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Constrained Periodic-Review Probabilistic Inventory Model with Increasing Holding Cost for Two Different Cases of the Relational Function

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Abstract: The periodic-review inventory process is a review of the level of stock for each item over a number of periods. The main problem with an inventory model is determining the optimal number of periods, the optimal maximum inventory level, and the minimum expected total inventory cost. This research deals with two different cases of relational function in periodic-review probabilistic inventory models, where the holding cost is an increasing function of the number of periods under nonlinear and linear constraints. The nonlinear constraint is the expected ordering cost and the linear constraint is storage space. The goal of this research is to find the minimum expected total cost for the two different probabilistic inventory models based on two different relational functions using a geometric programming approach. The classical inventory model without any constraints is derived as a special case. A numerical example is analyzed for each model.

Keywords: Inventory, relational function, increasing holding cost, constraints, geometric programming approach.

1 Introduction

An inventory model is a mathematical model that aims to determine the optimal level of inventories that should be maintained in a construction process to prevent the risk of stock running out. In the probabilistic inventory models, the demand rate is considered as a random variable and follows a known probability distribution with a known average. Multi-item probabilistic inventory models have been studied widely in the literature with and without constraints. An unconstrained multi-item probabilistic inventory model was investigated by [1], [2], and [3]. [4] introduces deterministic and probabilistic inventory models where classical optimization is used. [5] is the initial research into an optimized inventory model using a geometric programming approach (GPA).

[6] introduced GPA to solve non-linear cases. An Economic Order Quantity (EOQ) inventory model, where the production cost is independent of demand, was studied by [8] using GPA. [7] introduces a reliable production process with a fixed order cost in the EOQ model. An unconstrained inventory model was investigated by [9], again using GPA. [10] develops a periodic-review inventory model under the circumstances that demand in any periods is random. [11] determines the inventory policy variable where the order cost is a continuous function of the order quantity, again using GPA. [12] uses GPA to illustrate a multi-item EOQ inventory model where the holding cost is a continuous function of the order quantity under two constraints. [13] considers the order cost as an increased function of the number of periods with a constant relational function. [14] illustrates a mixed periodic-review inventory model. [15] introduces a periodic-review probabilistic inventory model where the order cost is an increased function of the number of periods.

[16] uses GPA to show the periodic-review safety stock model where the holding cost varies under only one constraint. [17] introduces an EOQ model with deteriorating items and time varying demand. [18] presents dynamic programming for a single-item periodic-review inventory model with a fixed lead time in a fluctuating environment. [19] uses GPA to solve a multi-item EOQ model where the holding cost is a decreasing continuous function of the production quantity. [20] adopts GPA to present an EOQ single-item inventory model where the order cost is a linear function of the order quantity.

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More recently, [21] discusses, via GPA, different cases of relational function for multi-product inventory models where the order cost is an increased function of the number of periods under three constraints.

In the literature of periodic-review inventory models consideration is mainly given to the varying order costs with constraints. However, in this research the holding cost is considered as an increasing continuous function of the number of periods, and the relational function can be either a constant or a rational function of the number of periods. This research is organized as follows: Section 2 proposes the model notations, assumptions, and two probabilistic inventory models, each model under two constraints. Model I considers the constant relational function, and a rational function of the number of periods is discussed in model II. The classical inventory models [4] is derived in Section 4. Finally, the comparison of the two models is illustrated by numerical examples in Section 5.

2 Model Assumptions

Let C_{pi}, C_{oi}, and $C_{hi}(N_i)$ be the purchase cost, order cost, and holding cost for the i^{th} item respectively. Let $E(PC)$, $E(OC)$, and $E(HC)$ be the expected purchase cost, expected order cost and the expected holding cost respectively. The expected total cost is denoted by $E(TC)$ which is the sum of $E(PC)$, $E(OC)$, and $E(HC)$ for each item. The demand is a random variable denoted by x_i for the i^{th} item during N_i, with $f(x_i)$ the probability density function of the demand, and the expected value of demand is $E(x_i) = \int_{x_{li}}^{x_{ui}} x_i f(x_i) dx_i$, where x_{ui} is the maximum values of x_i,and x_{li} is the minimum values of x_i. The annual demand rate for the i^{th} item per period is D_i, with expected annual demand $E(D_i)$. The expected level of inventory is $\bar{I}_i$, the maximum inventory level for the i^{th} item Q_{mi}. Let k_1 and k_2 be the limitation of the order cost and the storage space by square meter m^2 respectively. The following assumptions are considered in constructing the mathematical model:

–Consider that the maximum inventory level (maximum order quantity) Q_{mi} for the i^{th} item is associated with the expected order quantity $E(Q_i)$ during the cycle by the relation function $g(N_i)$, so $Q_{mi} = g(N_i)E(Q_i)$, where $E(Q_i) = N_i E(D_i)$.

