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Mathematical Model for Atmospheric Dispersion Equation (a Review)

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Abstract:First: - We estimated mathematical model for hermitized model in steady-state for the dispersion of air pollutants in low winds in three dimensions. The eddy diffusivities have been parameterized in terms of downwind distance for near source dispersion [1]. The solution has been used to simulate the field tracer data collected at IIT Delhi in low wind convective conditions.

Second: The analytical solution of advection-diffusion equation in unstable condition has been obtained taking power-law profiles for the mean wind speed and vertical eddy diffusivity coefficient. The comparison between the proposed model and data collected from nine experiments conducted at Egyptian Atomic Energy Authority has been done. The proposed model performs well with observed concentrations.

Keywords: Hermitized Model, Unstable Condition, Fourier Transform, Non-Gaussian, Copenhagen Experiment.

1 Introduction

The dispersion from the source is controlled by the downwind pollutants transport through the mean air flow and turbulent velocity fluctuation disperses the pollutant in all directions [2]. Along-wind diffusion is particularly important near the leading edge of the plume, where uncontaminated fluid from upwind mixes with the mass initially released [3]. In all part of the world and more specifically in tropical regions occur weak wind conditions. The low wind can be expected to occur 30-45% of the time at most sites [4, 5].

The Gaussian puff models are assumed to be the same as that of the plume, whereas actually puff dispersion and plume dispersion theories are quite different [6]. The turbulent diffusion in the direction of the mean flow is neglected relative to the transport due to advection, which implies that the model should be applied for average wind speeds of more than 1m/s [7]. From the mathematical point of view: initially eddy diffusivities are assumed to be constant for solving the diffusion equation.

They produce an unreasonable overestimation in low wind conditions [8, 9]. However, to overcome the problem of over prediction, various modifications in estimating dispersion coefficients have been suggested [e.g. split sigma and segmented plume methods [10], split sigma theta and short-term averaging methods [11], U_{min} approach [12] .

Various aspects of atmospheric dispersion in low winds have been recently been received by [13,14].

For nearly thirty years it has known that vertical concentration profiles from field and laboratory experiments of near-surface point source releases exhibit non-Gaussian distribution [15-17]. [18] have derived a puff formula for computing the concentration of smoke emitted from a point source in calm wind conditions by expressing the dispersion parameters as linear functions of time. The non-Gaussian shape has been attributed to the non-uniform turbulent mixing that occurs in boundary-layer flows. For elevated releases, [19, 20] have shown that the agreement between a Gaussian reflected-plume formula and measured vertical profiles becomes progressively worse at larger downwind distances.

Several researchers have highlighted the similarities between a non-Gaussian model and experimental data for surface releases [21, 22]. For elevated releases, [23] found favorable agreement between ground-level concentration measurements in a wind tunnel and a non-Gaussian model with a prescribed vertical diffusion coefficient σ_z, [24,25] have shown that a non-Gaussian model better simulates concentration profiles

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originating from surface sources than a Gaussian reflected-plume model.

Analytical models have been developed [26-29] for the atmospheric-dispersion in both bounded (inversion) and unbounded (infinite mixing layer) domains by expressing both eddy diffusivities and wind speed as power law functions of height. Tirabassi and his coworkers [30-32] have also used height dependent on wind speed as well as eddy diffusivities to obtain solutions of advection-diffusion equation.

Recently, it has been emphasized [33-36] that for the treatment of near source dispersion, eddy diffusivities are dependent on a linear function of downwind distance from the source. This has been used in dispersion models in convective [34] and stable weak wind conditions [35]. Analytical solutions of the advection–diffusion equation with variable vertical eddy diffusivity and wind speed using Hankel transform has been studied by [37].

It is recognized that the planetary boundary layer (PBL) is often capped by an inversion, which simply reflects back the material reaching the inversion base [38-39]. The earlier models [34,40,35,41] imply unrestricted diffusion of plume in the vertical direction. This does not occur in the real atmosphere where a finite layer of vanishing turbulence at the top of the PBL restricts vertical diffusion with the PBL.

In the first thing of the report, an attempt is made for generalized the atmospheric diffusion operator. This can be accomplished by employing the realizability procedure, to identify a surface operator, which ensures self-adjointness of the atmospheric diffusion operator. We have formulated a mathematical model for dispersion of air pollutants in low winds by taking into account the diffusion in all directions and advection along the mean wind. The eddy diffusivities are assumed to be linear functions of downwind distance. An analytical solution has been obtained for the resulting advection-diffusion equation with the physically relevant boundary conditions, from the series of field experiments (in tropical conditions) conducted at IIT Delhi sports ground [42], have been simulated by the solution obtained.

In the second thing of the report, we have formulated a mathematical model for dispersion of air pollutants in moderated winds by taking into account the diffusion in all directions and advection along the mean wind. The eddy diffusivities are assumed to be linear functions of downwind distance and power law in the vertical length. An analytical solution has been obtained for the resulting advection-diffusion equation with the physically relevant boundary conditions. The moderate data collected during the convective conditions. From nine experiments conducted at Inshas site, Cairo-Egypt [36] has been calculated by the solution obtained.

