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The Investigation of Exact Solutions and Conservation Laws of the Classical Boussinesq System Via the Lie Symmetry Method

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Abstract: In this paper, by applying the lie symmetry method with the aid of Maple, we study the classical Boussinesq (CB) system to investigate some new exact solutions. Using the infinitesimal generators and the linear combinations of the vector fields to convert the system to ordinary differential equations (ODEs) with a new dependent variable. Also, using the infinitesimal generators we can obtain the adjoint table and the commutator table of lie algebra. We employ the generalized tanh-function method to solve the ODEs. Hence, we obtain various travelling wave solutions for the CB system. We can investigate these solutions by using figures. Additionally, conservation laws for the CB system are obtained.

Keywords: Classical Boussinesq system; Lie point symmetry; Infinitesimal generator; Exact solutions.

1 Introduction

One of the most sensational advances of theoretical physics and nonlinear science has been the expansion of methods to seek exact solutions of nonlinear partial differential equations (NLPDEs). The study of the exact solutions has an exciting issue in both experimental and theoretical research, which can describe the nonlinear phenomena of fluid dynamics, solid state physics, fluid mechanics. In latest years, different kinds of exact solutions of NLPDEs have been obtained, such as periodic solutions, soliton solutions and rational solutions. Seeking the exact solutions of NLPDEs has been an intersting topic in physics and mathematics for a long time. Some in effect methods to investigate explicit traveling and solitary wave solutions of nonlinear evolution equations have been suggested, for instance the inverse scattering method [1], the tanh- function method [2], the tanh-coth method [3], the Extended tanh method [4], the Jacobi elliptic function method [5], the F-expansion method [6,7], the modified F-expansion method [8], the homogeneous balance method [9,10], $(\frac{G'}{G})$-expansion method [11,12,13], the lie point symmetry method [14,15,16], the generalized tanh-function method [17,18] and other methods [19,20, 21]. Lie group methods are perhaps the greatest powerful currently available in getting exact solutions of NLPDEs. This method has a deep impact on together pure and applied areas of mechanics, mathematics and physics , etc..

The aim of this paper is applying the lie symmetry method and the generalized tanh-function method to obtain the exact solutions of the CB system, that reported by many authors [22,23,24,25] and given as

$$\begin{aligned} &u_t+\tfrac{1}{4}v_{xxx}+[(1+u)v]_x=0,\\ &v_t+u_x+v\,v_x=0, \end{aligned} \tag{1}$$

where u and v are the elevation and the surface velocity of water wave, respectively. The system given in (1) was presented by Wu and Zhang in (1996), that is derived from Euler equation and they studied the run-up of ocean waves by using this system for instance tsunami waves on dykes and dams.

The plan of this paper arranged as the following: Firstly, we apply the Lie symmetry method of the CB system in section 2. Also, some special symmetry reductions are presented in this section to get exact solutions. In section 3, we construct the solutions of the reduced ODEs which obtained in section 2 via the generalized tanh-function method to get exact solutions of

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(1). Furthermore, in section 4, we get conservation laws for the CB system. Finally, conclusions of the paper are given in the last section.

2 Symmetry Analysis

In this section, firstly, we obtain the Lie point symmetries of (1) and by using them we get exact solutions of (1).

2.1 Lie Point Symmetries for the System

A Lie symmetry of differential equations is a transformations, that maps one solution to other solution. We can get the symmetries of CB system with the following vector field:

$$X=\xi(x,t,u,v)\,\partial_x+\tau(x,t,u,v)\,\partial_t+\eta^1(x,t,u,v)\,\partial_u+\eta^2(x,t,u,v)\,\partial_v. \quad (2)$$

Appling the prolongation $\mathrm{Pr}^{(3)}X$ into (1), we can obtain a system of linear partial differential equations which are named determining equations as follows:

$$\begin{aligned}
&\xi_u=\xi_v=\xi_{tx}=\xi_{xx}=\tau_u=\tau_v=\tau_x=0,\\
&\eta^1=-2(1+u)\,\xi_x,\ \eta^2=v\,\xi_x-v\,\tau_t+\xi_t,\\
&\eta^1_u=-2v\,\tau_x+\eta^2_v-\tau_t+\xi_x,\\
&\eta^1_v=-v^2\,\tau_x-\eta^2-v\,\tau_t+v\,\xi_x+\xi_t,\\
&\eta^1_x=-v\,\eta^2_x-\eta^2_t,\ \eta^2_u=\tau_x,\\
&\eta^2_{tv}=v\,\tau_{tx}-\xi_{tx},\ \eta^2_{vv}=v\,\tau_{vx}-\xi_{vx}+\tau_x,\\
&\eta^2_{vx}=v\,\tau_{xx}-\xi_{xx},\ \xi_u=v\,\tau_u-\tau_v.
\end{aligned} \quad (3)$$

Solving the system by Maple, we get the follwing infinitesimals:

$$\begin{aligned}
&\xi=c_1t+c_2x+c_3,\ \ \tau=2c_2t+c_4,\\
&\eta^1=-2c_2(1+u),\quad \eta^2=c_1-c_2v,
\end{aligned} \quad (4)$$

where c_1, c_2, c_3 and c_4 are constants. Using (2) and (4), we have the infinitesimal generators as

$$X_1=t\,\partial_x+\partial_v,\ \ X_2=x\,\partial_x+2t\,\partial_t-2(1+u)\,\partial_u-v\,\partial_v,\ X_3=\partial_x,\ X_4=\partial_t. \quad (5)$$

According to the commutator operators

$$[X_i,X_j]=X_iX_j-X_jX_i,$$

and the series

$$Ad(exp(\varepsilon X_i))X_j=X_j-\varepsilon\,[X_i,X_j]+\frac{1}{2!}\varepsilon^2[X_i,[X_i,X_j]]+\ldots,$$

we can obtain the commutator table and the adjoint table for the lie algebra (5) in table 1 and table 2, respectively.

