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Distributions of the Product and Ratio of Two Independent Pareto and Exponential Random Variables

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Abstract: The distributions of products and ratios of random variables are of interest in many areas of the sciences. In this paper, we find analytically the probability distributions of the product XY and the ratio X/Y , when X and Y are two independent random variables following Pareto and Exponential distributions, respectively.

Keywords: Product Distribution, Ratio Distribution, Pareto Distribution, Exponential Distribution, probability density function, Moment of order r, Survival function, Hazard function.

1 Introduction

Engineering, Physics, Economics, Order statistics, Classification, Ranking, Selection, Number theory, Genetics, Biology, Medicine, Hydrology, Psychology, these all applied problems depend on the distribution of product and ratio of random variables[1,2].

As an example of use of the product of random variables in physics, Sornette [3] mentions:
"...To mimic system size limitation, Takayasu, Sato, and Takayasu introduced a threshold x_c ...and found a stretched exponential truncating the power-law pdf beyond x_c . Frisch and Sornette recently developed a theory of extreme deviations generalizing the central limit theorem which, when applied to multiplication of random variables, predicts the generic presence of stretched exponential pdfs. The problem thus boils down to determining the tail of the pdf for a product of random variables ..."
Several authors have studied The product distributions for independent random variables come from the same family or different families, see [4] for t and Rayleigh families,[5] for Pareto and Kumaraswamy families, [6] for the t and Bessel families, and [7] for the independent generalized gamma-ratio family.

Examples of the use of the ratio of random variables include Mendelian inheritance ratios in genetics, mass to energy ratios in nuclear physics, target to control precipitation in meteorology, and inventory ratios in economics.
Several authors have studied The ratio distributions for independent random variables come from the same family or different families. The historical review, see [8,9] for the normal family, [10] for Student's t family, [11] for the Weibull family, [12] for the noncentral chi-squared family, [13] for the gamma family, [14] for the beta family, [15] for the logistic family, [16] for the Frechet family, [17] for the inverted gamma family, [18] for Laplace family, [19] for the generalized-F family, [20] for the hypoexponential family, [2] for the gamma and Rayleigh families, and [21] for gamma and exponential families.

In this paper, The analytical probability distributions are derived of XY and X/Y , when X and Y are two independent Pareto and Exponential distributions respectively. with probability density functions (p.d.f.s)

$$f_X(x) = \frac{ca^c}{x^{c+1}} \tag{1}$$

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$$f_Y(y) = \lambda e^{-\lambda y} \tag{2}$$

respectively, for $a \leq x < \infty$, $a > 0$, $c > 0$, $y > 0$, $\lambda > 0$.

Notations and Preliminaries

Recall some special mathematical functions, these will be used repeatedly throughout this article.

– The upper incomplete gamma function defined by

$$\Gamma(a,x) = \int_x^\infty \exp(-t)t^{a-1}dt \tag{3}$$

– The lower incomplete gamma function defined by

$$\gamma(a,x) = \int_0^x \exp(-t)t^{a-1}dt \tag{4}$$

– The Exponential integral is generalized to,
for $n = 0,1,2,...,x > 0$

$$E_n(x) = \int_1^\infty \frac{e^{-xt}}{t^n}dt \tag{5}$$

where n is then the order of the integral.

The calculations of this paper involve several Lemmas

Lemma 1 For $\alpha \geq 0$, $r \in \mathbb{R}$, and $b \in \mathbb{R}_+^*$

$$I(\alpha,r,b) = \int_\alpha^\infty x^r e^{-bx}dx = \frac{1}{b^{r+1}}\Gamma(r+1,b\alpha) \tag{6}$$

Proof Let $u = bx$, then

$$I(\alpha,r,b) = \int_{b\alpha}^{+\infty} \frac{u^r}{b^{r+1}}e^{-u}du = \frac{1}{b^{r+1}}\Gamma(r+1,b\alpha) \tag{7}$$

Lemma 2 For $t \in \mathbb{R}$,

$$\frac{d}{dx}\Gamma(t,v(x)) = -v(x)^{t-1}e^{-v(x)}\frac{d}{dx}v(x) \tag{8}$$

Proof

$$\frac{d}{dx}\Gamma(t,v) = \frac{d}{dv}\Gamma(t,v)\frac{dv}{dx} \tag{9}$$

Note that

$$\frac{d}{dv}\Gamma(t,v) = -v^{t-1}e^{-v}$$

Lemma 3 For $\alpha \geq 0$, $r \in \mathbb{R}$, and $b \in \mathbb{R}_+^*$

$$\int_0^\alpha x^r e^{-bx}dx = \frac{1}{b^{r+1}}\gamma(r+1,b\alpha) \tag{10}$$

Proof For $u = bx$

$$\int_0^{b\alpha} \frac{u^r}{b^{r+1}} e^{-u} du = \frac{1}{b^{r+1}} \gamma(r+1, b\alpha)$$

