k);\ \ k > \eta, \tag{40}$$

3. $L_3(\hat{\theta}, \theta; \eta, \gamma)$,

$$\hat{\theta} = \left(\frac{(\eta-\gamma)\frac{s^{-\eta}(k+\eta)}{\Gamma(k)} + \sqrt{(\eta-\gamma)^2\left(\frac{s^{-\eta}(k+\eta)}{\Gamma(k)}\right)^2 + \gamma(2\eta-\gamma)\frac{s^{-2\eta}(k+2\eta)}{\Gamma(k)}}}{2\eta-\gamma}\right)^{\frac{1}{\eta}};\ 2\eta > \gamma > 0, \text{ and}$$

$$R_{min}(\hat{\theta}) = \hat{\theta}^{-\eta}\left\{\hat{\theta}^2 - \frac{2k}{s}\hat{\theta} + \frac{k(k+1)}{s^2}\right\}, \tag{41}$$

4. $L_4(\hat{\theta}, \theta; \eta, \gamma)$, $\hat{\theta} = \frac{1}{s}\left(\frac{\Gamma(k+\gamma+\eta)}{\Gamma(k+\gamma)}\right)^{\frac{1}{\eta}}$; $k+\gamma+\eta > 0, k+\gamma > 0$, and

$$R_{min}(\hat{\theta}) = s^{-(\gamma+2\eta)}\left\{\frac{\Gamma(k+\gamma+2\eta)}{\Gamma(k)} - \frac{(\Gamma(k+\gamma+\eta))^2}{\Gamma(k)\Gamma(k+\gamma)}\right\}, k+\gamma+2\eta > 0 \tag{42}$$

$L_5(\hat{\theta}, \theta; p)$, $\hat{\theta} = G^{-1}_{\theta|\vec{x}}(p); 0 < p < 1$, i.e. $(100\,p)^{th}$ posterior percentile, and $R_{min}(\hat{\theta}) = \frac{k}{s}\left(p - G^*\left(G^{-1}(p)\right)\right)$, where G and $G^*$are distribution functions of $G\left(k, \frac{1}{s}\right)$ and $G\left(k+1, \frac{1}{s}\right)$ respectively. (43)

$L_6(\hat{\theta}, \theta; \eta; \gamma)$, $\hat{\theta} = \frac{-k}{\gamma+\eta}\log\left(\frac{1-\frac{\eta}{s}}{1+\frac{\gamma}{s}}\right)$; $0 < \eta < s$, $\gamma > 0$, and

$$R_{min}(\hat{\theta}) = \frac{(1-\eta/s)^{-k\gamma/(\gamma+\eta)}}{(1+\gamma/s)^{k\eta/(\gamma+\eta)}} - 1, \tag{44}$$

$L_7(\hat{\theta}, \theta; \gamma)$, $\hat{\theta} = \frac{-k}{\gamma}\log\left(1 - \frac{\gamma}{s}\right)$; $0 < \gamma < s$, and

$$R_{min}(\hat{\theta}) = -k\log(1-\gamma/s) - (\gamma\, k)/s. \tag{45}$$

Proof: To get the Bayes estimator $\hat{\theta}$ and the corresponding minimum risk $R_{min}(\hat{\theta})$ with respect to the loss function $L_1(\hat{\theta}, \theta)$ Brown (1968) , apply (20) to (2) and (22) to (3) to get (39), i.e.

$$\hat{\theta} = \exp\left(E_{\theta|\vec{x}}(\log(\theta))\right) = \frac{1}{s}\exp(\psi(k)), \text{ and } R_{min}(\hat{\theta}) = Var_{\theta|\vec{x}}\left(\log(\theta)\right) = \psi'(k).$$

$L_2(\hat{\theta}, \theta; \eta)$ Calabria and Pulcini (1994) , apply (16) to (4) and both (16) and (20) to (5) to get (40), i.e.

$\hat{\theta} = (E_{\theta|\vec{x}}(\theta^{-\eta}))^{-\frac{1}{\eta}} = \frac{1}{s}(\frac{\Gamma(k)}{\Gamma(k-\eta)})^{\frac{1}{\eta}};\ k-\eta > 0$, and

$$R_{min}(\hat{\theta}) = \log\left(E_{\theta|\vec{x}}(\theta^{-\eta})\right) + E_{\theta|\vec{x}}(\log(\theta^{\eta})) = \log\left(\frac{\Gamma(k-\eta)}{\Gamma(k)}\right) + \eta\,\psi(k);\ \ k > \eta\,,$$

$L_3(\hat{\theta}, \theta; \eta, \gamma)$ Norstrom (1996) , apply (16) to both (6) and (7) to get (41), i.e.

$$\hat{\theta} = \left(\frac{(\eta-\gamma)E_{\theta|\vec{x}}(\theta^{\eta}) + \sqrt{(\eta-\gamma)^2(E_{\theta|\vec{x}}(\theta^{\eta}))^2 + \gamma(2\eta-\gamma)E_{\theta|\vec{x}}(\theta^{2\eta})}}{2\eta-\gamma}\right)^{\frac{1}{\eta}};$$

$$= \left(\frac{(\eta-\gamma)\frac{s^{-\eta}(k+\eta)}{\Gamma(k)} + \sqrt{(\eta-\gamma)^2(\frac{s^{-\eta}(k+\eta)}{\Gamma(k)})^2 + \gamma(2\eta-\gamma)\frac{s^{-2\eta}(k+2\eta)}{\Gamma(k)}}}{2\eta-\gamma}\right)^{\frac{1}{\eta}}\ ;\ 2\eta > \gamma > 0, \text{ and}$$

$$R_{min}(\hat{\theta}) = \hat{\theta}^{-\eta}\{\hat{\theta}^2 - 2\hat{\theta}E_{\theta|\vec{x}}(\theta) + E_{\theta|\vec{x}}(\theta^2)\} = \hat{\theta}^{-\eta}\{\hat{\theta}^2 - \frac{2k}{s}\hat{\theta} + \frac{k(k+1)}{s^2}\}$$

$L_4(\hat{\theta}, \theta; \eta, \gamma)$ El - Sayyad (1967), apply (18) to (8) and both (16) and (18) to (9) to get (42), i.e.