2.2 Symmetry Group of the System

In this part, to get the group transformation $G_i:(x,t,u,v)\to(\hat{x},\hat{t},\hat{u},\hat{v})$ that is generated by the generator X_i for $i=1,2,3,4$. We want to solve the initial problems of ODEs that given as

$$\frac{d(\hat{x},\hat{t},\hat{u},\hat{v})}{d\varepsilon}=(\xi,\tau,\eta^1,\eta^2),$$

$$(\hat{x},\hat{t},\hat{u},\hat{v})|_{\varepsilon=0}=(x,t,u,v),$$

where $\xi=\xi(\hat{x},\hat{t},\hat{u},\hat{v}),\ \tau=\tau(\hat{x},\hat{t},\hat{u},\hat{v}),\ \eta^1=\eta^1(\hat{x},\hat{t},\hat{u},\hat{v}),\ \eta^2=\eta^2(\hat{x},\hat{t},\hat{u},\hat{v})$ and ε is a group parameter. Then the one-parameter symmetry groups G_i corresponding to the generators X_i that given in (5) can be obtained as follows:

$$\begin{aligned}
&G_1:(x,t,u,v)\to(x+\varepsilon t,t,u,v+\varepsilon),\\
&G_2:(x,t,u,v)\to(xe^{\varepsilon},te^{2\varepsilon},ue^{-2\varepsilon}+(e^{-2\varepsilon}-1),ve^{-\varepsilon}),\\
&G_3:(x,t,u,v)\to(x+\varepsilon,t,u,v),\\
&G_4:(x,t,u,v)\to(x,t+\varepsilon,u,v).
\end{aligned}$$

We observe that G_1 is a dependent and a space translation, whereas G_2 is a scaling and a translation. The transformations G_3 and G_4 are a space and a time translation, respectively.
Consider $u=U(x,t)$ and $v=V(x,t)$ is a solution of the system in (1), by using the one-parameter symmetry groups $G_i(i=1,2,3,4)$, we have a new solutions

$$\begin{aligned}
&u^{(1)}=U(x-\varepsilon t,t),\ v^{(1)}=V(x-\varepsilon t,t)+\varepsilon\\
&u^{(2)}=e^{-2\varepsilon}U(xe^{-\varepsilon},te^{-2\varepsilon})+(e^{-2\varepsilon}-1),\\
&v^{(2)}=e^{-\varepsilon}V(xe^{-\varepsilon},te^{-2\varepsilon})\\
&u^{(3)}=U(x-\varepsilon,t),\ v^{(3)}=V(x-\varepsilon,t)\\
&u^{(4)}=U(x,t-\varepsilon),\ v^{(4)}=V(x,t-\varepsilon)
\end{aligned}$$

2.3 Similarity Reductions

Here, we will use the infinitesimal generators that we obtained in the preceding section to get the symmetry variables and the symmetry solutions by applying the characteristic equation which is equivalent to solving the invariant surface condition

$$\eta^1(x,t,u,v)+\eta^2(x,t,u,v)-\xi(x,t,u,v)\,u_x-\tau(x,t,u,v)\,u_t=0. \quad (6)$$

Reduction 1: Using the generator $X_1=t\,\partial_x+\partial_v$ with (6), we get the similarity solutions and the similarity variable

$$u=f(r),\ v=\frac{x-g(r)}{t},\ r=t. \quad (7)$$

Putting (7) in (1), we obtain the reduction equations

$$\begin{aligned}
&\tfrac{df}{dt}+\tfrac{1}{t}f=-\tfrac{1}{t},\\
&\tfrac{dg}{dt}=0.
\end{aligned} \quad (8)$$

Table 1: Commutator table for the lie algebra in (5).

	X_1	X_2	X_3	X_4
X_1	0	$-X_1$	0	0
X_2	X_1	0	$-X_3$	$-2X_4$
X_3	0	X_3	0	0
X_4	0	$2X_4$	0	0

Table 2: Adjoint table for the lie algebra in (5).

	X_1	X_2	X_3	X_4
X_1	X_1	$X_2+\varepsilon X_1$	X_3	X_4
X_2	$X_1 e^{-\varepsilon}$	X_2	$X_3 e^{\varepsilon}$	$X_4 e^{2\varepsilon}$
X_3	X_1	$X_2-\varepsilon X_3$	X_3	X_4
X_4	X_1	$X_2-2\varepsilon X_4$	X_3	X_4

Obviously, $f=\frac{\beta_1-t}{t}$ and $g=\beta_2$. Therefore, equation (1) has a similarity solution given as

$$\begin{aligned} u(x,t) &= \tfrac{\beta_1-t}{t},\\ v(x,t) &= \tfrac{x-\beta_2}{t}, \end{aligned} \tag{9}$$

where β_1 and β_2 are constants.