Lemma 4 The Exponential integral is generalized to,
for $n = 0, 1, 2, ..., x > 0$

$$E_n(x) = \int_1^{\infty} \frac{e^{-xt}}{t^n} dt \tag{11}$$

where n is then the order of the integral. This expression (11) is closely related to the incomplete gamma function as follows, for $n = 0, 1, 2, ..., x > 0$

$$E_n(x) = x^{n-1} \Gamma(1-n, x) \tag{12}$$

Proof For $u = xt$

$$E_n(x) = x^{n-1} \int_x^{\infty} e^{-u} u^{-n} du = x^{n-1} \Gamma(-n+1)$$

2 Distribution of the Product X Y

Theorem 2.1. Suppose X and Y are independent and distributed according to (1) and (2), respectively. Then for $z > 0$ the cumulative distribution function *c.d.f.* of $Z = XY$ can be expressed as:

$$F_Z(z) = \begin{cases} 0 & \text{if } z \leq 0 \\ 1 - \frac{a^c}{z^c \lambda^c} \Gamma(c+1) + \frac{a^c}{z^c \lambda^c} \Gamma(c+1, \frac{\lambda z}{a}) - e^{-\frac{\lambda z}{a}} & \text{if } z > 0 \end{cases} \tag{13}$$

Proof the c.d.f. corresponding to (1) is $F_X(x) = 1 - (\frac{a}{x})^c$ Thus, one can write the c.d.f. of XY as

$$\begin{aligned} Pr(XY \leq z) &= \int_0^{\infty} F_X(\frac{z}{y}) f_Y(y) dy \\ &= \int_0^{z/a} (1 - (\frac{ay}{z})^c) f_Y(y) dy \end{aligned} \tag{14}$$

We can write $F_Z(z)$ as:

$$F_Z(z) = \int_0^{\infty} (1 - (\frac{ay}{z})^c) f_Y(y) dy - \int_{z/a}^{\infty} (1 - (\frac{ay}{z})^c) f_Y(y) dy \tag{15}$$

Let

$$I_1 = \int_0^{\infty} (1 - (\frac{ay}{z})^c) f_Y(y) dy \tag{16}$$

And

$$I_2 = \int_{z/a}^{\infty} (1 - (\frac{ay}{z})^c) f_Y(y) dy \tag{17}$$

Then

$$F_Z(z) = I_1 - I_2$$

.

Calculus of I_1

$$\begin{aligned} I_1 &= \int_0^{\infty} (1 - (\frac{ay}{z})^c) f_Y(y) dy \\ &= 1 - \frac{a^c}{z^c} \int_0^{\infty} y^c \lambda e^{-\lambda y} dy \\ &= 1 - \frac{\lambda a^c}{z^c} \int_0^{\infty} y^c e^{-\lambda y} dy \end{aligned} \tag{18}$$

Using **Lemma 1** (6) in the integral above

$$I_1 = 1 - \frac{a^c}{z^c \lambda^c} \Gamma(c+1) \tag{19}$$

Calculus of I_2

$$\begin{aligned} I_2 &= \int_{z/a}^{\infty} (1 - (\frac{ay}{z})^c) f_Y(y) dy \\ &= \int_{z/a}^{\infty} \lambda e^{-\lambda y} dy - \frac{a^c}{z^c} \int_{z/a}^{\infty} y^c \lambda e^{-\lambda y} dy \\ &= e^{-\lambda z/a} - \frac{a^c}{z^c} \int_{z/a}^{\infty} y^c \lambda e^{-\lambda y} dy \end{aligned} \tag{20}$$

Using **Lemma 1** (6) in the integral above

$$I_2 = e^{-\lambda z/a} - \frac{a^c}{z^c \lambda^c} \Gamma(c+1, \frac{\lambda z}{a})$$

And finally

$$F_Z(z) = \begin{cases} 0 & \text{if } z \le 0 \\ 1 - \frac{a^c}{z^c \lambda^c} \Gamma(c+1) + \frac{a^c}{z^c \lambda^c} \Gamma(c+1, \frac{\lambda z}{a}) - e^{-\frac{\lambda z}{a}} & \text{if } z > 0 \end{cases} \tag{21}$$

Corollary 2.2. Let X and Y are independent and distributed according to (1) and (2), respectively. Then for $z > 0$, the probability density function *p.d.f.* of $Z = XY$ can be expressed as:

$$f_Z(z) = \begin{cases} 0 & \text{if } z \le 0 \\ \frac{ca^c}{\lambda^c z^{c+1}} \left[\Gamma(c+1) - \Gamma(c+1, \frac{\lambda z}{a}) \right] & \text{if } z > 0 \end{cases} \tag{22}$$

Proof The probability density function $f_Z(z)$ in (22) follows by differentiation using **Lemma 2**

Corollary 2.3. Let X and Y are independent and distributed according to (1) and (2), respectively. Then for $c > r$, $z > \alpha$, $\alpha > 0$. the moment of order r of $Z = XY$ can be expressed as:

$$E[Z^r] = \left[\frac{ca^c \left[\Gamma(c+1) - \Gamma(c+1, \lambda\alpha/a) \right]}{\alpha^{c-r} \lambda^c (c-r)} + \frac{ca^c \Gamma(r+1, \lambda\alpha/a)}{\lambda^c (c-r)} \right] \tag{23}$$