2 First Case

2.1 Adjoint of the Atmospheric Diffusion Operator

We now give formal proof for the assertion that the conventional atmospheric diffusion operator in the steady state, in the simplest form; namely:

$$H = U\frac{\partial}{\partial x} - \frac{\partial}{\partial x}\left(K_x\frac{\partial}{\partial x}\right) - \frac{\partial}{\partial y}\left(K_y\frac{\partial}{\partial y}\right) - \frac{\partial}{\partial z}\left(K_z\frac{\partial}{\partial z}\right) \qquad (1)$$

It does not represent a hermitean operator, in which (K_x, K_y, K_z) are regular eddy diffusivity functions.

In view of the conventional second order Sturm-Liouville Differential operator [43,44] namely:

$$E = \frac{\partial}{\partial x}\left(K_x\frac{\partial}{\partial x}\right) + f(x) \qquad (2)$$

is not self-adjoint by itself, and its adjoint [43, 44], reads:

$$E^+ = E + \left[\frac{\overleftarrow{\partial}}{\partial x}, \tilde{\delta}(x)\right] \qquad (3)$$

Where $\frac{\overleftarrow{\partial}}{\partial x}$ and $\tilde{\delta}(x)$ are defining in the equations (A.8) and (A.9).

By virtue of the above two equations, the ad joint of equation (1) can be expressed as:

$$H^+ = H + \left[\frac{\overleftarrow{\partial}}{\partial x}, U\right] - \sum_{i=x,y,z}\left[\frac{\overleftarrow{\partial}}{\partial i}, \tilde{\delta}(i)K_i\right] \qquad (4)$$

H is not Hermitian (self-adjoint by itself) due to the presence of the four commutators, namely:

$$\left[\frac{\overleftarrow{\partial}}{\partial x}, U\right] = \frac{\overleftarrow{\partial}}{\partial x}U - U\frac{\partial}{\partial x} \qquad (5)$$

and

$$\sum_{i=x,y,z}\left[\frac{\overleftarrow{\partial}}{\partial i}, \tilde{\delta}(i)K_i\right] = \sum_{i=x,y,z}\left(\frac{\overleftarrow{\partial}}{\partial i}\tilde{\delta}(i)K_i - \tilde{\delta}(i)K_i\frac{\partial}{\partial i}\right) \qquad (6)$$

2.2 Reliability of the Atmospheric Diffusion Operator

Alternatively, and in view of equation (4), it is to be noted that, the atmospheric diffusion operator H can be rendered self-adjoint by adding any member of commentators as given in equations (5, 6).

In this respect, let us denote by $\tilde{H}$ the following operator:

$$\tilde{H} = \lambda H + \mu\left(\left[\frac{\overleftarrow{\partial}}{\partial x}, U\right] - \sum_{i=x,y,z}\left[\frac{\overleftarrow{\partial}}{\partial i}, \tilde{\delta}(i)K_i\right]\right) \qquad (7)$$

in which □□and□□ are expansion coefficient, which can be fixed upon evaluating its adjoint; namely:

$$\tilde{H}^{+}=\lambda^{*}H^{+}+\mu^{*}\left(\left[\frac{\overleftarrow{\partial}}{\partial x},U\right]-\sum_{i=x,y,z}\left[\frac{\overleftarrow{\partial}}{\partial i},\tilde{\delta}(i)K_i\right]\right)^{+} \quad (8)$$

By virtue of equation (5) and equation (6) and inserting equation (4) and equation (A.6) in equation (8); we get:

$$\tilde{H}^{+}=\lambda^{*}\left(H+\left[\frac{\overleftarrow{\partial}}{\partial x},U\right]-\sum_{i=x,y,z}\left[\frac{\overleftarrow{\partial}}{\partial i},\tilde{\delta}(i)K_i\right]\right)+\mu^{*}\left(-\left[\frac{\overleftarrow{\partial}}{\partial x},U\right]+\sum_{i=x,y,z}\left[\frac{\overleftarrow{\partial}}{\partial i},\tilde{\delta}(i)K_i\right]\right)$$

$$=\lambda^{*}H+\left(\lambda^{*}-\mu^{*}\right)\left(\left[\frac{\overleftarrow{\partial}}{\partial x},U\right]-\sum_{i=x,y,z}\left[\frac{\overleftarrow{\partial}}{\partial i},\tilde{\delta}(i)K_i\right]\right) \quad (9)$$

Employing the intrinsic self-adjointness:

$$\tilde{H} = H^{+} \quad (10)$$

There from, one sets the relationships:

$$\lambda = \lambda^{*}$$

and (11)

$$\lambda^{*}-\mu^{*}=\mu$$

together with the restrictions

$$|\lambda| = 1 \quad (12)$$

such set of algebraic equations has the solutions;

$$\lambda = 1$$

and (13)

$$\mu = 1/2$$

Consequently, the associate operator as given by equation (4) reads:

$$\tilde{H} = H + \sigma \quad (14)$$

In which σ stands for the surface operator:

$$\sigma=\frac{1}{2}\left(\left[\frac{\overleftarrow{\partial}}{\partial x},U\right]-\sum_{i=x,y,z}\left[\frac{\overleftarrow{\partial}}{\partial i},\tilde{\delta}(i)K_i\right]\right) \quad (15)$$

This surface operator not only ensures intrinsic self-adjointness of the atmospheric diffusion operator, but also is self-consistent, in view of the fact that it does not include the arbitrary boundary condition.