Reduction 2: For the infinitesimal generator $X_4=\partial_t$ and equation (6), we have the similarity solutions and the similarity variable as the following:

$$u=f(r),\ v=g(r),\ r=x. \tag{10}$$

Substituting (10) into (1), we obtain

$$\begin{aligned} & g\,f'+(f+1)\,g'+\tfrac{1}{4}\,g'''=0,\\ & g\,g'+f'=0. \end{aligned} \tag{11}$$

Equation (11) can be written in the form

$$g''-2\,g^3+4\,g=0, \tag{12}$$

with setting the constant of integration equal to zero.

Reduction 3: By the infinitesimal generator $X=\lambda X_3+\mu X_4$ with (6), we get the similarity solutions and the similarity variable

$$u=f(r),\ v=g(r),\ r=\mu\,x-\lambda\,t, \tag{13}$$

where λ and μ are constants. From (13) and (1), we have

$$\begin{aligned} & \tfrac{1}{4}\,\mu^3\,g'''+(\mu\,g-\lambda)\,f'+\mu\,(1+f)\,g'=0,\\ & \mu\,f'+(\mu\,g-\lambda)g'=0. \end{aligned} \tag{14}$$

We can write equation (14) as

$$\mu^4\,g''-2\,\mu^2\,g^3+6\,\lambda\,\mu\,g^2+4\,(\mu^2-\lambda^2)\,g=0. \tag{15}$$

Reduction 4: By using the infinitesimal generator $X=X_1+\Omega\,X_3+\delta\,X_4$ and (6), we get

$$u=f(r),\ v=\frac{1}{\delta}\,t-g(r),\ r=2\,(\delta\,x-\Omega\,t)-t^2, \tag{16}$$

where Ω and δ are arbitrary constants. Substituting (16) into (1), we obtain

$$\begin{aligned} & \delta^3\,g'''+(\delta\,g+\Omega)\,f'+\delta\,(1+f)\,g'=0,\\ & 2\,\delta^2\,f'+2\,\delta\,(\delta\,g+\Omega)g'+1=0. \end{aligned} \tag{17}$$

From (17) we can get

$$\begin{aligned} & 2\,\delta^5\,g'''-3\,\delta^3\,g^2\,g'-6\,\Omega\,\delta^2\,g\,g'+\delta\,[2\,(\delta^2-\Omega^2)-r]\,g'\\ & -\delta\,g-\Omega=0. \end{aligned} \tag{18}$$

Now, we construct solutions of the reduced ODEs that we obtained by applying the generalized tanh-function method.

3 Solutions of the Reduced Equations

In this section, using the generalized tanh-function method [17,18] to solve the equations (12), (15) and (18) . Hence, we obtain a new exact solutions of (1).

3.1 Solutions of (12)

Let the solution of (12) given in the form

$$g(r)=A_0+\sum_{i=1}^{M}A_iF^i(r)+B_iF^{-i}(r), \tag{19}$$

where $A_0, A_i, B_i\,(i=0,1,2,...,M)$ are constants, M is a positive integer that we can be calculated by balancing the nonlinear term (s) and the highest derivative term in (12) and $F(r)$ is a solution of Riccati equation

$$F'(r)=b+F^2(r), \tag{20}$$

where b is a constant and the solution of (20) are

$$\begin{cases} F(r)=-\sqrt{-b}\tanh(\sqrt{-b}r), & b<0,\\ F(r)=-\sqrt{-b}\coth(\sqrt{-b}r), & b<0,\\ F(r)=\sqrt{b}\tan(\sqrt{b}r), & b>0,\\ F(r)=\sqrt{b}\cot(\sqrt{b}r), & b>0,\\ F(r)=-\frac{1}{r}, & b=0. \end{cases} \tag{21}$$

Applying balancing procedure in (12), we get $M=1$. Then we can write the solution for it as

$$g(r)=A_0+A_1F(r)+\frac{B_1}{F(r)}. \tag{22}$$

Putting (22) into (12) and using (20), we get a polynomial in $F^i(r)$. Setting all coefficients of it equal to zero, we have a system of algebraic equations for A_0, A_1, B_1 and b. By using Maple to solve it, we have the results

Case 1

$$A_0=A_1=0, B_1=\pm 2, b=-2. \tag{23}$$

Case 2

$$A_0=B_1=0, A_1=\pm 1, b=-2. \tag{24}$$

Case 3

$$A_0=0, A_1=\pm 1, B_1=\pm\frac{1}{2}, b=-\frac{1}{2}. \tag{25}$$

Case 4

$$A_0=0, A_1=\pm 1, B_1=\pm 1, b=1. \tag{26}$$

Substituting these results into (22) with (21), we get the solutions of $g(r)$. From (11) and the solutions of $g(r)$, we obtain the solutions of $f(r)$. Putting these solutions in (10) we have the travelling wave solutions of the model (1) as

$$u=-\tanh^2(\sqrt{2}x),\ v=\pm\sqrt{2}\tanh(\sqrt{2}x), \tag{27}$$

$$u=-\coth^2(\sqrt{2}x),\ v=\pm\sqrt{2}\coth(\sqrt{2}x), \tag{28}$$

$$\begin{aligned} u&=\tfrac{-1}{2}\Big[\tan(x)+\cot(x)\Big]^2,\\ v&=\pm\Big[\tan(x)+\cot(x)\Big], \end{aligned} \tag{29}$$

$$\begin{aligned} u&=\tfrac{-1}{4}\Big[\tanh(\tfrac{1}{\sqrt{2}}x)+\coth(\tfrac{1}{\sqrt{2}}x)\Big]^2,\\ v&=\pm\tfrac{1}{\sqrt{2}}\Big[\tanh(\tfrac{1}{\sqrt{2}}x)+\coth(\tfrac{1}{\sqrt{2}}x)\Big]. \end{aligned} \tag{30}$$