$\hat{\theta} = (\frac{E_{\theta|\vec{x}}(\theta^{\eta+\gamma})}{E_{\theta|\vec{x}}(\theta^{\gamma})})^{\frac{1}{\eta}} = \frac{1}{s}(\frac{\Gamma(k+\gamma+\eta)}{\Gamma(k+\gamma)})^{\frac{1}{\eta}};\ k+\gamma+\eta > 0, k+\gamma > 0$, and

$$R_{min}(\hat{\theta}) = E_{\theta|\vec{x}}(\theta^{2\eta+\gamma}) - \frac{(E_{\theta|\vec{x}}(\theta^{\eta+\gamma}))^2}{E_{\theta|\vec{x}}(\theta^{\gamma})} = s^{-(\gamma+2\eta)}\left\{\frac{\Gamma(k+\gamma+2\eta)}{\Gamma(k)} - \frac{(\Gamma(k+\gamma+\eta))^2}{\Gamma(k)\Gamma(k+\gamma)}\right\},$$

where $k+\gamma+2\eta > 0$.

$L_5(\hat{\theta}, \theta; p)$ Varian(1975) , it is obvious from (10) that $\hat{\theta} = G^{-1}_{\theta|\vec{x}}(p)$; $0 < p < 1$, i.e. $(100\,p)^{th}$ posterior percentile. Apply (16) and (23) to (11) to get the (43) , i.e.

$R_{min}(\hat{\theta}) = p\,E_{\theta|\vec{x}}(\theta) - \int_{-\infty}^{G^{-1}_{\theta|\vec{x}}(p)} \theta\, g(\theta|\vec{x})d\theta = \frac{k}{s}(p - G^*(G^{-1}(p)))$, where G and G^* are distribution functions of $G(k, \frac{1}{s})$ and $G(k+1, \frac{1}{s})$ respectively.

$L_6(\hat{\theta}, \theta; \eta; \gamma)$ Higgins –Tsokos (1980) , apply (19) to both (12) and (13) to get (44), i.e.

$\hat{\theta} = \frac{1}{\gamma+\eta}\log(\frac{E_{\theta|\vec{x}}(e^{\eta\theta})}{E_{\theta|\vec{x}}(e^{-\gamma\theta})}) = \frac{-k}{\gamma+\eta}\log\left(\frac{1-\frac{\eta}{s}}{1+\frac{\gamma}{s}}\right);\ \ 0 < \eta < s,\ \gamma > 0$, and

$$R_{min}(\hat{\theta}) = (\frac{E_{\theta|\vec{x}}(e^{\eta\theta})}{E_{\theta|\vec{x}}(e^{-\gamma\theta})})^{-\frac{\eta}{\gamma+\eta}}E_{\theta|\vec{x}}(e^{\eta\theta}) - 1 = \frac{(1-\eta/s)^{-k\gamma/(\gamma+\eta)}}{(1+\gamma/s)^{k\eta/(\gamma+\eta)}} - 1.$$

$L_7(\hat{\theta}, \theta; \gamma)$ Varian (1975), apply (19) to (14) and both (16) and (19) to (15) to get (45), i.e.

$\hat{\theta} = \frac{1}{\gamma}\log\left(E_{\theta|\vec{x}}(e^{\gamma\theta})\right) = \frac{-k}{\gamma}\log\left(1 - \frac{\gamma}{s}\right);\ \ 0 < \gamma < s$, and

$$R_{min}(\hat{\theta}) = \log\left(E_{\theta|\vec{x}}(e^{\gamma\theta})\right) - \gamma\,E_{\theta|\vec{x}}(\theta) = -k\log(1-\gamma/s) - (\gamma\,k)/s.$$

Remark 3: Lemma 3 can be applied to determine $\hat{\theta}$ and $R_{min}(\hat{\theta})$ for each combination of a loss function and a prior distribution according to the following procedure. Select a prior distribution $p(\theta)$ and the corresponding values of k and s from its induced posterior distribution. Select a loss function $L(\hat{\theta}, \theta)$ an plug in the obtained values of k and s in the given forms of $\hat{\theta}$ and $R_{min}(\hat{\theta})$.

