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Some Contractive Mapping Theorems in Partially Ordered Metric Spaces and Application to Integral Equation

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Abstract: The purpose of this paper is to establish some coincidence point results for nonlinear contractive mappings with monotone property in a complete partially ordered metric space. Also some consequences of the results in terms of integral contractions of the mappings and two examples to support the findings are presented. Furthermore, we provided an application of the result to acquire the unique solution of an integral equation.

Keywords: Partially ordered metric spaces; rational contractions; compatible and weakly compatible mappings, coincidence point.

1 Introduction

In fixed point theory and approximation theory, the celebrated Banach contraction principle [1] plays a crucial role in acquiring the unique solution of many existing results. The contraction is one of the main tool to prove the existence and uniqueness of a fixed point in metric fixed point theory. It is most popular and powerful tool in finding solutions of many problems in nonlinear analysis and scientific applications. Most of the existed fixed point theorems of the mappings in metric spaces generalized and extended the underlying contraction condition in different ways. Several authors have contributed their work either by more general contractive conditions or by implementing some additional conditions on ambient spaces, some of which are in [2–11].

Numerous generalizations of usual metric space have been done on obtaining fixed point results namely rectangular metric spaces, pseudo metric spaces, fuzzy metric spaces, quasi metric spaces, probabilistic metric spaces, D-metric spaces, F-metric spaces, cone metric spaces, etc. First, the existence of fixed point for a mapping in partially sets was investigated by Ran and Reurings [28] in 2004 and applied their results to matrix equations. Later, Nieto et al. [25, 26] extended the results of [28] and provided some applications to ordinary differential equations. While notable work on the existence and uniqueness of a fixed point, coincidence point and common fixed point results of mappings in partially ordered metric spaces, cone metric spaces, rectangular metric spaces etc. with different topological properties are presented some in [12–35], which create natural interest to establish usable fixed point theorems.

In this paper, we prove the results of coincidence point and common fixed point for two self mappings satisfying a generalized rational contractive condition in ordered metric space. Few examples are illustrated to support our results and some consequences of the main result involving integral contractions are given. Further, an application to integral equation for finding unique solution is discussed. These results generalize and extend the result of Sharma and Yuel [9] in partially ordered metric space and the results from [20, 21, 25, 26, 32].

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2 Preliminaries

We use the following definitions frequently in our study.

Definition 21 *[31] The triple $(S,d,\preceq)$ is called a partially ordered metric space, if $(S,\preceq)$ is a partially ordered set together with (S,d) is a metric space.*

Definition 22 *[31] If (S,d) is a complete metric space, then triple $(S,d,\preceq)$ is called complete partially ordered metric space.*

Definition 23 *[29] Let $(S,\preceq)$ be a partially ordered set. A mapping $h: S \to S$ is said to be strictly increasing (strictly decreasing), if $h(x) \prec h(y)$ $(h(x) \succ h(y))$ for all $x,y \in S$ with $x \prec y$.*

Definition 24 *[31] A point $x \in A$, where A is a non-empty subset of a partially ordered set $(S,\preceq)$ is called a common fixed (coincidence) point of two self-mappings h and P, if $hx = Px = x$ $(hx = Px)$.*

Definition 25 *[30] The two self-mappings h and P defined over a subset A of a partially ordered set S are called commuting, if $hPx = Phx$ for all $x \in A$.*

Definition 26 *[30] Two self-mappings h and P defined over $A \subset S$ are compatible, if for any sequence $\{x_n\}$ with $\lim_{n\to+\infty} hx_n = \lim_{n\to+\infty} Px_n = \mu$ for some $\mu \in A$, then $\lim_{n\to+\infty} d(Phx_n, hPx_n) = 0$.*

Definition 27 *[31] Two self-mappings h and P defined over $A \subset S$ are said to be weakly compatible, if they commute only at their coincidence points (i.e., if $hx = Px$ then $hPx = Phx$).*

Definition 28 *[31] Let h and P be two self-mappings defined over a partially ordered set $(S,\preceq)$. A mapping P is called monotone h-nondecreasing, if*

$$hx \preceq hy \text{ implies } Px \preceq Py, \text{ for all } x,y \in X.$$

Definition 29 *[29] Let A be a non-empty subset of a partially ordered set $(S,\preceq)$. If every two elements of A are comparable then it is called well ordered set.*

3 Main Results

This section starts with the following coincidence point theorem.