3.2 Solutions of (15)

Consider the solution of (15) is the same solution as given in (19) and by using the balance of the nonlinear term (s) with the highest derivative term appearing in (15), we get $M=1$. Then the solution for it is the same solution in (22). Setting (22) in (15) and using (20), we have a polynomial in $F^i(r)$. Putting each coefficients of it equal to zero, we get a system of algebraic equations for A_0, A_1, B_1 and b. Using Maple to solve this system, we get the cases

Case 1

$$A_0=\frac{\lambda}{\mu}, A_1=0, B_1=\pm\frac{(\lambda^2+2\mu^2)}{\mu^3}, b=-\frac{(\lambda^2+2\mu^2)}{\mu^4}. \tag{31}$$

Case 2

$$A_0=\frac{\lambda}{\mu}, A_1=\pm\mu, B_1=0, b=-\frac{(\lambda^2+2\mu^2)}{\mu^4}. \tag{32}$$

Case 3

$$A_0=\frac{\lambda}{\mu}, A_1=\pm\mu, B_1=\pm\frac{(\lambda^2+2\mu^2)}{4\mu^3}, b=-\frac{(\lambda^2+2\mu^2)}{4\mu^4}. \tag{33}$$

Case 4

$$A_0=\frac{\lambda}{\mu}, A_1=\pm\mu, B_1=\pm\frac{(\lambda^2+2\mu^2)}{2\mu^3}, b=\frac{(\lambda^2+2\mu^2)}{2\mu^4}. \tag{34}$$

Setting these values into (22) with (21), we have the solutions of $g(r)$. Using (14) and the solutions of $g(r)$, we get the solutions of $f(r)$. Putting these solutions in (13), we construct the solutions of (1) as follows:

$$\begin{aligned} u&=(\tfrac{\lambda}{\mu})^2\pm\tfrac{\lambda\sqrt{\lambda^2+2\mu^2}}{\mu^2}\tanh\left(\tfrac{\sqrt{\lambda^2+2\mu^2}}{\mu^2}(\mu x-\lambda t)\right)\\ &-\tfrac{1}{2}\left[\tfrac{\lambda}{\mu}\pm\tfrac{\sqrt{\lambda^2+2\mu^2}}{\mu}\tanh\left(\tfrac{\sqrt{\lambda^2+2\mu^2}}{\mu^2}(\mu x-\lambda t)\right)\right]^2,\\ v&=\tfrac{\lambda}{\mu}\pm\tfrac{\sqrt{\lambda^2+2\mu^2}}{\mu}\tanh\left(\tfrac{\sqrt{\lambda^2+2\mu^2}}{\mu^2}(\mu x-\lambda t)\right). \end{aligned} \tag{35}$$

$$u=(\tfrac{\lambda}{\mu})^2\pm\frac{\lambda\sqrt{\lambda^2+2\mu^2}}{\mu^2}\coth\left(\frac{\sqrt{\lambda^2+2\mu^2}}{\mu^2}(\mu x-\lambda t)\right)$$
$$-\tfrac{1}{2}\left[\frac{\lambda}{\mu}\pm\frac{\sqrt{\lambda^2+2\mu^2}}{\mu}\coth\left(\frac{\sqrt{\lambda^2+2\mu^2}}{\mu^2}(\mu x-\lambda t)\right)\right]^2, \quad (36)$$
$$v=\frac{\lambda}{\mu}\pm\frac{\sqrt{\lambda^2+2\mu^2}}{\mu}\coth\left(\frac{\sqrt{\lambda^2+2\mu^2}}{\mu^2}(\mu x-\lambda t)\right).$$

$$u=(\tfrac{\lambda}{\mu})^2\pm\frac{\lambda\sqrt{\lambda^2+2\mu^2}}{2\mu^2}\left[\tanh\left(\frac{\sqrt{\lambda^2+2\mu^2}}{2\mu^2}(\mu x-\lambda t)\right)+\coth\left(\frac{\sqrt{\lambda^2+2\mu^2}}{2\mu^2}(\mu x-\lambda t)\right)\right]-\tfrac{1}{2}\left[\frac{\lambda}{\mu}\pm\frac{\sqrt{\lambda^2+2\mu^2}}{2\mu}\left[\tanh\left(\frac{\sqrt{\lambda^2+2\mu^2}}{2\mu^2}(\mu x-\lambda t)\right)+\coth\left(\frac{\sqrt{\lambda^2+2\mu^2}}{2\mu^2}(\mu x-\lambda t)\right)\right]\right]^2, \quad (37)$$
$$v=\frac{\lambda}{\mu}\pm\frac{\sqrt{\lambda^2+2\mu^2}}{2\mu}\left[\tanh\left(\frac{\sqrt{\lambda^2+2\mu^2}}{2\mu^2}(\mu x-\lambda t)\right)+\coth\left(\frac{\sqrt{\lambda^2+2\mu^2}}{2\mu^2}(\mu x-\lambda t)\right)\right].$$