Proof

$$\begin{aligned} E[Z^r] &= \int_0^{+\infty} z^r f_Z(z) dz \\ &= \lim_{\alpha \to 0^+} \int_\alpha^{+\infty} z^r \frac{ca^c}{\lambda^c z^{c+1}} \left[\Gamma(c+1) - \Gamma(c+1, \lambda z/a) \right] dz \\ &= \lim_{\alpha \to 0^+} \left[\int_\alpha^{\infty} \frac{ca^c}{\lambda^c z^{c-r+1}} \Gamma(c+1) dz - \int_\alpha^{\infty} \frac{ca^c}{\lambda^c z^{c+1-r}} \Gamma(c+1, \lambda z/a) dz \right] \end{aligned} \tag{24}$$

Let

$$I_1 = \int_\alpha^{\infty} \frac{ca^c}{\lambda^c z^{c-r+1}} \Gamma(c+1) dz$$

and

$$I_2=\int_{\alpha}^{\infty}\frac{ca^c}{\lambda^c z^{c+1-r}}\Gamma(c+1,\lambda z/a)dz$$

$$E[Z^r]=\Big[I_1-I_2\Big] \tag{25}$$

Calculus of I_1

$$I_1=\int_{\alpha}^{\infty}\frac{ca^c}{\lambda^c z^{c-r+1}}\Gamma(c+1)dz=\frac{ca^c\Gamma(c+1)}{\lambda^c\alpha^{c-r}(c-r)} \tag{26}$$

Calculus of I_2

$$I_2=\frac{ca^c}{\lambda^c}\int_{\alpha}^{\infty}z^{r-1-c}\Gamma(c+1,\lambda z/a)dz \tag{27}$$

Integration by parts implies of I_2:

$$I_2=-\frac{\alpha^{r-c}}{r-c}\Gamma(c+1,\lambda\alpha/a)+\frac{\lambda^{c+1}}{a^{c+1}(r-c)}\int_{\alpha}^{\infty}z^r e^{-\lambda z/a}dz \tag{28}$$

Using **Lemma 1** (6) in the integral above then

$$I_2=\frac{ca^c\Gamma(c+1,\lambda\alpha/a)}{\lambda^c(c-r)\alpha^{c-r}}-\frac{ca^c\Gamma(r+1,\lambda\alpha/a)}{\lambda^c(c-r)} \tag{29}$$

And finally

$$E[Z^r]=\left[\frac{ca^c\Big[\Gamma(c+1)-\Gamma(c+1,\lambda\alpha/a)\Big]}{\alpha^{c-r}\lambda^c(c-r)}+\frac{ca^c\Gamma(r+1,\lambda\alpha/a)}{\lambda^c(c-r)}\right] \tag{30}$$

Corollary 2.4. Let X and Y are independent and distributed according to (1) and (2), respectively. Then for $c>1$. the Expected value of $Z=XY$ can be expressed as:

$$E[Z]=\left[\frac{ca^c\Big[\Gamma(c+1)-\Gamma(c+1,\lambda\alpha/a)\Big]}{\alpha^{c-1}\lambda^c(c-1)}+\frac{ca^c\Gamma(2,\lambda\alpha/a)}{\lambda^c(c-1)}\right] \tag{31}$$

Corollary 2.5. Let X and Y are independent and distributed according to (1) and (2), respectively. Then for $c>2$. the Variance of $Z=XY$ can be expressed as:

$$\begin{aligned}\sigma^2=&\frac{ca^c\Big[\Gamma(c+1)-\Gamma(c+1,\lambda\alpha/a)\Big]}{\alpha^{c-2}\lambda^c(c-2)}+\frac{ca^c\Gamma(3,\lambda\alpha/a)}{\lambda^c(c-2)}\\ &-\left[\frac{ca^c\Big[\Gamma(c+1)-\Gamma(c+1,\lambda\alpha/a)\Big]}{\alpha^{c-1}\lambda^c(c-1)}+\frac{ca^c\Gamma(2,\lambda\alpha/a)}{\lambda^c(c-1)}\right]^2\end{aligned} \tag{32}$$

Proof By definition of variance :

$$\sigma^2=E[Z^2]-E[Z]^2$$

Corollary 2.6. Let X and Y are independent and distributed according to (1) and (2), respectively. Then for $z > 0$. the Survival function of $Z = XY$ can be expressed as:

$$S_Z(z) = \begin{cases} 1 & \text{if } z \leq 0 \\ \frac{a^c}{z^c \lambda^c} \Gamma(c+1) - \frac{a^c}{z^c \lambda^c} \Gamma(c+1, \frac{\lambda z}{a}) + e^{-\frac{\lambda z}{a}} & \text{if } z > 0 \end{cases} \tag{33}$$