Theorem 31 *Let $(S,d,\preceq)$ be a complete partially ordered metric space. Suppose the mappings $h,P: S \to S$ are continuous, P is a monotone h-nondecreasing, and $P(S) \subseteq h(S)$ satisfies*

$$\begin{aligned} d(Px,Py) \leq {} & \alpha \frac{d(hx,Px)\ [1+d(hy,Py)]}{1+d(hx,hy)} \\ & + \beta\,[d(hx,Px)+d(hy,Py)] \\ & + \gamma[d(hx,Py)+d(hy,Px)] + \delta d(hx,hy), \end{aligned} \quad (1)$$

for all x, y in S for which $h(x) \neq h(y)$ are comparable and there exist $\alpha,\beta,\gamma,\delta \in [0,1)$ such that $0 \leq \alpha + 2(\beta+\gamma) + \delta < 1$. If there exists $x_0 \in S$ such that $hx_0 \preceq Px_0$ and the mappings h and P are compatible, then h and P have a coincidence point in S.

Proof. Suppose there exists a point $x_0 \in S$ such that $hx_0 \preceq Px_0$. Form the hypotheses, choose a point $x_1 \in S$ such that $hx_1 = Px_0$. As $Px_1 \in h(S)$, then there exists a point $x_2 \in S$ such that $hx_2 = Px_1$. Thus, repeating the same process, we obtain a sequence $\{x_n\}$ in S such that $hx_{n+1} = Px_n$ for all $n \geq 0$.

Also, from the hypotheses we obtain that $hx_0 \preceq Px_0 = hx_1$ and then the monotone property of P implies that $Px_0 \preceq Px_1$. As from the above similar argument, we get

$$Px_0 \preceq Px_1 \preceq \ldots \preceq Px_n \preceq Px_{n+1} \preceq \ldots .$$

Now, we distinguish the following two cases:
Case:1 If for some n, $d(Px_n,Px_{n+1}) = 0$ then $Px_{n+1} = Px_n$. Thus, $Px_{n+1} = Px_n = hx_{n+1}$. Therefore, x_{n+1} is a coincidence point of P and h.
Case:2 If $d(Px_n,Px_{n+1}) \neq 0$ for all $n \in \mathbb{N}$, then from (1), we have

$$\begin{aligned} d(Px_{n+1},Px_n) \leq {} & \alpha \frac{d(hx_{n+1},Px_{n+1})\,[1+d(hx_n,Px_n)]}{1+d(hx_{n+1},hx_n)} \\ & + \beta\,[d(hx_{n+1},Px_{n+1})+d(hx_n,Px_n)] \\ & + \gamma[d(hx_{n+1},Px_n)+d(hx_n,Px_{n+1})] \\ & + \delta d(hx_{n+1},hx_n), \end{aligned}$$

this implies that

$$\begin{aligned} d(Px_{n+1},Px_n) \leq {} & \alpha d(Px_n,Px_{n+1}) \\ & + \beta\,[d(Px_n,Px_{n+1})+d(Px_{n-1},Px_n)] \\ & + \gamma[d(Px_n,Px_n)+d(Px_{n-1},Px_{n+1})] \\ & + \delta d(Px_n,Px_{n-1}). \end{aligned}$$

Therefore,

$$d(Px_{n+1},Px_n) \leq \left(\frac{\beta+\gamma+\delta}{1-\alpha-\beta-\gamma}\right) d(Px_n,Px_{n-1}).$$

Inductively, we obtain that

$$d(Px_{n+1},Px_n) \leq \left(\frac{\beta+\gamma+\delta}{1-\alpha-\beta-\gamma}\right)^n d(Px_1,Px_0).$$

Let $k = \frac{\beta+\gamma+\delta}{1-\alpha-\beta-\gamma} < 1$ and from triangular inequality for $m \geq n$, we have

$$\begin{aligned} d(Px_m,Px_n) & \leq d(Px_m,Px_{m-1}) + d(Px_{m-1},Px_{m-2}) + \ldots.. \\ & \quad \ldots\ldots + d(Px_{n+1},Px_n) \\ & \leq \left(k^{m-1}+k^{m-2}+\ldots+k^n\right) d(Px_1,Px_0) \\ & \leq \frac{k^n}{1-k} d(Px_1,Px_0), \end{aligned}$$

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as $m,n \to +\infty$, $d(Px_m,Px_n) \to 0$, this implies that $\{Px_n\}$ is a Cauchy sequence in S. Since S is complete then there exists some $\mu \in S$ such that $\lim_{n\to+\infty} Px_n = \mu$.