$$u=(\tfrac{\lambda}{\mu})^2\pm\frac{\lambda\sqrt{\lambda^2+2\mu^2}}{\sqrt{2}\mu^2}\left[\tan\left(\frac{\sqrt{\lambda^2+2\mu^2}}{\sqrt{2}\mu^2}(\mu x-\lambda t)\right)+\cot\left(\frac{\sqrt{\lambda^2+2\mu^2}}{\sqrt{2}\mu^2}(\mu x-\lambda t)\right)\right]-\tfrac{1}{2}\left[\frac{\lambda}{\mu}\pm\frac{\sqrt{\lambda^2+2\mu^2}}{\sqrt{2}\mu}\left[\tan\left(\frac{\sqrt{\lambda^2+2\mu^2}}{\sqrt{2}\mu^2}(\mu x-\lambda t)\right)+\cot\left(\frac{\sqrt{\lambda^2+2\mu^2}}{\sqrt{2}\mu^2}(\mu x-\lambda t)\right)\right]\right]^2, \quad (38)$$
$$v=\frac{\lambda}{\mu}\pm\frac{\sqrt{\lambda^2+2\mu^2}}{\sqrt{2}\mu}\left[\tan\left(\frac{\sqrt{\lambda^2+2\mu^2}}{\sqrt{2}\mu^2}(\mu x-\lambda t)\right)+\cot\left(\frac{\sqrt{\lambda^2+2\mu^2}}{\sqrt{2}\mu^2}(\mu x-\lambda t)\right)\right].$$

The traveling wave solutions and its position of (35), (36) and (38) are plotted when (+) sign is taken with the parameters $\lambda=2$ and $\mu=5$ as shown in Fig. (1), Fig. (2) and Fig. (3), respectively.

3.3 Solutions of (18)

Here, we also solve equation (18) by the generalized tanh-function method. By the same steps that we used in solving (12) and (15) , we can obtain

$$A_0=-\frac{\Omega}{\delta},\; A_1=B_1=0,\;\; b=b. \quad (39)$$

Substituting (39) into (22) with (21) and (16), we get the solution of (1) as

$$u=\tfrac{1}{2\delta^2}\left[t^2+\Omega^2-2(\delta x-\Omega t)\right],\quad v=\tfrac{1}{\delta}(t+\Omega). \quad (40)$$

4 Conservation Laws

Conservation laws play a vital role in physics and mathematics. Mathematical expressions of physical laws are the coservation laws, such as coservation of mass, energy and momentum. The coservation laws can be used to study the properties of the existance, uniqueness and stability of solutions. We investigate in this part the conservation laws for the system (1) by applying Ibragimove's theorem. Firstly, we simply present some notation used in this section. Suppose that a kth-order system of partial differential equations (PDEs) of m dependent variables $u=(u^1,u^2,.....,u^m)$ and of n independent variables $x=(x^1,x^2,.....,x^n)$, define as

$$F_\alpha(x,u,u_{(1)},...,u_{(k)})=0,\; \alpha=1,2,...,m, \quad (41)$$

where, $u_{(1)},u_{(2)},...,u_{(k)}$ represent the collections of all first, second,...,kth-order partial derivatives. This means that, $u_i^\alpha=D_i(u^\alpha)$, $u_{ij}^\alpha=D_jD_i(u^\alpha)$, ..., respectively, where the total derivative operator with respect to x^i given as

$$D_i=\frac{\partial}{\partial x^i}+u_i^\alpha\frac{\partial}{\partial u^\alpha}+u_{ij}^\alpha\frac{\partial}{\partial u_j^\alpha}+...,\; i=1,2,...,n. \quad (42)$$

Also, we can define the symmetry operator and the adjoint equation for the system (41), respectively as

$$X=\xi^i\frac{\partial}{\partial x^i}+\eta^\alpha\frac{\partial}{\partial u^\alpha} \quad (43)$$

$$\frac{\delta(v^iF^i)}{\delta u^\alpha}=F_\alpha^*(x,u,v,u_{(1)},v_{(1)},...,u_{(k)},v_{(k)})=0,\; \alpha=1,2,...,m. \quad (44)$$

Theorem 1 [26]: One Lie-Bäcklund, Lie point and non-local symmetry X, that is define in (43) admitted by the system (41) provides a conservation law for (41) and its adjoint (44), then the conserved vectors T^i are calculated by

$$T^i=\xi^iL+W^\alpha\left[\tfrac{\partial L}{\partial u_i^\alpha}-D_j\left(\tfrac{\partial L}{\partial u_{ij}^\alpha}\right)+D_jD_k\left(\tfrac{\partial L}{\partial u_{ijk}^\alpha}\right)-...\right]+$$
$$D_j(W^\alpha)\left[\tfrac{\partial L}{\partial u_{ij}^\alpha}-D_k\left(\tfrac{\partial L}{\partial u_{ijk}^\alpha}\right)+D_kD_r\left(\tfrac{\partial L}{\partial u_{ijkr}^\alpha}\right)-...\right]+ \quad (45)$$
$$D_jD_k(W^\alpha)\left[\tfrac{\partial L}{\partial u_{ijk}^\alpha}-D_r\left(\tfrac{\partial L}{\partial u_{ijkr}^\alpha}\right)+...\right]+...,$$

where, $W^\alpha=\eta^\alpha-\xi^iu_i^\alpha$ and $L=\sum_{i=1}^m v^iF^i$ are the Lie characteristic function and the formal Lagrangian, respectively. Now we obtain the conservation laws for (1), first we can define the Lagrangian formal for the system (1) as

$$L=\left[\bar{u}\left(u_t+v_x+uv_x+vu_x+\frac{1}{4}v_{xxx}\right)+\bar{v}\left(v_t+vv_x+u_x\right)\right], \quad (46)$$