Proof By definition of the Survival function $S_Z(z) = 1 - F_Z(z)$

Corollary 2.7. Let X and Y are independent and distributed according to (1) and (2), respectively. Then for $z > 0$. the Hazard function of $Z = XY$ can be expressed as:

$$h_Z(z) = \begin{cases} 0 & \text{if } z \leq 0 \\ \frac{ca^c \left[\Gamma(c+1) - \Gamma(c+1, \lambda z/a) \right]}{za^c \left[\Gamma(c+1) - \Gamma(c+1, \lambda z/a) \right] + z^{c+1} \lambda^c e^{-\lambda z/a}} & \text{if } z > 0 \end{cases} \tag{34}$$

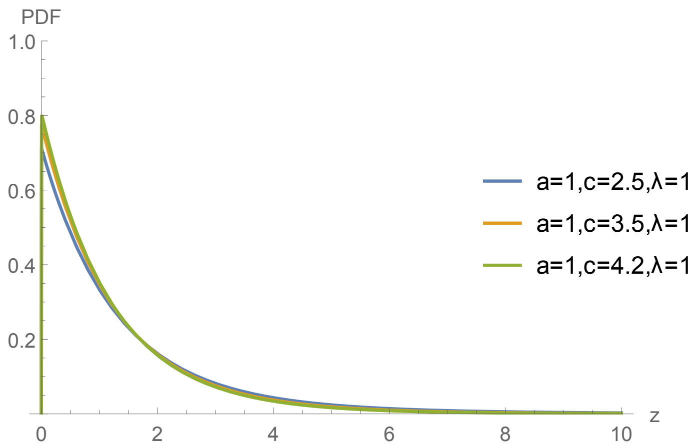

Fig. 1: Plots of the pdf (22) for $a = 1, c = 2.5, 3.5, 4.2, \lambda = 1$

Figure 1 illustrates possible shapes of the pdf (22) of the product distribution XY for a range of values of the scale parameter $c = 2.5, 3.5, 4.2,\ \ a = 1$ and $\lambda = 1$.
Note that the pdf's curves are positive, continuous and decreasing in the interval $0 < z < +\infty$, in addition the area between the density curve and horizontal X-axis is equal to 1, $\int_0^{+\infty} f_Z(z) = 1$, moreover the shapes are uni-modal and the parameter c control the scale of the pdf's curves.

Figure 2 illustrates possible shapes of the pdf (22) of the product distribution XY for a range of values of the location parameter $a = 1, 2, 3,\ \ c = 2$ and $\lambda = 1$.
Note that the pdf's curves are positive, continuous and decreasing in the interval $0 < z < +\infty$, in addition the area between the density curve and horizontal X-axis is equal to 1, $\int_0^{+\infty} f_Z(z) = 1$, moreover the shapes are uni-modal and the value of the parameter a largely dictates the behaviour of the pdf near $z = 0$.

Figure 3 shows the shape of the hazard function of the product distribution XY for different values of the shape parameter $c = 3, 5.6, a = 1, \lambda = 1$. Note that the hazard's curves are positive, continuous and decreasing in the interval $0 < z < +\infty$ and $\lim_{z \to \infty} h_Z(z) = 0$, moreover the shapes are uni-modal and the parameter c control the scale of the hazard's curves.