3 Simulation Study

Since theoretical comparisons of estimators and their properties, we will apply a simulation study to achieve our goals. Assume that we have a class of r estimation methods and we have simulated N samples of same size n from the underlying distribution. Based on these circumstances, for the j^{th} method of estimation, we obtained N estimators for θ w.r.t. loss function L_j. These estimators are denoted by $\hat{\theta}_{j,i}$; $i = 1, 2, \ldots, N$, and $j = 1, 2, \ldots, r$. Let us denote the representative (the average) of j^{th} class of estimators by $\hat{\theta}_j$. To explore the behavior of the obtained estimators, we adopt the following measures of accuracy of simulated estimators: loss , absolute relative error, mean square error, relative error, and risk, that are defined respectively by, $L_j(\hat{\theta}_j) = \frac{1}{N}\sum_{t=1}^{N} L_j(\hat{\theta}_{j,t}, \theta)$, $ARE(\hat{\theta}_j) = \frac{1}{N}\sum_{t=1}^{N} |\frac{\hat{\theta}_{j,t}}{\theta} - 1|$, $MSE(\hat{\theta}_j) = \frac{1}{N}\sum_{t=1}^{N}(\hat{\theta}_{j,t} - \theta)^2$, $RE(\hat{\theta}_j) = \frac{\sqrt{MSE(\hat{\theta}_j)}}{\theta} = \sqrt{\frac{1}{N}\sum_{t=1}^{N}(\frac{\hat{\theta}_{j,t}}{\theta} - 1)^2}$, and risk $R_j(\hat{\theta}_j) = \frac{1}{N}\sum_{t=1}^{N} R_j(\hat{\theta}_{j,t}, \theta)$. An estimator $\hat{\theta}_i$ is better than (dominates) $\hat{\theta}_j$ according to a generic accuracy measure, AM, if $\frac{AM(\hat{\theta}_j)}{AM(\hat{\theta}_i)} > 1$. Moreover, we will use Pitman (1937) closeness to compare two estimators that are defined as $Pr(\hat{\theta}_j, \hat{\theta}_i) = P(|\hat{\theta}_j - \theta| > |\hat{\theta}_i - \theta|)$. According to this measure, $\hat{\theta}_i$ is better than (dominates) $\hat{\theta}_j$ whenever, $Pr(\hat{\theta}_j, \hat{\theta}_i) > \frac{1}{2}$. Finally, for a given prior and given estimation method, the optimal scheme within a given collection of d schemes is the one that maximizes $\frac{AM(\hat{\theta}_{com})}{AM(\hat{\theta}_{sch\,(j)})}$; $j = 1, \ldots, d$, where $\hat{\theta}_{com}$ and $\hat{\theta}_{sch(j)}$ are the estimates obtained from complete sample and the j^{th} censoring scheme.

Based on 20000 simulated samples of size $n = 20$ when $\theta = 0.01$, eight sampling schemes are used in this study, namely, complete sample $sch1 = (0^{20})$, 5-stage uniform removal $sch2 = (3^5)$, non-zero removal $sch3 = (2^2, 4,1,6)$, type II left censoring $sch4 = (15,0^4)$, type II right $sch5 = (0^4, 15)$, middle removal $sch6 = (0^2, 15,0^2)$, two extremes with more removals at first step $sch7 = (8,0^3, 7)$, two extremes with more removals at last step $sch8 = (7,0^3, 8)$, where h^v means that h units have been removed v consecutive times.

We have designed a Mathematica10 code that implements all required simulations and computations that are needed to achieve our goals of this study. It also computes values of estimators, descriptive statistics of estimators, matrices of relative efficiencies, based on each optimality criterion, with respect to each estimator together with Pitman closeness.

An exploration run of this code is implemented to MLE and 59 Bayes estimators based on seven classes of loss function with a total of 59 loss functions as defined in Table 1.

Table 2 shows the ports highest two relative frequencies of local optimal estimators (within each class) classified by loss classes, prior, scheme, and optimality criterion.

A second run of the code is implemented to the 14 selected loss functions in addition to MLE to select the global optimal estimators. Table 3 reports the obtained results. It turned out that the optimal estimators are $\hat{\theta}_7, \hat{\theta}_{24}, \hat{\theta}_{39}$, and $\hat{\theta}_{40}$.It should be noted that these estimators have been optimal in all simulated cases with relative frequency equals 0.907.

Tables 4a and 4b indicates relative efficiencies w.r.t. optimal estimator. Table 5 shows the properties of the estimators. Table 6 reports relative efficiency of censored schemes w.r.t. complete sample.

4 Discussion

We emphasize that this discussion is based on the full output of the simulation code even though only a representative sample of the output is reported in the appendix of this paper. Moreover, we suggest using the following two thumb rules. First for any optimality criterion, if any relative efficiency of an estimator to the optimal one belongs to the interval [1, 1.1), then they are considered equivalent. Second, an estimator is equivalent to the Pitman optimal one, if the probability of closeness belongs to the interval [0.5, 0.55).

In the light of these rules, investigating the output of the simulation study, (see e.g. Tables 3a and 3b), we observe from the values of relative efficiencies and Pitman closeness that the optimal estimator based on loss or on risk is the Bayes number 39 in all cases, regardless of prior and sampling scheme estimator with respect to loss function

Based on Pitman closeness criterion, $\hat{\theta}_7$ is equivalent to the MLE regardless of the sampling scheme for all assumed prior distributions except in the case of gamma prior. In this case, the MLE is equivalent to $\hat{\theta}_{24}$. Thus, one may claim that either one of MLE or $\hat{\theta}_7$ is optimal with respect to Pitman criterion.

Based on MSE, $\hat{\theta}_{24}$ is optimal when the prior is MJ, J, H, and IL for all sampling schemes. In these cases $\hat{\theta}_{24}$ is equivalent to $\hat{\theta}_{40}$. However, only $\hat{\theta}_{40}$ is optimal when the prior is either uniform or gamma. Moreover, it may be noted that $\hat{\theta}_7$ is equivalent to $\hat{\theta}_{24}$ in few cases. Hence, one may claim that either one of $\hat{\theta}_{24}$ or $\hat{\theta}_{40}$ is optimal with respect to MSE. Almost the same results hold based on RE and ARE. Moreover, $\hat{\theta}_{40}$ is optimal with respect to sd, criterion.

Table 4 (for $\hat{\theta}_7$, $\hat{\theta}_{24}$) shows the following;

For all priors and estimators, the bias in case of complete sample is much less than the bias in cases of censored schemes. Moreover, values of ARE and RE in cases of censored samples are much higher than those in case of complete sample. This is expected to be the price of censoring that reduces the accuracy of the estimators.