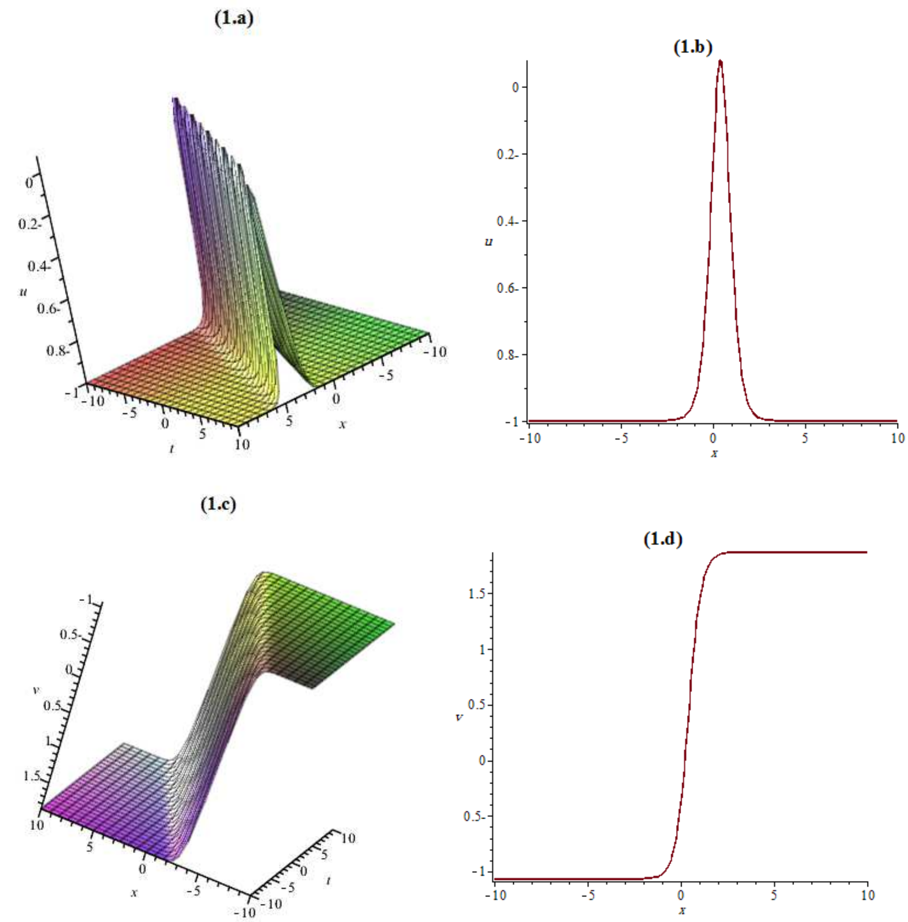

Fig. 1: Different shapes of traveling wave solutions of (35) are plotted when (+) sign is taken with the parameters $\lambda = 2$ and $\mu = 5$: (1.a) Traveling wave solution of u and (1.b) its position at $t = 1$. (1.c) and (1.d) the solution of v and its position at $t = 1$, respectively.

where, $\bar{v}$ and $\bar{u}$ are two a new dependent variables. By using (45) and (46), we obtain

$$\begin{aligned} T^x &= \xi L + W^1(\bar{u}v + \bar{v}) + W^2(\bar{u} + u\bar{u} + v\bar{v} + \tfrac{1}{4}\bar{u}_{xx}) - \\ &\tfrac{1}{4}\bar{u}_x D_x(W^2) + \tfrac{1}{4}\bar{u} D_x^2(W^2), \\ T^t &= \tau L + W^1\bar{u} + W^2\bar{v}. \end{aligned} \tag{47}$$

From the symmetry operators given in (5) with (47), we get the following cases for the conservation laws:

Case 1: We consider the symmetry operator $X_1 = t\,\partial_x + \partial_v$, we have $\xi = t$, $\tau = \eta^1 = 0$, $\eta^2 = 1$ and the Lie characteristic functions corresponding to this symmetry are $W^1 = -t\,u_x$ and $W^2 = 1 - t\,v_x$. Hence, the associated conserved vectors are

$$\begin{aligned} T^x &= \bar{u}(tu_t + u + 1) + \bar{v}(tv_t + v) + \tfrac{1}{4}[t(\bar{u}_x v_{xx} - v_x\bar{u}_{xx}) + \bar{u}_{xx}], \\ T^t &= -t\,u_x\,\bar{u} + (1 - t\,v_x)\,\bar{v}. \end{aligned} \tag{48}$$