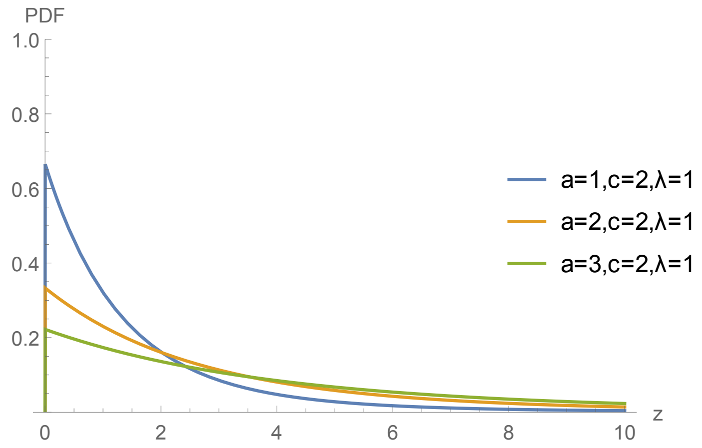

Fig. 2: Plots of the pdf (22) for $a = 1,2,3, c = 2, \lambda = 1$

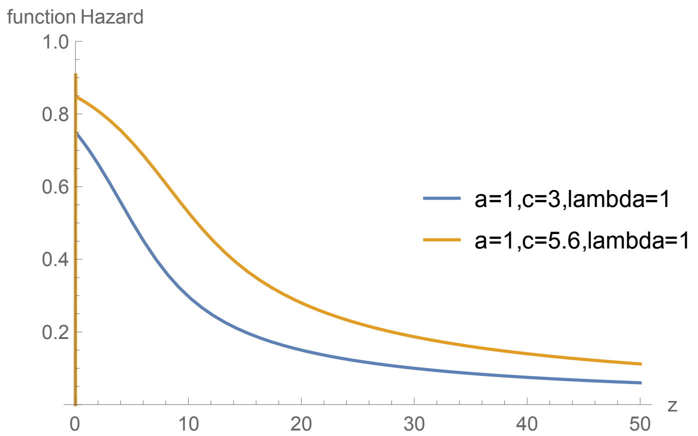

Fig. 3: Plots of the Hazard function (34) for $a = 1, c = 3, 5.6, \lambda = 1$

3 Distribution of the Ratio X/Y

Theorem 3.1. Suppose X and Y are independent and distributed according to (1) and (2), respectively. Then for $z > 0$ the cumulative distribution function *c.d.f.* of $Z = X/Y$ can be expressed as:

$$F_Z(z) = \begin{cases} 0 & \text{if } z \leq 0 \\ e^{-\lambda a/z} - \frac{a^c \lambda^c}{z^c} \Gamma(-c+1, \lambda a/z) & \text{if } z > 0, c < 1 \\ e^{-\lambda a/z} - \frac{a\lambda}{z} E_c(\lambda a/z) & \text{if } z > 0, c = 1,2,3,\ldots \end{cases} \tag{35}$$

Proof the c.d.f. corresponding to (1) is $F_X(x) = 1 - (\frac{a}{x})^c$

$$\begin{aligned} F_Z(z) &= pr(X/Y \le z) \\ &= \int_0^\infty F_X(zy) f_Y(y) dy \\ &= \int_{a/z}^\infty (1 - (\frac{a}{yz})^c) f_Y(y) dy \\ &= \int_{a/z}^\infty f_Y(y) dy - \frac{a^c}{z^c} \int_{a/z}^\infty \frac{f_Y(y)}{y^c} dy \end{aligned} \tag{36}$$

Let

$$I_1 = \int_{a/z}^\infty \lambda e^{-\lambda y} dy = e^{-\lambda a/z}$$

And

$$I_2 = \int_{a/z}^\infty \frac{\lambda e^{-\lambda y}}{y^c} dy$$

Then we can write

$$F_Z(z) = I_1 - I_2$$

Calculus of I_2

$$\begin{aligned} I_2 &= \lambda \int_{a/z}^\infty \frac{e^{-\lambda y}}{y^c} dy \\ &= \lambda \int_{a/z}^\infty y^{-c} e^{-\lambda y} dy \end{aligned}$$

if we substitute $u = \lambda y$, Then

$$\begin{aligned} I_2 &= \lambda \int_{\lambda a/z}^\infty e^{-u} (\frac{u}{\lambda})^{-c} \frac{du}{\lambda} \\ &= \lambda^c \int_{\lambda a/z}^\infty u^{-c} e^{-u} du \\ &= \lambda^c \Gamma(1 - c, \lambda a/z) \end{aligned} \tag{37}$$

Finally we get

$$F_Z(z) = -\frac{a^c \lambda^c}{z^c} \Gamma(-c + 1, \frac{\lambda a}{z}) + e^{-\lambda \frac{a}{z}} \tag{38}$$

For $c < 1$.

For $c = 1, 2, 3, \ldots$ using **Lemma 4** we have

$$\Gamma(1 - c, \lambda a/z) = (\frac{\lambda a}{z})^{1-c} E_c(\lambda a/z)$$

, and

$$F_Z(z) = e^{-\lambda a/z} - \frac{a\lambda}{z} E_c(\lambda a/z) \tag{39}$$

Where c is the order of the integral.

Corollary 3.2. Let X and Y are independent and distributed according to (1) and (2), respectively. Then for $z > 0$, the probability density function *p.d.f.* of $Z = X/Y$ can be expressed as:

$$f_Z(z) = \begin{cases} 0 & \text{if } z \le 0 \\ c(a\lambda)^c \frac{\Gamma(1-c, a\lambda/z)}{z^{c+1}} & \text{if } z > 0, c < 1 \\ \frac{\lambda a}{z^2} e^{-\lambda a/z} - \frac{(a\lambda)^2}{z^3} E_{c-1}(\lambda a/z) + \frac{a\lambda}{z^2} E_c(\lambda a/z) & \text{if } z > 0, c = 1, 2, 3, \ldots \end{cases} \tag{40}$$

Proof The probability density function $f_Z(z)$ in (40) follows by differentiation using **Lemma 2**(8), and

$$\frac{d}{dz} E_n(z) = -E_{n-1}(z) \tag{41}$$

For $n = 1, 2, 3, \ldots$

Corollary 3.3. Let X and Y are independent and distributed according to (1) and (2), respectively. Then for $z > 0$.the Survival function of $Z = X/Y$ can be expressed as:

$$S_Z(z) = \begin{cases} 1 & \text{if } z \le 0 \\ 1 - e^{-\lambda a/z} + \frac{a^c \lambda^c}{z^c} \Gamma(-c+1, \lambda a/z) & \text{if } z > 0, c < 1 \\ 1 - e^{-\lambda a/z} + \frac{a\lambda}{z} E_c(\lambda a/z) & \text{if } z > 0, c = 1, 2, 3, \ldots \end{cases} \tag{42}$$