2.3 Model Formulation of Hermitized Diffusion Equation

By virtue of equation (1) together with the two equations (14) and (15), considering the intrinsically self-adjoint diffusion equation in the steady state and the mean wind U is in the direction of x-axis [40]; namely:

$$U\frac{\partial C}{\partial x}=\frac{\partial}{\partial x}\left(K_x\frac{\partial C}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_y\frac{\partial C}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_z\frac{\partial C}{\partial z}\right)+\frac{1}{2}\left(\left[\frac{\overleftarrow{\partial}}{\partial x},U\right]-\sum_{i=x,y,z}\left[\frac{\overleftarrow{\partial}}{\partial i},\tilde{\delta}(i)K_i\right]\right)C(x,y,z)$$

...(16)

Upon, choosing K_s [34] in the above equation as:

$$K_x=\alpha Ux, \qquad K_y=\beta Ux, \qquad K_z=\gamma Ux \quad (17)$$

One gets:

$$U\frac{\partial C}{\partial x}=\alpha U\left(\frac{\partial C}{\partial x}+x\frac{\partial^2 C}{\partial x^2}\right)+\beta Ux\frac{\partial^2 C}{\partial y^2}+\gamma Ux\frac{\partial^2 C}{\partial z^2}+\frac{1}{2}\left\{\left(\frac{\overleftarrow{\partial}}{\partial x}U-U\frac{\partial}{\partial x}\right)-\left(\frac{\overleftarrow{\partial}}{\partial x}\alpha Ux\tilde{\delta}(x)-\alpha Ux\tilde{\delta}(x)\frac{\partial}{\partial x}\right)\right.$$
$$\left.-\left[\frac{\overleftarrow{\partial}}{\partial y},\beta Ux\tilde{\delta}(y)\right]-\left[\frac{\overleftarrow{\partial}}{\partial z},\gamma Ux\tilde{\delta}(z)\right]\right\}C(x,y,z) \quad (18)$$

Expressing $\frac{\overleftarrow{\partial}}{\partial x}$ in terms of $\frac{\partial}{\partial x}$ as given by equation (A.8) and canceling the terms containing $x\tilde{\delta}(x)$ (Messiah 1962) , equation(18) becomes:

$$\alpha x\frac{\partial^2 c}{\partial x^2}+(\alpha-z)\frac{\partial c}{\partial x}+\frac{\tilde{\delta}(x)c}{L}+\beta x\frac{\partial^2 c}{\partial y^2}+\gamma x\frac{\partial^2 c}{\partial z^2}-$$
$$\frac{1}{2}x\left\{\left[\frac{\overleftarrow{\partial}}{\partial y},\beta\check{\delta}(y)\right]+\left[\frac{\overleftarrow{\partial}}{\partial z},\gamma\check{\delta}(y)\right]\right\}c(x,y,z)=0 \quad (19)$$

This partial differential equation under the boundary conditions:

i. Far away from the source, the concentration decreases to zero, i.e.

$$C\to 0 \quad , \quad \text{as} \quad x,|y|,\ z\to\infty \quad (20)$$

Ground surface is assumed impermeable to the pollutants, i.e.

$$\frac{\partial C}{\partial z}=0 \quad \text{at} \quad z=0 \quad (21)$$

Eq. (19) has been solved analytically using the method of integral transforms [29], to obtain:

$$C(x,y,z)=\frac{Q}{U\pi\sqrt{\beta\gamma}\,x^2}\left[1+\frac{\alpha}{x^2}\left(\frac{y^2}{\beta}+\frac{z^2}{\gamma}\right)\right]^{-\left(\frac{1}{\alpha}+1\right)} \quad (22)$$

By virtue of the relation $(1+x)^{1/x}\to e$, as , $x\to 0$, the above equation when $\alpha\to 0$ becomes:

$$C(x,y,z)=\frac{Q}{\pi U\sqrt{\beta x^2}\sqrt{\gamma x^2}}\exp-\left(\left[\frac{y^2}{\beta x^2}+\frac{z^2}{\gamma x^2}\right]\right) \quad (23)$$

which, is similar to Gaussian plume solution with ($\sigma_y^2=\beta x^2,\sigma_z^2=\gamma x^2$); namely:

$$C(x,y,z)=\frac{Q}{\pi U\sigma_y\sigma_z}\exp-\frac{y^2}{2\sigma_y^2}\exp-\frac{z^2}{2\sigma_z^2} \quad (24)$$

2.4 Parameterization

For practical application of solution (22), one needs to specify the turbulence parameters α, β and γ. These parameterized can be identified as squares of turbulence intensities using Taylor's statistical theory of diffusion [1] i.e.

$$\alpha = (\sigma_{u/U})^2;\ \beta = (\sigma_{v/U})^2;\ \gamma = (\sigma_{w/U})^2 \quad (25)$$

When the measurements of intensities of turbulence are available, the turbulence parameters should be computed directly from the above relations. However, in the absence of direct measurements of σ_u, σ_v and σ_w, they can be parameterized through the use of empirical similarity relations for the planetary boundary layer (PBL)[6,45-49,1]

For the convective boundary layer (CBL), mixed layer similarity scaling and empirical turbulence data suggested that $\sigma_u = \sigma_v \approx a w_*$ (a=0.56) [50] and $\sigma_w = b\, w_*$, where w_* is the convective velocity scale. The constant b can take values from 0.4 to 0.6, depending on the dimensionless height z/z_i, where z_i is the convective mixing height. As a good approximation, one can take b≈0.4 for modeling dispersion in the surface layer and b≈0.6 for dispersion in the mixed layer. With the above parameterization of turbulence in the surface layer in convective conditions, equation (25) can be expressed as:

$$\alpha = \beta = 0.31(w_*/U)^2;\quad \gamma = 0.16(w_*/U)^2 \quad (26)$$

The relations (26), for convective conditions, have been used in solution (22) for estimating the diffusion experiments conducted at IIT Delhi for ground-level releases during low wind conditions.