Further, the continuity of P implies that

$$\lim_{n\to+\infty} P(Px_n) = P\left(\lim_{n\to+\infty} Px_n\right) = P\mu.$$

Since, $hx_{n+1} = Px_n$ then $\lim_{n\to+\infty} hx_{n+1} = \mu$.

Further, from the compatibility of a pair of mappings (P,h), we have

$$\lim_{n\to+\infty} d(Phx_n,hPx_n) = 0.$$

Moreover, the triangular inequality of d, we have

$$d(P\mu,h\mu) = d(P\mu,Phx_n) + d(Phx_n,hPx_n) + d(hPx_n,h\mu).$$

On taking $n \to +\infty$ in the above inequality and from the continuity of P, h, we obtain that $d(P\mu,h\mu) = 0$, which implies that $P\mu = h\mu$. Hence, μ is a coincidence point of P and h in S .

We have the following corollaries from Theorem 31.

Corollary 32 *Let $(S,d,\preceq)$ be a complete partially ordered metric space. The mappings $h,P : S \to S$ are continuous, P is monotone h-nondecreasing, $P(S) \subseteq h(S)$ satisfies*

$$d(Px,Py) \le \alpha \frac{d(hx,Px)\,[1+d(hy,Py)]}{1+d(hx,hy)} + \beta\,[d(hx,Px)+d(hy,Py)] + \delta d(hx,hy),$$

for all x, y in S for which $h(x) \ne h(y)$ are comparable and where $\alpha,\beta,\delta \in [0,1)$ such that $0 \le \alpha+2\beta+\delta < 1$. If there exists $x_0 \in S$ such that $hx_0 \preceq Px_0$ and the mappings h and P are compatible, then h and P have a coincidence point in S.

Proof. The required proof can be obtained by setting $\gamma = 0$ in Theorem 31.

Corollary 33 *Let $(S,d,\preceq)$ be a complete partially ordered metric space. Suppose the mappings $h,P : S \to S$ are continuous, P is monotone h-nondecreasing, $P(S) \subseteq h(S)$ satisfies*

$$d(Px,Py) \le \alpha \frac{d(hx,Px)\,[1+d(hy,Py)]}{1+d(hx,hy)} + \gamma[d(hx,Py)+d(hy,Px)] + \delta d(hx,hy),$$

for all $x,y \in S$ for which $h(x) \ne h(y)$ are comparable and there exist $\alpha,\gamma,\delta \in [0,1)$ with $0 \le \alpha+2\gamma+\delta < 1$. If there exists $x_0 \in S$ such that $hx_0 \preceq Px_0$ and the mappings h and P are compatible, then h and P have a coincidence point in S.

Proof. The proof follows from Theorem 31 by setting $\beta = 0$.

Corollary 34 *Let $(S,d,\preceq)$ be a complete partially ordered metric space. Suppose that $P : S \to S$ be a mapping such that for all comparable $x,y \in S$, the contraction condition in Theorem 31 (or Corollaries 32 and 33) is satisfied.*

Assume that P satisfies the following hypotheses:

(i).P is continuous,
(ii).$P(Px) \preceq Px$ for all $x \in S$.

If there exists a point $x_0 \in S$ such that $x_0 \preceq Px_0$, then P has a fixed point in S.

Proof. Follow from Theorem 31 by taking $h = I_S$ (the identity map).