Case 2: Using the symmetry operator $X_2 = x\,\partial_x + 2t\,\partial_t - 2(1+u)\,\partial_u - v\,\partial_v$, we have $\xi = x$, $\tau = 2t$, $\eta^1 = -2(1+u)$, $\eta^2 = -v$ and the Lie characteristic functions become $W^1 = -2(1+u) - x\,u_x - 2t\,u_t$ and $W^2 = -v - x\,v_x - 2t\,v_t$. So, the associated conserved vectors given as

$$\begin{aligned} T^x &= \tfrac{1}{4}\,[\bar{u}\,(2t\,v_{tx} - 2t\,v_{txx} - 3\,v_{xx}) + \bar{u}_x\,(2\,v_x + x\,v_{xx}) \\ &-\bar{u}_{xx}\,(x\,v_x + 2t\,v_t + v)] + \bar{u}\,[x\,u_t - (3\,u + 2t\,u_t + 3)v \\ &-2t\,(1+u)\,v_t] + \bar{v}\,[x\,v_t - 2u - 2t\,u_t - v^2 - 2t\,v\,v_t - 2], \\ T^t &= 2\,\bar{u}\,[t\,(v_x + u\,v_x + v\,u_x) - u - 1] \\ &+\bar{v}\,[2t\,(v\,v_x + u_x) - v - x\,v_x] + \tfrac{1}{2}\,\bar{u}\,(t\,v_{xxx} - 2x\,u_x). \end{aligned} \tag{49}$$

Case 3: For the symmetry operator $X_3 = \partial_x$, we have $\xi = 1$, $\tau = \eta^1 = \eta^2 = 0$ and the Lie characteristic functions written as $W^1 = -u_x$ and $W^2 = -v_x$. So, the conserved vectors are

$$\begin{aligned} T^x &= \bar{u}\,u_t + \bar{v}\,v_t + \tfrac{1}{4}\,(\bar{u}_x\,v_{xx} - v_x\,\bar{u}_{xx}), \\ T^t &= -\,(u_x\,\bar{u} + v_x\,\bar{v}). \end{aligned} \tag{50}$$

Case 4: Using the symmetry operator $X_4 = \partial_t$, we have $\tau = 1$, $\xi = \eta^1 = \eta^2 = 0$ and we get $W^1 = -u_t$, $W^2 = -v_t$.

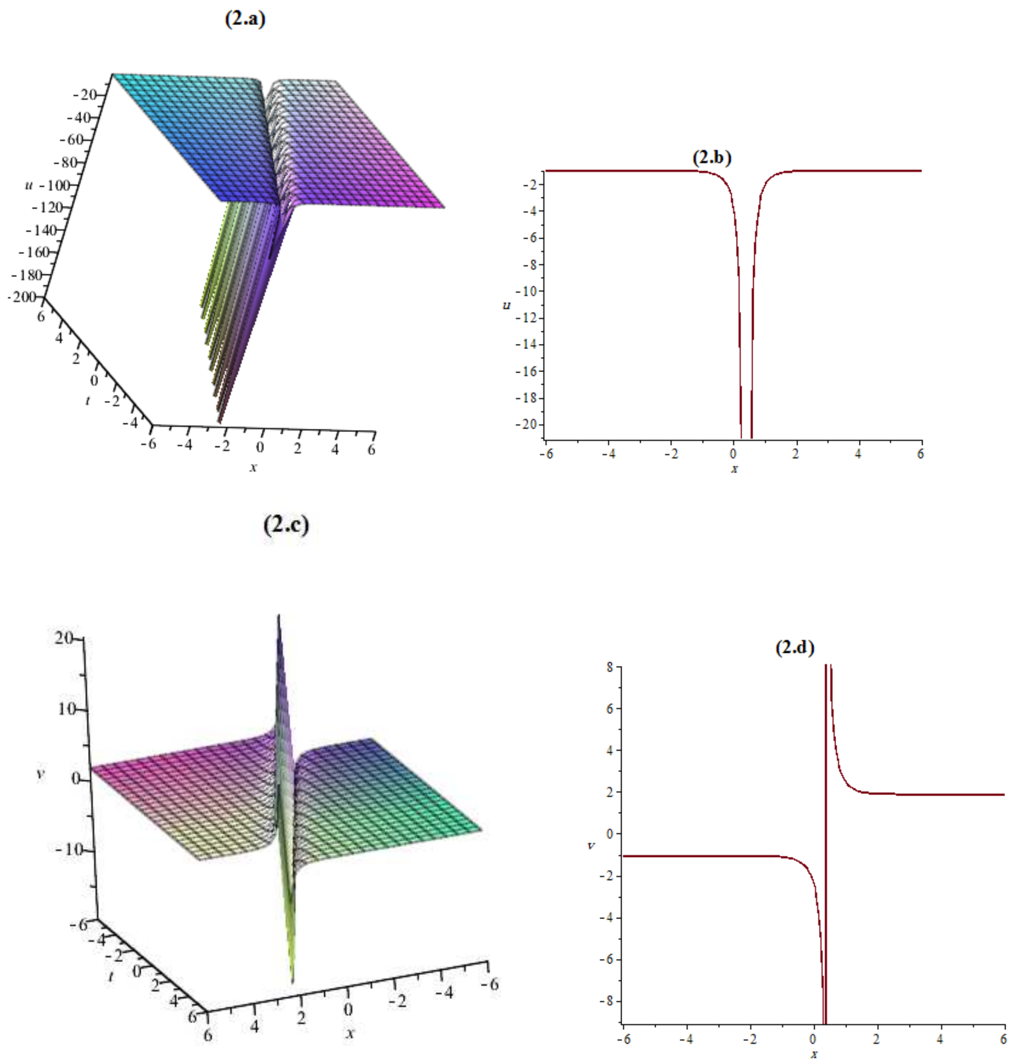

Fig. 2: Traveling wave solutions of (36) are plotted when (+) sign is taken with the parameters $\lambda = 2$ and $\mu = 5$: (1.a) Traveling wave solution of u and (1.b) its position at $t = 1$. (1.c) and (1.d) the solution of v and its position at $t = 1$, respectively.