2.5 The Field Tracer Data

The diffusion data chosen for the simulation were collected during SF_6–tracer experiments in low wind and unstable conditions at IIT Delhi sports ground. During each test run, the tracer was released for an hour at a height of about 1 m and the air samples were collected during the latter half of the release period, at a height of about 0.5m. Twenty sampler pumps for collecting air samples were placed on three circular arc of radii 50,100, and 150m with the centre as the release point in most of the cases. The air samples thus collected were later analyzed in the Air Pollution Lab (Dry), CAS, IIT Delhi,

Table1: Relevant experimental details of the convective test runs conducted at IIT Delhi sports ground in February 1991.

Run no.	Sampling time (h)	Wind speed (m/s)	w_* (m/s)	z_i (m)	P-G Stab.
1	1200-1230	1.36	2.37	1570	A-B
2	1530-1600	0.74	2.26	1240	B
6	1000-1030	1.40	2.04	1070	B
7	1245-1315	1.54	2.28	1240	B
8	1645-1715	0.89	1.09	943	B
11	1000-1030	1.07	1.83	1070	A-B
12	1215-1245	1.55	2.32	1325	B
13	1530-1600	1.08	1.72	1070	B

Using electron–capture gas chromatography [42]. Meteorological inputs have been provided by the measurements done at 1, 2, 4, 8, 15, and 30m levels at a 30m micrometeorological tower located about 300m south-east of the release point. Table 1 gives the relevant information about the diffusion tests and the wind vectors. In addition, it includes values of w_* and z_i. The data from these 8 unstable test runs have been utilized for the following analysis.

2.6 Results and Discussion

Solution (22) for a ground-level source is adopted here to calculate concentration. Its usage requires the specification of mean wind speed U, source strength Q and the turbulence parameters α, β and γ. In convective conditions α and β are nearly the same, as $\sigma_u \approx \sigma_v$. This is reflected in equation (26). Therefore, solution (22) simplifies to Sodar measurements done by the National Physical Laboratory at a slightly different location (only a few kilometers away) in Delhi. All these factors could affect the model predictions. The concentrations are computed at z=0.5m which is sampling height.

$$C(x,y,z) = \frac{Q}{U\pi x^2\sqrt{\alpha\gamma}}\left[1+\frac{1}{x^2}\left(y^2+\frac{\alpha}{\gamma}z^2\right)\right]^{-\left(\frac{1}{\alpha}+1\right)} \quad (27)$$

The peak concentrations obtained from solution (27) on 50 and 100m arc are tabulated along with observations in Tables 2 and 3, respectively. The tables reveal that the present model (eq. (27)) gives an under predicting trend on both the arcs. The peak values are under predicted, roughly by a factor of 3 to 10.

Tables 2 and 3 include the results from slender plume approximation (α→0) which is the same as the Gaussian plume formula with σ's based on the above-mentioned similarity scaling. Theoretically the downwind diffusion is important away from the plume centerline [51,52].

Table 2: Peak values of tracer concentration (ppt) observed and predicted by various cases at 50m downwind of the source.

Run no.	Obs-erved	Present Model	Sharan Model (1996)	Similarity Present model	Gaussian Similarity Sharan
1	832	118	123	155	133
2	1068	74	67	85	76
6	1101	85	90	121	97
7	248	73	77	103	81
8	1282	304	333	484	354

11	616	85	90	113	91
12	759	117	125	165	139
13	1060	155	164	212	178
Run no.	Obs-erved	Present Model	Sharan Model (1996)	Similarity Present model	Gaussian Similarity Sharan
1	832	118	123	155	133
2	1068	74	67	85	76
6	1101	85	90	121	97

Table 3: Peak values of tracer concentration (ppt) observed and predicted by various cases at 100m downwind of the source.

Run no.	Obs-erved	Present Model	Sharan Model (1996)	Similarity Present model	Gaussian Similarity Sharan
1	345	37	38	40	41
2	460	20	20	21	19
6	176	23	23	25	24
7	288	18	18	20	20
8	345	83	83	93	88
11	162	24	24	26	23
12	222	31	31	34	35
13	215	40	40	44	44