For each fixed loss function, there is an effect of prior distribution on the values of bias, ARE and RE. This effect depends on the loss function. For example the minimum and maximum values of each of these measures occur for the priors (J,G), (U,IL), (H,G), and (G,H) in cases of loss functions numbers 7, 24, 39, and 40 respectively. As expected there is no effect when using MLE. MLE and Bayes estimators with respect to loss function number 39 are overestimated in all cases.

Bayes estimators with respect to loss number 40 are underestimated while those with respect to loss number 24 are underestimated in all cases except in case of gamma prior. Estimators with respect to loss number 7 are underestimated in case of Hartigan prior, and only on two censoring schemes, i.e., type II right and left censoring schemes in case of Jefferys prior.

Finally, Table 5 shows that, for each fixed prior and fixed estimation method, the values of the relative efficiencies of censored schemes to complete sample are almost the same. This observation is consisting of with the well known observation that when the parent distribution is exponential, all schemes are optimal.

5 Conclusion

The present paper illustrated that the optimal estimators are only $\hat{\theta}_7, \hat{\theta}_{24}, \hat{\theta}_{39}, \hat{\theta}_{40}$. The loss functions that produced these estimators are $L_2(\hat{\theta}, \theta; 1), L_4\left(\hat{\theta}, \theta; \frac{1}{4}, -2\right), L_4(\hat{\theta}, \theta; 1,1)$ and $L_5(\hat{\theta}, \theta; 0.1)$, respectively. According to discussion, we recommend that:

1) If someone is interested in using a loss function from only one class, he/she may use the most frequent optimal estimator as given in Table 1.
2) One may use $L_2(\hat{\theta}, \theta; 1)$ since it is scale invariant, asymmetric and is the natural balanced entropy loss function that takes goodness of fit into consideration, i.e. it is the Kullback- Leibler (1951) directed divergence between $f(x, \theta)$ and $f(x, \hat{\theta})$ when f is the pdf of exponential distribution. The estimators wrt this loss function are optimal according to Pitman closeness and ARE criteria. Moreover, $\hat{\theta}_7$ is almost as efficient as the MLE.
3) The loss functions $L_4\left(\hat{\theta}, \theta; \frac{1}{4}, -2\right)$, and $L_4(\hat{\theta}, \theta; 1,1)$ are weighted generalized quadratic loss functions. The loss due to the second one is very much smaller than that of the first one. This explains why $\hat{\theta}_{39}$ is always optimal according to both loss and risk criteria. Whereas, $\hat{\theta}_{24}$ is optimal with respect to both MSE and RE. Moreover, the first one puts higher penalty on under-estimation, while the second one puts it on overestimation. Thus, the use of one of them depends on the personal judgment of selecting optimality criterion and the type of penalty to be put on the over or under estimation.
4) The loss function $L_5(\hat{\theta}, \theta; 0.1)$ is always optimal according to sd criterion and sometimes optimal according to MSE and RE. It puts higher penalty on overestimated values.
5) If one has no prior information, he/she may use MLE.

Conflict of Interest

The authors declare that they have no conflict of interest.

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Appendix

Table 1. Classes of loss functions with their parameters.

Class of Loss	Tuning Parameters of Loss Functions	index of estimator $\hat{\theta}$ #
	MLE	1
L_1	Log-Quadratic Loss	2
$L_2(\boldsymbol{\eta})$	$\boldsymbol{\eta} = -1, -\frac{1}{2}, \frac{1}{3}, \frac{1}{2}, 1$	3-7
$L_3(\boldsymbol{\eta},\boldsymbol{\gamma})$	$(\boldsymbol{\eta},\boldsymbol{\gamma}) = \left(1,\frac{1}{4}\right), \left(1,\frac{1}{3}\right), \left(1,\frac{1}{2}\right), (1,1), \left(1,\frac{3}{2}\right), \left(\frac{3}{2},\frac{1}{3}\right), \left(\frac{3}{2},\frac{1}{2}\right), \left(\frac{3}{2},1\right), \left(\frac{3}{2},2\right), \left(\frac{3}{2},\frac{5}{2}\right), \left(2,\frac{1}{2}\right), (2,1), \left(2,\frac{3}{2}\right), (2,2), \left(2,\frac{5}{2}\right), \left(2,\frac{7}{2}\right)$	8-23
$L_4(\boldsymbol{\eta},\boldsymbol{\gamma})$	$(\boldsymbol{\eta},\boldsymbol{\gamma}) = \left(\frac{1}{4},-2\right), \left(\frac{1}{3},-2\right), \left(\frac{1}{2},-2\right), (1,-2), \left(\frac{1}{4},-1\right), \left(\frac{1}{3},-1\right), \left(\frac{1}{2},-1\right), (1,-1), \left(\frac{1}{4},0\right), \left(\frac{1}{3},0\right), \left(\frac{1}{2},0\right), (1,0), \left(\frac{1}{4},1\right), \left(\frac{1}{3},1\right), \left(\frac{1}{2},1\right), (1,1),$	24-39
$L_5(\boldsymbol{p})$	$\boldsymbol{p} = 0.1, 0.25, 0.5, 0.75, 0.9$	40-44
$L_6(\boldsymbol{\eta},\boldsymbol{\gamma})$	$(\boldsymbol{\eta},\boldsymbol{\gamma}) = \left(\frac{1}{3},\frac{1}{3}\right), \left(\frac{1}{2},\frac{1}{2}\right), (1,1), \left(\frac{3}{2},\frac{3}{2}\right), \left(0.2,\frac{1}{3}\right), \left(0.2,\frac{1}{2}\right), (0.2,1), \left(0.2,\frac{3}{2}\right), (0.3,2), (0.5,2), (0.7,2), (0.9,2)$	45-56
$L_7(\boldsymbol{\gamma})$	$\boldsymbol{\gamma} = -1, -\frac{1}{2}, \frac{1}{2}, 1$	57-60

Table 2. Highest two relative frequencies of local optimal estimators within each class of loss functions, classified by class, scheme, criterion, and prior distribution.