In Theorem 31, the continuity criteria of P is not necessary to obtain a fixed point in the space. If S satisfies the following condition then P has a fixed point.

for any nondecreasing sequence $\{x_n\} \subset S$ such that $x_n \to x$ then $x_n \preceq x$ for $n \in \mathbb{N}$. (2)

Theorem 35 *In Theorem 31, assume that S satisfies condition (2). If h(S) is a complete subset of S, then P and h have a coincidence point in S. Further, if P and h are weakly compatible, then P and h have a common fixed point in S. Moreover, the set of common fixed points of P and h are well ordered if and only if P and h have one and only one common fixed point in S.*

Proof. Assume that $h(S)$ is a complete subset of S. From the proof of Theorem 31, we have a Cauchy sequence $\{hx_n\} \subset h(S)$ and for some $hu \in h(S)$ such that

$$\lim_{n\to+\infty} Px_n = \lim_{n\to+\infty} hx_n = hu.$$

Notice that the sequences $\{Px_n\}$ and $\{hx_n\}$ are nondecreasing from which we get $Px_n \preceq hu$ and $hx_n \preceq hu$ and, also the monotone property of P implies that $Px_n \preceq Pu$ for all n. Hence, by limiting case of it, we obtain that $hu \preceq Tu$.

Suppose that $hu \prec Pu$. Construct a sequence $\{u_n\} \subset S$ by $u_0 = u$ and $hu_{n+1} = Pu_n$ for all $n \in \mathbb{N}$. From the proof of Theorem 31, the sequence $\{hu_n\}$ is nondecreasing and Cauch sequence such that $\lim_{n\to+\infty} h(u_n) = \lim_{n\to+\infty} Pu_n = hv$ for some $v \in S$. Thus from the hypotheses, we have $\sup hu_n \preceq hv$ and $\sup Pu_n \preceq hv$ for all $n \in \mathbb{N}$.

Therefore,

$$hx_n \preceq hu \preceq hu_1 \preceq \ldots \preceq hu_n \preceq \ldots \le hv.$$

Now, we have the following cases:
Case:1 If $hx_{n_0} = hu_{n_0}$ for some $n_0 \ge 1$ then

$$hx_{n_0} = hu = hu_{n_0} = hu_1 = Pu.$$

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Thus, u is a coincidence point of P and h.
Case:2 For all $n \in \mathbb{N}$, $hx_{n_0} \neq hu_{n_0}$ then from (1), we have

$$\begin{aligned} d(hx_{n+1}, hu_{n+1}) &= d(Px_n, Pu_n) \\ &\leq \alpha \frac{d(hx_n, Px_n)\,[1+d(hu_n, Pu_n)]}{1+d(hx_n, hu_n)} \\ &\quad + \beta\,[d(hx_n, Px_n)+d(hu_n, Pu_n)] \\ &\quad + \gamma[d(hx_n, Pu_n)+d(hu_n, Px_n)] \\ &\quad + \delta d(hx_n, hu_n). \end{aligned}$$

Taking limit as $n \to +\infty$ in the above inequality, we obtain that

$$\begin{aligned} d(hu, hv) &\leq (2\gamma+\delta) d(hu, hv) \\ &< d(hu, hv), \text{ since } 2\gamma+\delta < 1. \end{aligned}$$

Therefore,

$$hu = hv = hu_1 = Pu,$$

which shows that u is a coincidence point of P and h.

Now, assume that P and h are weakly compatible and let w be a coincidence point. Then we have

$$Pw = Phz = hPz = hw, \text{ since } w = Pz = hz, \text{ for some } z \in S.$$

From (1), we have

$$\begin{aligned} d(Pz, Pw) &\leq \alpha \frac{d(hz, Pz)\,[1+d(hw, Pw)]}{1+d(hz, hw)} \\ &\quad + \beta\,[d(hz, Pz)+d(hw, Pw)] \\ &\quad + \gamma[d(hz, Pw)+d(hw, Pz)] + \delta d(hz, hw) \\ &\leq (2\gamma+\delta) d(Pz, Pw). \end{aligned}$$

As $2\gamma+\delta < 1$, we get $d(Pz, Pw) = 0$ this implies that $Pz = Pw = hw = w$. Therefore, w is a common fixed point of P and h.