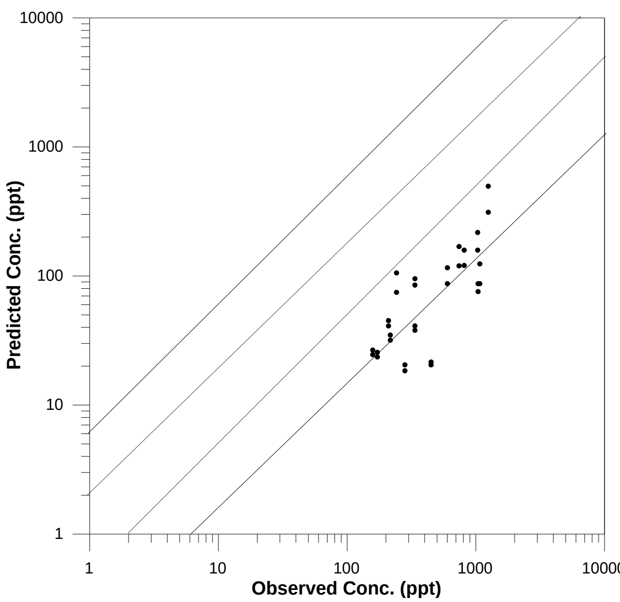

Fig. 1: Scatter diagram of the peak model predictions for convective cases and the corresponding observations. dashed lines indicate a factor of two and solid a factor of six.

The results from the present model for convective cases are shown in Fig. 1. The figure gives a scatter diagram of the peak of the predicted and observed concentrations from all the test runs. It may be seen that although there's a clear under predicting trend, the number of predictions within a factor of 6 are reasonable. A similar trend has been observed for the results from model [1].

3 Second Case

3.1 Model Formulation

The dispersion of pollutants in the atmosphere is governed by the basic atmospheric diffusion equation. Under the assumption of incompressible flow, atmospheric diffusion equation based on the Gradient transport theory can be written in the rectangular coordinate system as:

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}+w\frac{\partial C}{\partial z}=\frac{\partial}{\partial x}\left(k_x\frac{\partial C}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y\frac{\partial C}{\partial y}\right)+\frac{\partial}{\partial z}\left(k_z\frac{\partial C}{\partial z}\right)+S+R \qquad (28)$$

where C is the mean concentration of a pollutant (Bq/m^3); S and R are the source and removal terms, respectively; (u,v,w) and (k_x , k_y, k_w) are the components of wind and diffusivity vectors in x , y and w directions , respectively , in an Eulerian frame of reference.

The following assumptions are made in order to simplify equation (29):

1) Steady -state conditions are considered, i.e. $\partial C/\partial t = 0$
2) As the vertical velocity is much smaller than the horizontal one, the term $w(\partial C/\partial z)$ is neglected.
3) x-axis is oriented in the direction of mean wind (u=U and v=0).
4) Source and removal (physical / chemical) pollutants are ignored so that S=0 and R=0.

With the above assumptions, equation (28) reduces to:

$$u\frac{\partial C}{\partial x}=\frac{\partial}{\partial x}\left(k_x\frac{\partial C}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y\frac{\partial C}{\partial y}\right)+\frac{\partial}{\partial z}\left(k_z\frac{\partial C}{\partial z}\right) \qquad (29)$$

Here U is taken to be function of power law of z:

$$U=\alpha z^{p} \quad z\neq 0 \quad \text{and} \quad U=U_0 \quad \text{at } z=0 \qquad (30)$$

$$k_x=\frac{U}{3\alpha x}\frac{d\sigma_x^2}{dx}, \quad \sigma_x^2=\alpha x^3$$
$$k_y=\frac{U}{3\alpha x}\frac{d\sigma_y^2}{dx}, \quad \sigma_y^2=\beta x^3$$
$$k_z=\frac{z^n}{3x}\frac{d\sigma_z^2}{dx}, \quad \sigma_z^2=\gamma x^3 \qquad (31)$$

or simply:

$$K_x=xU, \quad K_y=\frac{\beta}{\alpha}xU, \quad K_z=\gamma x z^{n}$$

Where α, β, and γ represent turbulence parameters and depend on atmospheric stability. Using above parameterization of diffusivities, equation (29) becomes:

$$\gamma z^n \frac{\partial^2 C}{\partial z^3} + n\gamma z^{n-1}\frac{\partial C}{\partial z} + \alpha z^p \frac{\partial^2 C}{\partial x^2} + \beta z^p \frac{\partial^2 C}{\partial y^2} = 0 \qquad (32)$$

We are going to solve the above equation together with the following boundary conditions:

A continuous point source with strength Q is assumed to be located at the point (0,0,0), i.e.

$$UC = Q\,\delta(x)\delta(y) \quad \text{as} \quad z=0 \qquad (33)$$

Where $\delta(\ldots)$ is Dirac's delta function.

- Far away from the source, the concentration decreases to zero, i.e.

$$C \rightarrow 0 \ \text{as} \quad x.y, z \rightarrow \infty \qquad (34)$$

One can solve the power three-dimensional partial differential equation (32) analytically by using the method of integral transformation, this can be accomplished upon using cosine transform in x and cosine transform in y, to obtain a partial differential equation in one-dimensional only z, namely:

$$\gamma z^n \frac{\partial^2 C_*}{\partial z^2} + n\gamma z^{n-1}\frac{\partial C_*}{\partial z} - \beta\lambda_2^2 z^p C_* - \alpha z^p \lambda_1^2 C_* = 0 \quad (35)$$

this, is simply reads:

$$z^2 \frac{\partial^2 C_*}{\partial z^2} + nz\frac{\partial C_*}{\partial z} - b^2 a^2 z^{2a} C_*(\lambda_1,\lambda_2,z) = 0 \quad (36)$$

where, $C_* = C_*$ (λ_1, λ_2, z) is transformed variable and is related to C(x,y,z) as:

$$C_*(\lambda_1, \lambda_2, z) = \int_0^\infty\int_0^\infty C(x,y,z)\cos\lambda_1 x \ \cos\lambda_2 y \ dx \ dy \quad (37)$$

and a, b is given by:

$$a = \frac{p-n+2}{2} \qquad (38)$$

where p and n are constants which takes values 0.15-0.28 and 0.5-0.9 respectively for the smooth boundary layer [22].