Classification by Class

Class #	Loss #	Relative frequency (r.f.)	Class #	Loss #	Relative frequency	Class #	Loss #	Relative frequency
2	7	0.622	4	24	0.323	6	53	0.699
	5	0.309		39	0.284		49	0.285
3	8	0.714	5	40	0.613	7	57	0.842
	18	0.285		41	0.252		58	0.077

Classification by sampling scheme

(0^{20})		(3^5)		$(2^2,4,1,6)$		$(15,0^4)$		$(0^4,15)$		$(0^2,15,0^2)$		$(8,0^3,7)$		$(7,0^3,8)$	
$\hat{\theta}$#	r.f.	$\hat{\theta}$#	r.f.	$\hat{\theta}$#	r.f.	$\hat{\theta}$#	r.f.	$\hat{\theta}$#	r.f.	$\hat{\theta}$#	r.f.	$\hat{\theta}$#	r.f.	$\hat{\theta}$#	r.f.
57	0.142	57	0.139	57	0.143	57	0.139	57	0.143	57	0.139	57	0.139	57	0.139
53	0.119	8	0.119	53	0.119	8	0.119	53	0.119	8	0.119	8	0.119	8	0.119

Classification by optimality criterion

ARE		Loss		sd		MSE		Risk		RE		Pitman	
57	0.167	49	0.167	57	0.166	57	0.167	57	0.166	57	0.167	8	0.166
53	0.167	39	0.167	53	0.166	53	0.167	49	0.166	53	0.167	57	0.149

Classification by prior distribution

Uniform		M Jefferys		Jefferys		Hartigan		I Levy		Gamma	
57	0.142	57	0.143	57	0.143	57	0.128	57	0.143	57	0.143
40	0.125	53	0.119	53	0.119	8	0.119	53	0.119	53	0.119

Table 3. Highest two relative frequencies of global optimal estimators among estimators of phase II, classified by prior, criterion, and scheme

Optimal Loss	Relative frequency	Optimal Loss	Relative frequency	Optimal Loss	Relative frequency	Optimal Loss	Relative frequency	Optimal Loss	Relative frequency	Optimal Loss	Relative frequency	Optimal Loss	Relative frequency	Optimal Loss	Relative frequency
classified by prior distributions															
Uniform		M Jefferys		Jefferys		Hartigan		I Levy		Gamma					
40	0.344	39	0.286	39	0.286	7	0.321	24	0.394	40	0.518				
39	0.25	24	0.286	24	0.286	39	0.286	39	0.286	39	0.286				
classified by optimality criteria															
ARE		Loss		sd		MSE		Risk		RE		Pitman			
7	0.5	39	1	40	1	24	0.583	39	1	24	0.583	7	0.285		
24	0.354					40	0.313			40	0.313	2	0.286		
classified by sampling schemes															
(0^20)		(3^5)		(3^2,4,1,6)		(15,0^4)		(0^4,15)		(0^2,15,0^2)		(8,0^3,7)		(7,0^3,8)	
24	0.326	40	0.302	39	0.279	39	0.279	39	0.279	39	0.279	39	0.279	39	0.279
39	0.279	39	0.279	40	0.256	40	0.256	40	0.256	40	0.256	40	0.256	40	0.256

Table 4a. Relative efficiency w.r.t. optimal estimator & Pitman closeness classified by optimality criteria and sampling schemes when Jeffreys is prior for θ.

	Optimal	# of Compotators estimators ($\hat{\theta}$ #)					Optimal	# of Compotators estimators ($\hat{\theta}$ #)				
		1	7	24	39	40		1	7	24	39	40
Criteria		Complete (0^20)						(3^5)				
ARE	7	1.043	*	1.024	1.135	1.438	7	1.198	*	1.151	1.583	1.221
Loss	39	----	329223	4.59*10^7	*	4687.3	39	----	123631	2.71*10^7	*	805.43
sd	40	1.376	1.308	1.213	1.445	*	40	2.055	1.644	1.085	2.466	*
MSE	24	1.216	1.049	*	1.493	1.65	24	2.14	1.221	*	3.701	1.07
Risk	39	----	360032	5.44*10^7	*	5136	39	----	106317	3.27*10^7	*	687.18
RE	24	1.103	1.024	*	1.222	1.284	24	1.463	1.105	*	1.923	1.034
Pitman	1	*	0.51	0.57	0.57	0.71	1	*	0.53	0.66	0.64	0.68
		(2^2,4,1,6)						(15,0^4)				
ARE	7	1.197	*	1.13	1.575	1.194	7	1.202	*	1.126	1.582	1.191
Loss	39	----	116249	2.47*10^7	*	760.42	39	----	119467	2.53*10^7	*	779.94
sd	40	2.055	1.644	1.085	2.466	*	40	2.055	1.644	1.085	2.466	*
MSE	24	2.284	1.306	*	3.9	1.056	24	2.263	1.287	*	3.889	1.061
Risk	39	----	86947.8	2.67*10^7	*	567.1	39	----	98622	3.02*10^7	*	644.17
RE	24	1.511	1.143	*	1.974	1.027	24	1.504	1.134	*	1.972	1.03
Pitman	1	*	0.53	0.66	0.64	0.68	1	*	0.52	0.66	0.64	0.68
		(0^4,15)						(0^2,15,0^2)				
ARE	7	1.206	*	1.13	1.589	1.195	7	1.197	*	1.129	1.57	1.193
Loss	39	----	119707	2.54*10^7	*	784.51	39	----	119790	2.54*10^7	*	777.97
sd	40	2.055	1.644	1.085	2.466	*	40	2.055	1.644	1.085	2.466	*
MSE	24	2.263	1.284	*	3.896	1.062	24	2.262	1.294	*	3.868	1.058