Lastly, assume that the set of common fixed points of P and h is well ordered. Next, to show that the common fixed point of P and h is unique. Let $u \neq v$ be two common fixed points of P and h. Then from (1), we have

$$\begin{aligned} d(u, v) &\leq \alpha \frac{d(hu, Pu)\,[1+d(hv, Pv)]}{1+d(hu, hv)} \\ &\quad + \beta\,[d(hu, Pu)+d(hv, Pv)] \\ &\quad + \gamma[d(hu, Pv)+d(hv, Pu)] + \delta d(hu, hv) \\ &\leq (2\gamma+\delta)\; d(u, v) \\ &< d(u, v), \text{ since } 2\gamma+\delta < 1, \end{aligned}$$

which is a contradiction. Thus, $u = v$. Conversely, suppose P and h have only one common fixed point then the set of common fixed points of P and h being a singleton is well ordered.

Besides, in Corollary 32 and Corollary 33 by relaxing the continuity criteria on T and satisfying the hypothesis of Theorem 35, one can obtains the coincidence point, common fixed point and its uniqueness of P and h in S.

From Theorem 35, we have the following corollary.

Corollary 36 *Let $(S, d, \preceq)$ be a complete partially ordered metric space. Suppose that $P : S \to S$ be a mapping such that for all comparable $x, y \in S$, the contraction condition (1) is satisfied.*

Suppose that the following hypotheses are satisfied

(i).if $\{x_n\}$ is a nondecreasing sequence in S with respect to $\preceq$ such that $x_n \to x \in S$ as $n \to +\infty$, then $x_n \preceq x$, for all $n \in \mathbb{N}$ and
(ii).$P(Px) \preceq Px$ for all $x \in S$.

If there exists a point $x_0 \in S$ such that $x_0 \preceq Px_0$, then P has a fixed point in S.

Proof. Follow from Theorem 35 by taking $h = I_S$ (the identity map).

Some other consequences of the main result for the self mappings involving in the integral type contractions are as follows:

Let Θ denote the set of all functions $\zeta : [0, +\infty) \to [0, +\infty)$ satisfying the following hypotheses:

(a).each ζ is Lebesgue integrable function on every compact subset of $[0, +\infty)$ and
(b).for any $\varepsilon > 0$, we have $\int_0^\varepsilon \zeta(t)dt > 0$ for $t \in [0, +\infty)$.

Corollary 37 *Let $(S, d, \preceq)$ be a complete partially ordered metric space. Suppose that the mappings $P, h : S \to S$ are continuous, P is a monotone h-nondecreasing, $P(S) \subseteq h(S)$ satisfies*

$$\begin{aligned} \int_0^{d(Px,Py)} \zeta(t)dt &\leq \alpha \int_0^{\frac{d(hx,Px)[1+d(hy,Py)]}{1+d(hx,hy)}} \zeta(t)dt \\ &\quad + \beta \int_0^{d(hx,Px)+d(hy,Py)} \zeta(t)dt \\ &\quad + \gamma \int_0^{d(hx,Py)+d(hy,Px)} \zeta(t)dt \\ &\quad + \delta \int_0^{d(hx,hy)} \zeta(t)dt, \end{aligned} \tag{3}$$

for all x, y in S for which $hx \neq hy$ are comparable, $\zeta \in \Theta$ and where $\alpha, \beta, \gamma, \delta \in [0,1)$ such that $0 \leq \alpha + 2(\beta+\gamma) + \delta < 1$. If there exists a point $x_0 \in S$ such that $hx_0 \preceq Px_0$ and the mappings h and P are compatible, then h and P have a coincidence point in S.

We have the following consequences from Corollary 37 by setting $\gamma = 0$ and $\beta = 0$.

Corollary 38 *Let $(S, d, \preceq)$ be a complete partially ordered metric space. Suppose that the self-mappings h, P on S are continuous, P is a monotone h-nondecreasing, $P(S) \subseteq h(S)$ such that*

$$\begin{aligned} \int_0^{d(Px,Py)} \zeta(t)dt &\leq \alpha \int_0^{\frac{d(hx,Px)[1+d(hy,Py)]}{1+d(hx,hy)}} \zeta(t)dt \\ &\quad + \beta \int_0^{d(hx,Px)+d(hy,Py)} \zeta(t)dt \\ &\quad + \delta \int_0^{d(hx,hy)} \zeta(t)dt, \end{aligned} \tag{4}$$

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for all x, y in S for which $hx \neq hy$ are comparable, $\zeta \in \Theta$ and for some $\alpha,\beta,\delta \in [0,1)$ with $0 \le \alpha+2\beta+\delta < 1$. If there exists a point $x_0 \in S$ such that $hx_0 \preceq Px_0$ and the mappings P and h are compatible, then h and P have a coincidence point in S.