$$b = \left[\frac{\alpha\lambda_1^2}{\gamma a^2} + \frac{\beta\lambda_2^2}{\gamma a^2}\right]^{\frac{1}{2}} \qquad (39)$$

Equation (37) can be reducible to Bessel's equation [53], namely:

$$z_*^2 \frac{\partial^2 C_{**}}{\partial z_*^2} + z_*\frac{\partial C_{**}}{\partial z_*} - \left[\left(bz_*\right)^2 + \nu^2\right]C_{**} = 0 \qquad (40)$$

On changing the dependent (C_*) and independent (z) variables by means of the substitutes:

$$\left.\begin{aligned} &\text{And} \quad C_* = C_{**}\, z^{\frac{1-n}{2}} \\ &z_* = z^a \end{aligned}\right\} \qquad (41)$$

in which ν is given by:

$$\nu = \frac{1-n}{2a} \qquad (42)$$

Equation (40) is a modified Bessel equation and has a solutions a modified Bessel function $I_\nu\left(bz_*\right)$ of first kind and $K_\nu\left(bz_*\right)$ of second kind of order $\nu = (1-n)/2a$.

The general solution of equation (36) is thus:

$$C_*(\lambda_1,\lambda_2,z) = C_{**}\, z^{\frac{1-n}{2}} = \left[AI_\nu\left(bz^a\right) + Bk_\nu\left(bz^a\right)\right]z^{a\nu} \quad (43)$$

where A and B are arbitrary constants of integration, and can be determent upon using the boundary conditions (34), and boundary condition (33) which imply respectively that:

A=0

and

$$B = \frac{Q\ 2^{1-\nu}}{U_0\ \pi}\Gamma\left(1-\nu\right)\sin\nu\pi \ \ b^\nu$$

Consequently; equation (44) can be expressed explicitly as:

$$C_*(\lambda_1,\lambda_2,z) = \Lambda b^\nu k_\nu\left(bz^a\right) \qquad (44)$$

Under the conditions:

$$\nu > 0 \quad , \quad n < 1 \quad \& \quad p > n-2$$

where, Λ is a function in z namely:

$$\Lambda = \frac{Q\ 2^{1-\nu}}{U_0\ \pi}\Gamma\left(1-\nu\right)\sin\nu\pi \ \ z^{a\nu} \qquad (45)$$

Consequently; Eq. (44) can be expressed explicitly as:

$$C_*(\lambda_1,\lambda_2,z) = \Lambda b^\nu k_\nu\left(bz^a\right) \qquad (46)$$

Under the conditions:

$$\nu > 0 \quad , \quad n < 1 \quad \& \quad p > n-2$$

where, Λ is a function in z namely:

$$\Lambda = \frac{Q\ 2^{1-\nu}}{U_0\ \pi}\Gamma\left(1-\nu\right)\sin\nu\pi \ \ z^{a\nu} \qquad (47)$$

Now, inverting Eq. (45) with respect to the parameters λ_1 and λ_2, one gets:

$$C(x,y,z) = \frac{2^2}{\pi^2}\Lambda\int_0^\infty\int_0^\infty b^\nu K_\nu\left(bz^a\right)\cos\lambda_1 x \ \cos\lambda_2 y \ d\lambda_1\, d\lambda_2 \quad (48)$$

Upon, introducing in the above equation, the explicit expression of b as given in equation (40) which is a function in λ_1 and λ_2, one can evaluate the double integral in equation (47), where the integration on λ_1 is given by [54]:

$$\int_0^\infty b^\nu K_\nu\left(bz^a\right)\cos\lambda_1 x \ d\lambda_1 = \Lambda_*\lambda_2^{\nu+\frac{1}{2}}K_{\nu+\frac{1}{2}}\left[\lambda_2\Lambda_{xz}\right] \quad (49)$$

where, Λ_* and Λ_{xz} are functions in x and z and independent of λ_2, namely:

$$\Lambda_* = a\left(\frac{\beta}{\gamma a^2}\right)^{\nu+\frac{1}{2}} \sqrt{\frac{\gamma}{\alpha}}\ \Lambda_{xz}^{-\left(\nu+\frac{1}{2}\right)}\quad z^{a\nu}$$

and

$$\Lambda_{xz} = \sqrt{x_* + z_*} \qquad (50)$$

in which

$$x_* = \frac{\beta}{\alpha}x^2 \quad \& \quad z_* = \frac{\beta}{a^2\gamma}z^{2a}$$

Upon, inserting equation (49) in equation (48), one gets:

$$C(x,y,z) = \frac{4}{\pi^2}\Lambda\Lambda_*\int_0^\infty \lambda_2^{\nu+\frac{1}{2}} K_{\nu+\frac{1}{2}}\left(\lambda_2\Lambda_{xz}\right)\cos\lambda_2 y\, d\lambda_2 \qquad (51)$$