Risk	39	----	99870	3.06*10^7	*	653.06	39	----	96564	2.99*10^7	*	628.78
RE	24	1.504	1.133	*	1.973	1.03	24	1.504	1.137	*	1.966	1.028
Pitman	1	*	0.52	0.66	0.64	0.68	1	*	0.53	0.66	0.63	0.68
		(8,0^3,7)						(7,0^3,8)				
ARE	7	1.19	*	1.135	1.561	1.199	7	1.201	*	1.131	1.584	1.197
Loss	39	----	121336	2.59*10^7	*	787.05	39	----	120348	2.56*10^7	*	784.71
sd	40	2.055	1.644	1.085	2.466	*	40	2.055	1.644	1.085	2.466	*
MSE	24	2.227	1.277	*	3.81	1.06	24	2.246	1.278	*	3.861	1.062
Risk	39	----	98257.6	3.06*10^7	*	637.31	39	----	100394	3.09*10^7	*	654.66
RE	24	1.492	1.13	*	1.951	1.029	24	1.498	1.13	*	1.965	1.03
Pitman	1	*	0.53	0.66	0.63	0.68	1	*	0.52	0.65	0.64	0.68

Table 4b. Relative efficiency w.r.t. optimal estimator & Pitman closeness classified by optimality criteria and sampling schemes when Gamma is prior for θ .

Criteria →	Optimal→	Scheme → Complete (0^20)					Optimal→	(3^5)				
		# of Compotators estimators						# of Compotators estimators				
		1	7	24	39	40		1	7	24	39	40
ARE	24	1.024	1.11	*	1.407	1.134	40	1.22	1.568	1.103	2.616	*
Loss	39	----	249260	2.63*10^7	*	3761.5	39	----	94376.3	6.85*10^6	*	810.79
sd	40	1.235	1.292	1.208	1.415	*	40	1.318	1.54	1.189	2.054	*
MSE	24	1.074	1.303	*	2.068	1.025	40	1.944	3.16	1.483	7.444	*
Risk	39	----	271995	3.52*10^7	*	4508.74	39	----	38861.8	5.51*10^6	*	456.91
RE	24	1.036	1.141	*	1.438	1.012	40	1.394	1.777	1.218	2.728	*
Pitman	24	0.51	0.55	*	0.63	0.65	24	0.51	0.6	*	0.74	0.58
		(2^2,4,1,6)						(15,0^4)				
ARE	40	1.22	1.573	1.101	2.633	*	40	1.23	1.592	1.107	2.672	*
Loss	39	----	92647.5	6.69*10^6	*	792.283	39	----	93167.6	6.68*10^6	*	796.06
sd	40	1.32	1.54	1.189	2.054	*	40	1.323	1.54	1.189	2.054	*
MSE	40	1.946	3.142	1.483	7.356	*	40	1.959	3.168	1.485	7.47	*
Risk	39	----	37110.3	5.24*10^6	*	437.061	39	----	36990.7	5.198*10^6	*	435.18
RE	40	1.395	1.772	1.218	2.712	*	40	1.399	1.78	1.218	2.733	*
Pitman	24	0.52	0.6	*	0.75	0.58	24	0.53	0.61	*	0.75	0.57
		(0^4,15)						(0^2,15,0^2)				
ARE	40	1.232	1.593	1.109	2.669	*	40	1.224	1.578	1.103	2.642	*
Loss	39	----	94247.4	6.76*10^6	*	807.157	39	----	92463.1	6.69*10^6	*	786.97
sd	40	1.318	1.54	1.189	2.054	*	40	1.32	1.54	1.189	2.054	*
MSE	40	1.959	3.195	1.491	7.548	*	40	1.941	3.131	1.479	7.334	*
Risk	39	----	38726.7	5.43*10^6	*	457.287	39	----	37154.8	5.27*10^6	*	436.17
RE	40	1.399	1.787	1.221	2.747	*	40	1.393	1.769	1.216	2.708	*
Pitman	24	0.52	0.61	*	0.75	0.57	24	0.52	0.6	*	0.75	0.58
		(8,0^3,7)						(7,0^3,8)				
ARE	40	1.218	1.567	1.1	2.62	*	40	1.228	1.586	1.106	2.661	*

Loss	39	----	92915	6.76*10^6	*	793.091	39	----	90649	6.49*10^6	*	773.42
sd	40	1.321	1.54	1.189	2.054	*	40	1.328	1.54	1.189	2.054	*
MSE	40	1.938	3.122	1.476	7.323	*	40	1.965	3.128	1.483	7.295	*
Risk	39	----	37190.8	5.30*10^6	*	435.358	39	----	33538.7	4.71*10^6	*	395.26
RE	40	1.392	1.767	1.215	2.706	*	40	1.401	1.768	1.217	2.701	*
Pitman	24	0.52	0.6	*	0.74	0.58	24	0.52	0.61	*	0.75	0.57

Table 5. Estimates, biases , ARE's , and ER's classified by schemes , priors and loss functions #7 and #24