Corollary 39*Let $(S,d,\preceq)$ be a complete partially ordered metric space. Suppose that the self-mappings h,P on S are continuous, P is a monotone h-nondecreasing, $P(S) \subseteq h(S)$ such that*

$$\int_0^{d(Px,Py)} \zeta(t)dt \le \alpha \int_0^{\frac{d(hx,Px)\ d(hy,Py)}{d(hx,hy)}} \zeta(t)dt + \gamma \int_0^{d(hx,Py)+d(hy,Px)} \zeta(t)dt + \delta \int_0^{d(hx,hy)} \zeta(t)dt, \quad (5)$$

for all x, y in S for which $hx \neq hy$ are comparable, $\zeta \in \Theta$ and for some $\alpha,\gamma,\delta \in [0,1)$ with $0 \le \alpha+2\gamma+\delta < 1$. If there exists a point $x_0 \in S$ such that $hx_0 \preceq Px_0$ and the mappings P and h are compatible, then h and P have a coincidence point in S.

Now, we give the examples for the main Theorem 31.

Example 310*Define a metric $d : S\times S \to [0,+\infty)$ by $d(x,y) = |x-y|$, where $S = [0,1]$ with usual order $\le$. Let P and h be two self mappings on S such that $Px = \frac{x^2}{2}$ and $hx = \frac{2x^2}{1+x}$, then P and h have a coincidence point in S.*

*Proof.*Note that (S,d) is a complete metric space and thus, $(S,d,\le)$ be a complete partially ordered metric space with respect to usual order $\le$. Let $x_0 = 0 \in S$ then $hx_0 \le Px_0$ and also note that P and h are continuous, P is monotone h-nondecreasing and $P(S) \subseteq h(S)$.

Now consider the following for any x, y in S with $x < y$,

$$\begin{aligned} d(Px,Py) &= \frac{1}{2}|x^2-y^2| = \frac{1}{2}(x+y)|x-y| \\ &\le \frac{2(x+y+xy)}{(1+x)(1+y)}|x-y| \\ &\le \alpha\frac{2x^2|3-x|\left[(1+y)+y^2|3-y|\right]}{4(1+x)(1+y)+2|x-y|(x+y+xy)} \\ &+\frac{\beta}{2}\frac{x^2(1+y)|x-3|+y^2(1+x)|y-3|}{(1+x)(1+y)} \\ &+\gamma\frac{(1+y)|4x^2-y^2(1+x)|+(1+x)|4y^2-x^2(1+y)|}{2(1+x)(1+y)} \\ &+\delta\frac{2(x+y+xy)}{(1+x)(1+y)}|x-y|, \end{aligned}$$

which implies that

$$\begin{aligned} d(Px,Py) &\le \alpha\frac{\frac{x^2|x-3|}{2(1+x)}\cdot\frac{2(1+y)+y^2|3-y|}{2(1+y)}}{1+\frac{2|x-y|(x+y+xy)}{(1+x)(1+y)}} + \beta\left[\frac{x^2|x-3|}{2(1+x)}+\frac{y^2|y-3|}{2(1+y)}\right] \\ &+\gamma\left[\left|\frac{x^2}{(1+x)}-\frac{y^2}{2}\right|+\left|\frac{2y^2}{(1+y)}-\frac{x^2}{2}\right|\right] \\ &+\delta\frac{2(x+y+xy)}{(1+x)(1+y)}|x-y| \\ &\le \alpha\frac{d(hx,Px)\left[1+d(hy,Py)\right]}{1+d(hx,hy)} + \beta\left[d(hx,Px)+d(hy,Py)\right] \\ &+\gamma[d(hx,Py)+d(hy,Px)]+\delta d(hx,hy). \end{aligned}$$

Then, the contraction condition in Theorem 31 holds by selecting proper values of $\alpha,\beta,\gamma,\delta$ in $[0,1)$ such that $0 \le \alpha+2(\beta+\gamma)+\delta < 1$. Therefore, P and h have a coincidence point $0 \in S$.