Proof By definition of the Survival function $S_Z(z) = 1 - F_Z(z)$

Corollary 3.4. Let X and Y are independent and distributed according to (1) and (2), respectively. Then for $z > 0$. the Hazard function of $Z = X/Y$ can be expressed as:

$$h_Z(z) = \begin{cases} 0 & \text{if } z \le 0 \\ \frac{c(a\lambda)^c \Gamma(1-c, a\lambda/z)}{z(a\lambda)^c \Gamma(1-c, a\lambda/z) + z^{c+1}(1 - e^{-\lambda a/z})} & \text{if } z > 0, c < 1 \\ \frac{\frac{\lambda a}{z^2} e^{-\lambda a/z} - \frac{(a\lambda)^2}{z^3} E_{c-1}(\lambda a/z) + \frac{a\lambda}{z^2} E_c(\lambda a/z)}{1 - e^{-\lambda a/z} + \frac{a\lambda}{z} E_c(\lambda a/z)} & \text{if } z > 0, c = 1, 2, \ldots \end{cases} \tag{43}$$

Proof By definition of the hazard function $h_Z(z) = \frac{f_Z(z)}{S_Z(z)}$

Figure 4 illustrates possible shapes of the pdf (40) of the ratio distribution X/Y for a range of values of the hazard rate λ, $a = 1$, $c = 0.5$ and $\lambda = 1, 2, 5$.
Note that the pdf's curves are positive and continuous, in addition these curves are increasing then decreasing in the interval $0 < z < +\infty$, in addition the area between the density curve and horizontal X-axis is equal to 1, $\int_0^{+\infty} f_Z(z) = 1$, moreover the shapes are uni-modal and the value of the parameter λ control the scale of the pdf's curves.

Figure 5 illustrates possible shapes of the pdf (40) of the ratio distribution X/Y for a range of values of the scale parameter $c = 2, 3, 4$, $a = 1$ and $\lambda = 1$.
Note that the pdf's curves are positive, continuous and increasing in the interval $0 < z \le 0.6$ then decreasing in the interval $0.6 < z < +\infty$, in addition the area between the density curve and horizontal X-axis is equal to 1, $\int_0^{+\infty} f_Z(z) = 1$, moreover the shapes are uni-modal and the parameter c control the scale of the pdf's curves.

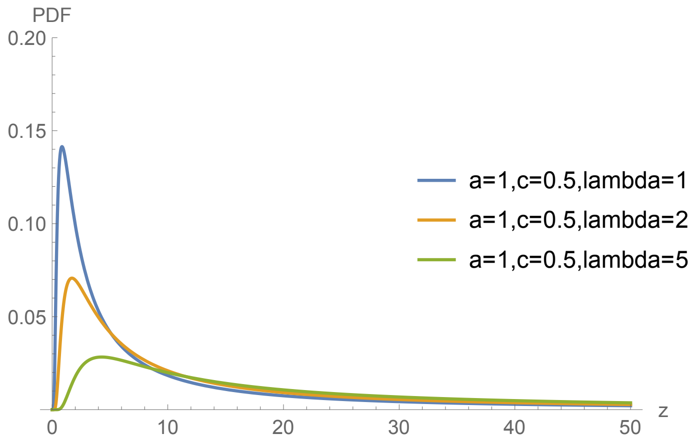

Fig. 4: Plots of the probability density function (40) for $a = 1, c = 0.5, \lambda = 1, 2, 5$

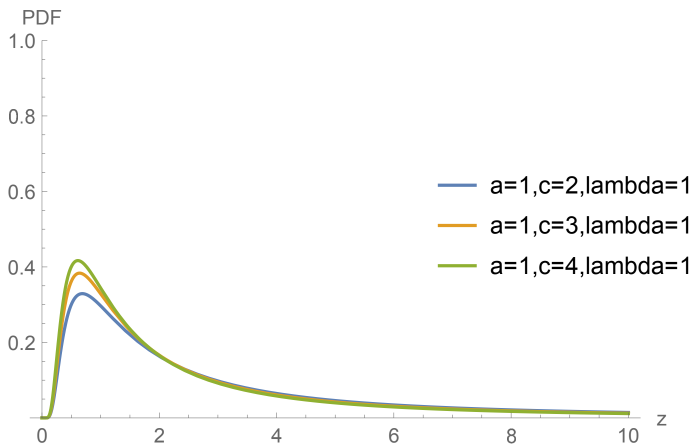

Fig. 5: Plots of the probability density function (40) for $a = 1, c = 2, 3, 4, \lambda = 1$