The integration in Eq. (49) can be evaluated from [54] to give:

$$C(x,y,z) = \frac{4}{\pi^2}\Lambda\Lambda_*\left[\sqrt{\pi}\, 2^{\nu-\frac{1}{2}}\Lambda_{xz}^{\nu+\frac{1}{2}}\Gamma(\nu+1)\left(y^2+\Lambda_{xz}^2\right)^{-\nu-1}\right] \qquad (52)$$

which, upon employing the value of Λ_* as given in Eq. (48), the mean concentration of a pollutant C(x,y,z), is finally expressed explicitly as:

C(x,y,z)=

$$\frac{Q\ 2^{5/2}}{U_0\ \pi^{5/2}}\Gamma(1\text{-}\nu)\Gamma(1+\nu)\sin\nu\pi\ \ a^{-2\nu}\gamma^{-\nu}\alpha^{-1/2}\beta^{\nu+1/2}z^{2a\nu}\left(\frac{\beta}{\alpha}x^2+y^2+\frac{\beta}{\gamma a^2}z^{2a}\right)^{-\nu-1}$$

-----------(53)

The solution given by Eq. (33) can be rewritten as:

C(x,y,z)=

$$\frac{Q\ 2^{5/2}}{U_0\ \pi^{3/2}}\nu\left(\frac{\beta}{\alpha}\right)^{1/2}\left(\frac{\beta}{\gamma a^2}\right)^{-1} z^{-2a}\left(\left[1+\frac{\gamma a^2}{\beta z^{2a}}\left(\frac{\beta}{\alpha}x^2+y^2\right)\right]^{1/\left(\frac{\gamma a^2}{\beta z^{2a}}\left(\frac{\beta}{\alpha}x^2+y^2\right)\right)}\right)^{-(\nu+1)\frac{\gamma a^2}{\beta z^{2a}}\left(\frac{\beta}{\alpha}x^2+y^2\right)}$$

-----------(54)

If K_x, K_y, K_z are functions in xU(z) (i.e. p=n), the mean concentration of a pollutant C(x,y,z), reads:

$$C(x,y,z) = \frac{Q\ 2^{5/2}}{U_0\ \pi^{3/2}}\nu\left(\frac{\beta}{\alpha}\right)^{1/2}\left(\frac{\beta}{\gamma}\right)^{-1} z^{-2}e^{-(\nu+1)\frac{\gamma}{\beta z^2}\left(\frac{\beta}{\alpha}x^2+y^2\right)} \qquad (55)$$

Assuming that at the ground surface the wind velocity U_o equals 2.0m/s. While α=3, p=0.12-0.28 and n=0.5-0.9. [22]. The turbulence parameters have been obtained [29] as follows:

$$\beta = 0.31(w_*/U)^2 \quad \textit{and} \quad \gamma = 0.16(w_*/U)^2 \qquad (56)$$

where w_* is the convective scale vertical velocity.

The relation (56) for convective conditions has been used in solution (55) and (54) for estimating the diffusion of nine experiments worked at Inshas, Cairo for elevated releases during moderate wind condition.

3.2 Source

Source in EAEA. The study area is flat, dominated by sandy soil with poor vegetation cover. The air samples collected were analyzed in Radiation Protection Department, NRC, EAEA, using a high-volume air sampler with 220V /50Hz bias [14]. Meteorological data have been provided by the measurements done at 10 and 60 m. Table 4. gives the data information about the diffusion tests and the wind vectors. In addition, it contains values of vertical velocity scale (w*) and mixing height (zi). The data from these nine unstable test runs have been utilized for the following analysis.

Table 4: Meteorological data of the nine convective test runs at Inshas site in March and May 2006.

Run no.	Working Hours	Wind speed	W* (ms-1)	Emission rate Bq	zi (m)	P-G Stab.
1	48	4	2.27	1028571	600.85	A
2	49	4	3.05	1050000	801.13	A
3	1.5	6	1.61	42857.14	973	B
4	22	4	1.23	471428.6	888	C
5	23	4	0.958	492857.1	921	A
6	24	4	1.3	514285.7	443	D
7	28	4	1.51	1007143	1271	C
8	48.7	4	1.64	1043571	1842	C
9	48.45	4	2.1	1033929	1642	A

3.3 Model Parameters

For the concentration computations, we require the knowledge of wind speed, wind direction, source strength, the dispersion parameters, mixing height and the vertical scale velocity. Wind speeds are greater than 3m/s most of the time even at 10m level. Further the variation wind direction with time is also visible. Thus in the present study, we have adopted dispersion parameters for urban terrain which are based on power law functions. The analytical expressions depend upon downwind distance, vertical distance and atmospheric stability. The atmospheric stability has been calculated from Monin-Obukhov length scale (1/L) [55] based on friction velocity, temperature, and surface heat flux.

3.4 Results and Discussion

The concentration is computed using data collected at vertical distance of a 30m multi-level micrometeoro-logical tower. In all a test runs were conducted for the purpose of computation. The concentration at a receptor can be computed in the following two ways:

(i) Applying formula (54) which contains eddy diffusivities as function with power law at y=0.0 for half hourly averaging.