Sch		Estimate	Bias	ARE	RE		Estimate	Bias	ARE	RE
1	$\widehat{\theta_7}$	0.0105	0.0005	0.19	0.2538	$\widehat{\theta_7}$	0.010183	0.000183	0.183941	0.241334
2	Uniform	0.0124	0.0024	0.4603	0.7479	M Jefferys	0.010849	0.000849	0.407472	0.63138
3		0.0125	0.0025	0.4682	0.7578		0.010891	0.000891	0.411156	0.649111
4		0.0124	0.0024	0.4656	0.7342		0.010837	0.000837	0.407531	0.630813
5		0.0125	0.0025	0.4647	0.7542		0.010848	0.000848	0.412472	0.645385
6		0.0126	0.0026	0.4709	0.7668		0.010812	0.000812	0.408588	0.641486
7		0.0125	0.0025	0.4695	0.7696		0.010887	0.000887	0.411278	0.634186
8		0.0125	0.0025	0.47	0.761		0.010849	0.000849	0.406188	0.62992
1	θ_{24}	0.0098	-0.0002	0.1813	0.2323	θ_{24}	0.00946	-0.00054	0.183932	0.229977
2	Uniform	0.009	-0.001	0.3799	0.5241	M Jefferys	0.007439	-0.00256	0.411821	0.499634
3		0.0091	-0.0009	0.3864	0.5279		0.007468	-0.00253	0.413746	0.508412
4		0.009	-0.001	0.3829	0.5123		0.007431	-0.00257	0.412388	0.499806
5		0.0091	-0.0009	0.3832	0.527		0.007438	-0.00256	0.415713	0.508016
6		0.0091	-0.0009	0.3863	0.533		0.007414	-0.00259	0.414195	0.507201
7		0.0091	-0.0009	0.3856	0.5358		0.007465	-0.00253	0.412506	0.499644
8		0.0091	-0.0009	0.3861	0.5289		0.007439	-0.00256	0.410636	0.498777
1	$\widehat{\theta_7}$	0.010007	6.54E-06	0.182582	0.236466	$\widehat{\theta_7}$	0.009819	-0.00018	0.179284	0.228734
2	Jefferys	0.010031	3.07E-05	0.392864	0.588723	Hartigan	0.009199	-0.0008	0.391018	0.545359
3		0.010057	5.65E-05	0.391305	0.574066		0.009094	-0.00091	0.383405	0.530909
4		0.009975	-2.5E-05	0.393901	0.580041		0.00918	-0.00082	0.386886	0.539158
5		0.00994	-6E-05	0.382563	0.555627		0.009083	-0.00092	0.381455	0.519989
6		0.010045	4.51E-05	0.392841	0.576906		0.009178	-0.00082	0.384401	0.539844
7		0.010014	1.42E-05	0.394983	0.583982		0.00921	-0.00079	0.38794	0.564423
8		0.010029	2.87E-05	0.391151	0.574156		0.009183	-0.00082	0.38515	0.532231
1	θ_{24}	0.009284	-0.00072	0.187108	0.230783	θ_{24}	0.009097	-0.0009	0.18863	0.229754
2	Jefferys	0.006619	-0.00338	0.444032	0.514996	Hartigan	0.005791	-0.00421	0.489307	0.540822
3		0.006636	-0.00336	0.442236	0.506598		0.005725	-0.00428	0.490259	0.539648
4		0.006582	-0.00342	0.447127	0.513134		0.005779	-0.00422	0.48834	0.539162
5		0.00656	-0.00344	0.440421	0.502778		0.005718	-0.00428	0.487549	0.535883
6		0.006629	-0.00337	0.442453	0.508503		0.005778	-0.00422	0.48646	0.539541
7		0.006608	-0.00339	0.445939	0.513362		0.005798	-0.0042	0.489421	0.548047
8		0.006618	-0.00338	0.442487	0.507875		0.005781	-0.00422	0.487226	0.536312
1	$\widehat{\theta_7}$	0.010269	0.000269	0.184369	0.242682	$\widehat{\theta_7}$	0.011037	0.001037	0.207018	0.281411
2	I Levy	0.011224	0.001224	0.419417	0.656231	Gamm	0.014877	0.004877	0.607976	0.971267
3		0.011267	0.001267	0.421934	0.67091		0.014902	0.004902	0.607901	0.985476
4		0.011224	0.001224	0.41517	0.649544		0.014886	0.004886	0.60513	0.969781
5		0.011253	0.001253	0.421098	0.670083		0.014941	0.004941	0.610102	0.969978
6		0.011287	0.001287	0.419303	0.660272		0.014854	0.004854	0.604225	0.980278
7		0.011224	0.001224	0.421955	0.674492		0.014812	0.004812	0.602363	0.974684
8		0.011314	0.001314	0.431005	0.769828		0.014913	0.004913	0.606515	0.994549
1	θ_{24}	0.009546	-0.00045	0.182425	0.228769	θ_{24}	0.010315	0.000315	0.186444	0.246529
2	I Levy	0.007825	-0.00217	0.401559	0.499323	Gamma	0.011489	0.001489	0.427738	0.665491
3		0.007855	-0.00214	0.402074	0.506938		0.011508	0.001508	0.425684	0.677175
4		0.007825	-0.00217	0.397285	0.495058		0.011496	0.001496	0.420783	0.663951
5		0.007845	-0.00215	0.402393	0.506997		0.011538	0.001538	0.424946	0.662667

6		0.007869	-0.00213	0.399081	0.499277		0.011471	0.001471	0.42257	0.673918
7		0.007825	-0.00217	0.405323	0.511023		0.011438	0.001438	0.422604	0.670168
8		0.007888	-0.00211	0.411789	0.569443		0.011516	0.001516	0.423031	0.684779

Table 6. Relative efficiency to the estimators from complete sample classified by prior, scheme and criteria