Example 311*Define a distance function $d : S\times S \to [0,+\infty)$ by $d(x,y) = |x-y|$, where $S = [0,1]$ with usual order $\le$. Let P and h be two self mappings on S such that $Px = x^3$ and $hx = x^4$, then P and h have two coincidence points 0, 1 in S with $x_0 = \frac{1}{4}$.*

4 Applications

Now our aim is to give an existence theorem for a solution of the following integral equation.

$$\hbar(x) = \int_0^M \mu(x,y,\hbar(y))dy + g(x), \quad x \in [0,M], \quad (6)$$

where $M > 0$. Let $S = C[0,M]$ be the set of all continuous functions defined on $[0,M]$. Now, define $d : S\times S \to \mathbb{R}^+$ by

$$d(u,v) = \sup_{x\in[0,M]} \{|u(x)-v(x)|\}$$

then, $(S,\le)$ is a partially ordered set. Now, we prove the following result.

Theorem 41*Suppose the following hypotheses holds:*

(i).$\mu : [0,M]\times[0,M]\times\mathbb{R}^+ \to \mathbb{R}^+$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous,
(ii).for each $x,y \in [0,M]$, we have

$$\mu\left(x,y,\int_0^M \mu(x,z,\hbar(z))dz+g(x)\right) \le \mu(x,y,\hbar(y)),$$

(iii).there exists a continuous function $N : [0,M]\times[0,M] \to [0,+\infty]$ such that

$$|\mu(x,y,a)-\mu(x,y,b)| \le N(x,y)|a-b| \text{ and}$$

(iv).

$$\sup_{x\in[0,M]} \int_0^M N(x,y)dy \le \gamma$$

for some $\gamma < 1$. Then, the integral equations (6) has a solution $a \in C[0,M]$.

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Proof. Define $P : C[0,M] \to C[0,M]$ by

$$Pw(x) = \int_0^M \mu(x,y,w(x))dx + g(x),\ x \in [0,M].$$

Now, we have

$$\begin{aligned} P(Pw(x)) &= \int_0^M \mu(x,y,Pw(x))dx + g(x) \\ &= \int_0^M \mu(x,y,\int_0^M \mu(x,z,w(z))dz + g(x))dx + g(x) \\ &\leq \int_0^M \mu(x,y,w(z))dz + g(x) \\ &= Pw(x) \end{aligned}$$

Thus, we have $P(Px) \leq Px$ for all $x \in C[0,M]$. For any $x^*, y^* \in C[0,M]$ with $x \leq y$, we have

$$\begin{aligned} d(Px^*, Py^*) &= \sup_{x\in[0,M]} |Px^*(x) - Py^*(y)| \\ &= \sup_{x\in[0,M]} |\int_0^M \mu(x,y,x^*(x)) - \mu(x,y,y^*(x))dx| \\ &\leq \sup_{x\in[0,M]} |\int_0^M \mu(x,y,x^*(x)) - \mu(x,y,y^*(x))|dx \\ &\leq \sup_{x\in[0,M]} |\int_0^M N(x,y)|x^*(x) - y^*(x)|dx \\ &\leq \sup_{x\in[0,M]} |x^*(x) - y^*(x)| \sup_{x\in[0,M]} \int_0^M N(x,y)dx \\ &= d(x^*,y^*) \sup_{x\in[0,M]} \int_0^M N(x,y)dx \\ &\leq \gamma d(x^*,y^*). \end{aligned}$$

Moreover, $\{x_n^*\}$ is a nondecreasing sequence in $C[0,M]$ such that $x_n^* \to x^*$, then $x_n^* \leq x^*$ for all $n \in \mathbb{N}$. Thus all the required hypotheses of Corollary 36 are satisfied. Therefore, there exists a solution $a \in [0,M]$ of the integral equation (6).

5 Conclusions

Some coincidence point and common fixed point results of nonlinear contractive mappings in a complete partially ordered metric space are proved, which generalize and extend the known results in the literature. Further, some consequences of the main result, numerical examples are discussed. Finally, an application to the results is provided towards obtaining a unique solution to the integral equation.

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