(ii) Applying formula (55) with eddy diffusivities are function in linear downwind distance at y=0.0 for half hourly averaging.

Table 5: Observed and predicted concentrations for Run 9 experiments.

Test	Down wind distance (m)	Z (m)l	Obs. Conc. (Bq /m^3)	Predicted Conc. Eq.(54) (Bq/m^3)	Pred. Conc. Eq.(55) (Bq/m^3)
1	100	5	0.025	0.020	0.030
2	98	10	0.037	0.039	0.135
3	115	5	0.091	0.078	0.253
4	135	5	0.197	0.121	0.499
5	99	2	0.272	0.292	0.011
6	184	11	0.188	0.124	0.134
7	165	12	0.447	0.280	0.105
8	134	7.5	0.123	0.121	0.151
9	96	5.0	0.032	0.031	0.109

As an illustration, results computed from these approaches are shown in Table 5, for nine typical test. This Table shows that the observed and predicted concentrations for I135 using Eq. (54) with power law of the wind speed and eddy diffusivities are nearer to each other. While the predicted concentrations are estimated from Eq. (55) since the wind speed is function in power law of "z" are greater three times than the observed concentrations of I135.

Fig. 2 shows the variation of predicted and observed concentration of I135 with the downwind distance. One gets good agreement between observed and predicted concentration which is estimated from Eq. (55), while the concentration using Eq. (56) are a factor of four with the corresponding observations. Fig. 3 shows that the predicted concentrations which are estimated from Eq. (54) are a factor of two with the observed concentration while the predicted concentrations which is estimated from Eq. (55) are in factor of four with the corresponding observations.

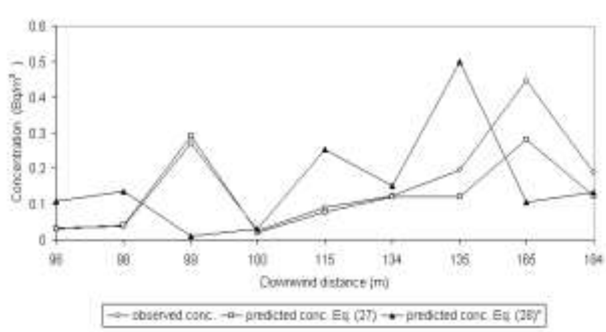

Fig. 2: Maximum computed concentrations compared with observed maximum value for each test run.

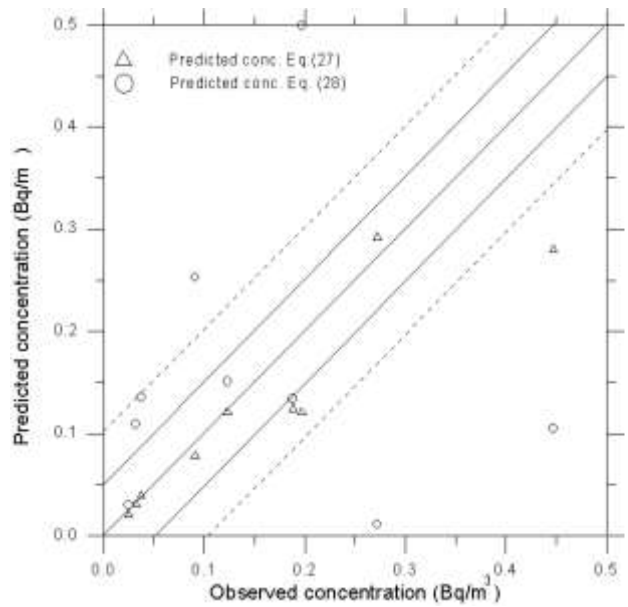

Fig. 3: Diagram of Predicted model for two Eqns. (54) and (55) with corresponding observation. Solid lines indicate a factor of two and dashed lines a factor of three.

From this study, one can get when the wind speed is function in power law of "z" and the eddy diffusivities are linear function and power law in downwind and vertical distances respectively, the predicted concentrations are in a best agreement with the corresponding observation than the predicted concentrations from Eq. (55) since the wind speed is function in power law of the vertical distance and eddy diffusivities are function in linear downwind distance.

4 Conclusions

An analytical solution of the mathematical model for hermitized atmospheric dispersion of a pollutant has been obtained for the steady-state form of advection-diffusion equation with linearly varying eddy diffusivities.

The slender plume approximation which gives concentration close to the plume centerline is shown to be analogous to the Gaussian plume solution.

The turbulence parameters in the model have been identified as squares of intensities of turbulence. They have been parameterized in terms of empirical relations using similarity theory.

Using the solution of equation (22) for a ground-level source, IIT-SF_6 convective diffusion tests have been simulated.

The present model simulations are found to be low relative to the observations and the Gaussian plume simulations.

The simulations are close to those based on Sharan's and Arya's models and Gaussian approach using parameterization in terms of convective velocity.

The solution described in this study has a practical limitation that it doesn't give the concentration field in the region upstream of the source, although the upstream diffusion may be expected near the source under low wind convective conditions [1].

From second study, one can get when the wind speed is function in power law of "z" and the eddy diffusivities are linear function and power law in downwind and vertical distances, the predicted concentrations are in a best agreement with the corresponding observation than the predicted concentrations from Eq. (55) with the eddy diffusivities are linear function in downwind since the wind speed is function in power law of the vertical distance.

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