So, we obtain the conserved vectors as

$$T^x = -\bar{u}\,(v\,u_t + u\,v_t + v_t) - \bar{v}\,(u_t + v\,v_t) - \frac{1}{4}$$
$$(v_t\,\bar{u}_{xx} + \bar{u}\,v_{txx} - \bar{u}_x\,v_{tx}), \qquad (51)$$
$$T^t = \bar{u}(v_x + u\,v_x + v\,u_x + \frac{1}{4}\,v_{xxx}) + \bar{v}(v\,v_x + u_x).$$

5 Conclusion

In this paper, we have used the Lie point symmetry method for the CB system to obtain the symmetries and similarity reduction. This reduction leads us to transform the system to nonlinear ODEs with a new dependent variable. Solving the nonlinear ODEs via the generalized tanh-function methods, we construct a new exact solutions for this system. Our new solutions are soliton, periodic solutions and rational solutions which we investigated some of them by using figures. Also, we get the adjoint table, the commutator table and the symmetry group of the system. Moreover, the conservation laws are obtained. Insure that the Lie point symmetry is an actual powerful method and is worthy of studying further. The computer systems like as Maple is used to easy and solve the complecated algebraic equations. Also, by Maple software we have checked the solutions that we obtained in this paper.

Disclosure Statement

No potential conflict of interest was reported by the authors.

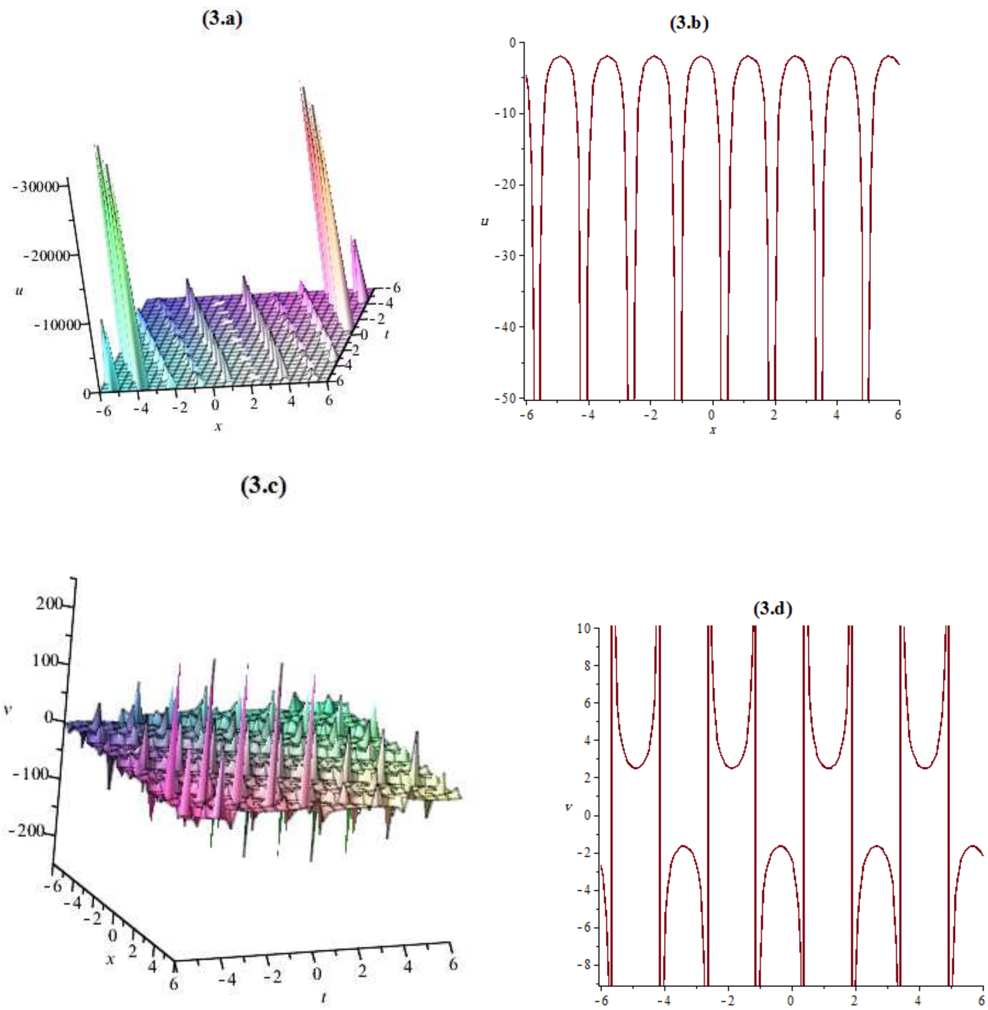

Fig. 3: The solutions of (38)are plotted when (+) sign is taken with the parameters $\lambda = 2$ and $\mu = 5$: (1.a) 3D plots the solution of u and (1.b) its position at $t = 1$. (1.c) and (1.d) the solution of v and its position at $t = 1$, respectively.

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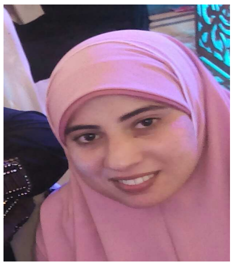

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