Prior	Sch	$(\hat{\theta}\,\#\,)39$					$(\hat{\theta}\,\#\,)40$				
		ARE	RE	MSE	Loss	Risk	ARE	RE	MSE	Loss	Risk
Uniform	2	0.2287	0.2114	0.0447	0.0447	0.038	0.6028	0.4945	0.2445	0.2246	0.4008
	3	0.228	0.209	0.0436	0.0436	0.0363	0.6049	0.4859	0.2361	0.2246	0.4001
	4	0.2284	0.2116	0.0447	0.0447	0.0362	0.615	0.4958	0.2458	0.2292	0.4005
	5	0.2269	0.2113	0.0446	0.0446	0.0379	0.6103	0.4978	0.2478	0.2255	0.399
	6	0.2293	0.21	0.0441	0.0441	0.0363	0.6104	0.4876	0.2378	0.2284	0.4014
	7	0.2304	0.2111	0.0445	0.0445	0.0364	0.6084	0.4898	0.2399	0.229	0.4025
	8	0.228	0.2075	0.043	0.043	0.0328	0.6113	0.4804	0.2308	0.2268	0.3998
M. Jefferys	2	0.2124	0.1965	0.0386	0.0386	0.0358	0.6508	0.5252	0.2758	0.212	0.3942
	3	0.2117	0.1942	0.0377	0.0377	0.0342	0.653	0.5161	0.2664	0.2121	0.3935
	4	0.2122	0.1967	0.0386	0.0386	0.0341	0.664	0.5266	0.2774	0.2164	0.394
	5	0.2107	0.1964	0.0385	0.0385	0.0357	0.6588	0.5287	0.2796	0.2129	0.3925
	6	0.2129	0.1952	0.0381	0.0381	0.0343	0.6589	0.5179	0.2682	0.2156	0.3948
	7	0.214	0.1962	0.0385	0.0385	0.0343	0.6568	0.5202	0.2706	0.2162	0.3959
	8	0.2117	0.1928	0.0372	0.0372	0.0309	0.6599	0.5103	0.2604	0.2141	0.3933
Jefferys	2	0.2044	0.1892	0.0358	0.0358	0.0347	0.6772	0.5427	0.2945	0.2069	0.3906
	3	0.2038	0.187	0.0349	0.0349	0.0331	0.6796	0.5333	0.2844	0.2069	0.3899
	4	0.2042	0.1893	0.0358	0.0358	0.033	0.691	0.5442	0.2961	0.2112	0.3903
	5	0.2029	0.1891	0.0357	0.0357	0.0346	0.6856	0.5463	0.2985	0.2077	0.3889
	6	0.2049	0.1879	0.0353	0.0353	0.0332	0.6857	0.5352	0.2864	0.2104	0.3912
	7	0.206	0.1889	0.0357	0.0357	0.0332	0.6835	0.5375	0.2889	0.211	0.3923
	8	0.2038	0.1857	0.0344	0.0344	0.0299	0.6867	0.5273	0.278	0.209	0.3896
Hartigan	2	0.1951	0.1791	0.032	0.032	0.0333	0.7034	0.5589	0.3124	0.1997	0.3864
	3	0.1945	0.1771	0.0313	0.0313	0.0318	0.7059	0.5493	0.3017	0.1997	0.3858
	4	0.1949	0.1793	0.0321	0.0321	0.0317	0.7177	0.5604	0.3141	0.2038	0.3862
	5	0.1936	0.1791	0.032	0.032	0.0331	0.7121	0.5627	0.3166	0.2004	0.3848
	6	0.1956	0.1779	0.0316	0.0316	0.0318	0.7122	0.5511	0.3038	0.203	0.387
	7	0.1965	0.1789	0.032	0.032	0.0318	0.7099	0.5536	0.3065	0.2036	0.3881
	8	0.1944	0.1758	0.0309	0.0309	0.0287	0.7133	0.5431	0.2949	0.2016	0.3855
Inverse Levy	2	0.2149	0.1992	0.0396	0.0396	0.0363	0.6373	0.5157	0.266	0.2139	0.3959
	3	0.2143	0.1969	0.0387	0.0387	0.0347	0.6396	0.5068	0.2569	0.2139	0.3952
	4	0.2147	0.1994	0.0397	0.0397	0.0346	0.6503	0.5171	0.2674	0.2182	0.3957
	5	0.2133	0.1991	0.0396	0.0396	0.0362	0.6452	0.5192	0.2696	0.2147	0.3942
	6	0.2155	0.1979	0.0391	0.0391	0.0347	0.6453	0.5086	0.2586	0.2175	0.3965
	7	0.2166	0.1989	0.0395	0.0395	0.0348	0.6432	0.5108	0.2609	0.218	0.3977
	8	0.2143	0.1955	0.0382	0.0382	0.0313	0.6463	0.5011	0.2511	0.216	0.395
Gamma	2	0.2586	0.2378	0.0565	0.0565	0.0416	0.5457	0.457	0.2089	0.2624	0.411
	3	0.2578	0.2351	0.0552	0.0552	0.0397	0.5476	0.4491	0.2017	0.2625	0.4103
	4	0.2583	0.238	0.0566	0.0566	0.0396	0.5568	0.4583	0.21	0.2678	0.4107
	5	0.2566	0.2377	0.0565	0.0565	0.0415	0.5525	0.4601	0.2117	0.2635	0.4092
	6	0.2593	0.2363	0.0558	0.0558	0.0398	0.5526	0.4507	0.2031	0.2669	0.4116
	7	0.2606	0.2375	0.0564	0.0564	0.0398	0.5507	0.4527	0.2049	0.2676	0.4128
	8	0.2578	0.2334	0.0545	0.0545	0.0359	0.5534	0.4441	0.1972	0.265	0.41