4 Hazard function

It is useful to think about real phenomena and how their hazard functions might be shaped. For example, if T denotes the age of a car when it first has a serious engine problem, then one might expect the corresponding hazard function $h(t)$ to be increasing in t; that is, the conditional probability of a serious engine problem in the next month, given no problem so far, will increase with the life of the car. In contrast, if one were studying infant mortality in a region of the world where there was poor nutrition, one might expect $h(t)$ to be decreasing during the first year of life. This is known to be due to selection during the first year of life. Finally, in some applications (such as when T is the lifetime of a light bulb or the time to which you won a BIG lottery), the hazard function will be approximately constant in t. This means that the chances of failure in the next short time interval, given that failure hasn't yet occurred, does not change with t; e.g., a 1-month old bulb has the same probability of burning out in the next week as does a 5-year old bulb. As we will see below, this 'lack of aging' or 'memory less' property uniquely defines the exponential distribution, which plays a central role in survival analysis. The hazard function may assume more a complex form. For example, if T denote the age of death, then the hazard function $h(t)$ is expected to be decreasing at first and then gradually increasing in the end, reflecting higher hazard of infants and elderly.

Let take an example, for $Z = X/Y$ quotient of Pareto and Exponential random variables, For $z > 0$

$$h_Z(z) = \frac{c(a\lambda)^c \Gamma(1-c, a\lambda/z)}{z(a\lambda)^c \Gamma(1-c, a\lambda/z) + z^{c+1}(1 - e^{-\lambda a/z})} \tag{44}$$

1. **Continuity of $h_Z(z)$:**
for $z > 0$

$$h_Z(z) = \frac{u(z)}{v(z)}$$

Where

$$u(z) = c(a\lambda)^c \Gamma(1-c, a\lambda/z) \tag{45}$$

And

$$v(z) = z(a\lambda)^c \Gamma(1-c, a\lambda/z) + z^{c+1}(1-e^{-\lambda a/z}) \tag{46}$$

We have $\Gamma(1-c, a\lambda/z)$ is continuous, then $u(z)$ is a continuous function , and for $z > 0$, $(1-e^{-\lambda a/z}) > 0$ and continuous, then the function $v(z) > 0$ is positive and continuous. finally we get that, $h_Z(z) = \frac{u(z)}{v(z)}$ is a continuous function.

2. **Derivative of $h_Z(z)$**
the derivative of $h_Z(z)$ is given as

$$\frac{d}{dz}h_Z(z) = \frac{c(1+c)(1-e^{-a\lambda/z})z^c(a\lambda)^c\Gamma(1-c,a\lambda/z) + (a\lambda)^{2c}c(\Gamma(1-c,a\lambda/z))^2}{\left[z(a\lambda)^c\Gamma(1-c,a\lambda/z) + z^{c+1}(1-e^{-\lambda a/z})\right]^2} + \frac{(a\lambda)ce^{-a\lambda/z}}{z^{-c+2}\left[(1-e^{-a\lambda/z})z^{1+c} + (a\lambda)^c z\Gamma(1-c,a\lambda/z)\right]} \tag{47}$$

Proof

$$\frac{d}{dz}\left(\frac{u(z)}{v(z)}\right) = \frac{\frac{du(z)}{dz}v(z) - u(z)\frac{dv(z)}{dz}}{v(z)^2} \tag{48}$$

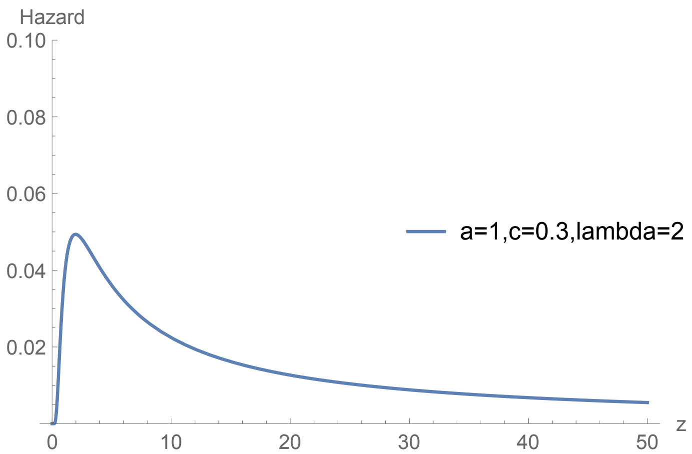

Fig. 6: Plot of the Hazard function (43) for $a = 1, c = 0.3$ *and* $\lambda = 2$.

Figure 6 shows the shape of the hazard function of the ratio distribution X/Y for the parameters $c = 0.3, a = 1, \lambda = 2$. Note that the hazard's curve is positive and continuous, in addition the hazard curve is increasing then decreasing in the interval $0 < z < +\infty$, and $\lim_{z\to\infty} h_Z(z) = 0$.

5 Applications and simulations

5.1 Replenishment of a Life Support system

With the advent of long-duration manned space flights, estimating the changes that will occur in the quantities of certain substances in the desired ecological system has become more complex.
Ideally, a mathematical model for general system analysis of an ecological system for.lung-duration flights will provide for:

1.A formulation of the control problem from which optimal control functions car be determined.
2.A preliminary design for the ecological system in respect to system stability and control considerations.
3.A method for determining the time required to restore the Išystem to a suitable balance in the case of a mishap and the time required initially to put the system into operation.
4.The determination of resupply requirements.

One aspect of a preliminary model designed to include the above considerations has resulted in the requirement for evaluating a product of random variables. It may be stated in the following manner:

Consider the amount of oxygen in a cabin atmosphere. The amount is affected by leakage, crew consumption, and resupply from a storage capacity. Let:

$X(t)$=amount of oxygen (in moles) in the cabin atmosphere at time t.

L= proportion of rate of loss of oxygen due to leakage from the cabin atmosphere per time period.

Y_1=rate of increase in oxygen content in the cabin atmosphere from storage per time period.

K=rate of decrease in the oxygen content in the cabin atmosphere due to crew consumption per time period.

The estimated amount of oxygen at time t is:

$$X(t) = \exp(Lt)(Y_1 - K) + X(0) \tag{49}$$

In this study, $W = exp(Lt)$ and $Z = (Y_1 - K)$ represent random variables which are functions of time. A knowledge of combining random Variables in product forms is required for solving this problem. Suppose Z is a random variable follows Pareto distribution with parameter $c = 2$ and $a = 1$, and W is a random variable follows Exponential distribution with parameter $\lambda = 1$, then The estimated amount of oxygen at time t is:

$$X(t) = \begin{cases} 0 & \text{if } t < 0 \\ X(0) & \text{if } t = 0 \\ \frac{2[\Gamma(3)-\Gamma(3,t)]}{t^3} & \text{if } t > 0 \end{cases} \tag{50}$$

5.2 Measurement or Radiation by Electronic Counters

Proportional, Geiger, and scintillation counters are often used to ’ detect X and γ radiation, as well as other charged particles such as electrons and α particles. Design of these counters and their associated circuits depends to some extent on what is to be detected. A device common to all counters is a scalar. This electronic device counts pulses produced by the counter. Once the number of pulses over a measured period of time is known, the average counting rate is obtained by simple division. If the rate of pulses is too high for a mechanical device, it is necessary to scale down the pulses by a known factor before feeding them to a mechanical counter. There are two kinds of scalars: the binary scalar in which the scalar factor is some power of 2, and the decade scalar in which the scaling factor is some power of 10.

A typical binary scalar has several scaling factors ranging from 2^0 to 2^{14}. The scaling circuit is made up of a number of identical ”stages” connected in series, the number of stages being equal to n, where 2^n is the desired scaling factor, Each stage is composed of a number of vacuum tubes, capacitors, and resistors, connected so that only one pulse of current is transmitted for every two pulses received, Since the output of one stage is connected to the input of another, this division by two is repeated as many times as there are stages. The output of the last stage is connected to a mechanical counter that will register one count for every pulse transmitted to it by the last stage. Thus, if N pulses from a counter are passed through a circuit of n stages, only $\frac{N}{2^n}$ will register on the mechanical counter. Because arrival of X-ray quanta in the counter is random in time, the accuracy of a counting rate measurement is governed by the laws of probability. Two counts of the same X-ray beam for identical periods of time will not be precisely the same because of the random spacing between pulses, even though the counter and scaler are functioning perfectly. Clearly, the accuracy of a rate measurement of this kind improves as the time of counting is prolonged. It is therefore; important to know how long to count in order to attain a specified degree of accuracy. This problem is complicated when additional background causes contamination in the counting process. This unavoidable background is due to cosmic rays and may be augmented, particularly in some laboratories, by nearby radioactive materials. Suppose we want to estimate the diffraction background in the presence of a fairly large unavoidable background. Let N be the number of pulses ! counted in a given time from a radiation source; Let N_b be the number counted in the same time with radiation source removed. The N_b counts are due to unavoidable background and $(N-N_b)$ to the diffract able background being measured. The relative probable error in $(N-N_b)$ is

$$E_{N-N_b} = \frac{67\sqrt{N+N_b}}{N-N_b}$$

percent.

since N and N_b are random variables, the desirability of obtaining the density function of the above quotient from of random variable is apparent.
For instance, if $\sqrt{N+N_b}$ is a random variable follows Pareto distribution with parameter $c=0.5$ and $a=1$, and $N-N_b$ is a random variable follows Exponential distribution with parameter $\lambda=1$, then by using our result The relative probable error in $(N-N_b)$ is

$$E_{N-N_b} = \begin{cases} 0 & \text{if } z \le 0 \\ \frac{33.5\Gamma(0.5,1/z)}{z^{1.5}} & \text{if } z > 0 \end{cases} \tag{51}$$

6 Conclusion

This paper has derived The analytical expressions of the PDF, CDF, the moment of order r, the survival function, and the hazard function, for the distributions of XY and X/Y when X and Y are Pareto and Exponential random variables distributed independently of each other. we illustrate our results in some graphics of the distributions of product and ratio. Finally we have discussed two examples of engineering applications for the distribution of the product and ratio.
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Conflict of interest

The authors declare that they have no conflict of interest.

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