–To maintain Q_{mi} for any cycle N_i the safety stock is reviewed for every N_i which can help to prevent the risk of stock-out.

–The holding cost is an increasing function of N_i which takes the form $C_{hi}(N_i) = C_{hi} N_i^{\beta}, \quad C_{hi} > 0, \ , \ 0 \leq \beta \leq 1$. The holding cost $C_{hi}(N_i)$ is an increasing function in the number of periods N_i for all values of β, and can be reduced to the constant value C_{hi} if the value of β is equal to zero.

3 Probabilistic Inventory Model with Increasing Holding Cost under two constraints

The annual expected total cost consists of the sum of three components which are as follows

$$E(TC) = E(PC) + E(OC) + E(HC),$$

where $E(PC)$ is defined as,

$$E(PC) = \sum_{i=1}^{n} C_{pi} E(D_i),$$

and $E(OC)$ has the following form

$$E(OC) = \sum_{i=1}^{n} \frac{C_{0i}}{N_i}.$$

$E(HC)$ is given by

$$E(HC) = \sum_{i=1}^{n} \frac{C_{hi}(N_i)\bar{I}}{N_i}.$$

The expected level of inventory $\bar{I}_i = N\left[Q_{mi} - \frac{E(Q_i)}{2}\right]$, then $\bar{I}_i = E(D_i)N_i^2[\frac{2g(N_i)-1}{2}]$, and the expected holding cost is given by

$$E(HC) = \sum_{i=1}^{n} \frac{C_{hi}(N_i)E(D_i)N_i[2g(N_i) - 1]}{2}.$$

According to the model assumption the expected total cost is

$$E(TC) = \sum_{i=1}^{n} \Big[C_{pi}E(D) + \frac{C_{oi}}{N_i} + \frac{C_{hi}N_i^{\beta+1}E(D_i)}{2}(2g(N_i)-1)\Big]. \tag{1}$$

Under non-linear and linear constraints:

$$\left.\begin{array}{r}\sum_{i=1}^{n} \frac{C_{oi}}{N_i} \leq k_1 \\ \sum_{i=1}^{n} SE(D_i)N_i \leq k_2\end{array}\right\} \tag{2}$$

As mentioned earlier the relational function $g(N_i)$ can takes either form constant or rational function.

3.1 Model I

In this model the constant case of the relational function is considered, where $g(N_i) = \gamma, \quad \gamma > \frac{1}{2}$, where γ represents the proportion of one period's consumption that is held as safety stock. Substituting the value of $g(N_i)$ in equation (1) gives the expected total cost as follows

$$E(TC) = \sum_{i=1}^{n} \Big[C_{pi}E(D_i) + \frac{C_{oi}}{N_i} + \frac{C_{hi}E(D_i)N_i^{\beta+1}(2\gamma-1)}{2}\Big]. \tag{3}$$

The first term of the above equation $\sum_{i=1}^{n} C_{pi}E(D_i)$ can be removed without any effect on the solution to the optimization problem, because it is not dependent on N_i, but it has an effect on the calculation of expected total cost. Therefore, the minimum expected total cost is

$$\min E(TC) = \sum_{i=1}^{n} \Big[\frac{c_{oi}}{N_i} + \frac{C_{hi}E(D_i)N_i^{\beta+1}(2\gamma-1)}{2}\Big] \tag{4}$$

subject to

$$\left.\begin{array}{l}\sum_{i=1}^{n} \frac{C_{oi}}{N_i k_1} \leq 1 \\ \sum_{i=1}^{n} \frac{SE(D_i)N_i}{k_2} \leq 1\end{array}\right\} \tag{5}$$

Applying GPA to equation (4) and equation(5), the primal geometric function is obtained as follows:

$$\begin{aligned} G(W) &= \prod_{i=1}^{n} \Big[\frac{C_{oi}}{N_i w_{1i}}\Big]^{w_{1i}} \Big[\frac{C_{hi}E(D_i)N_i^{\beta+1}}{2w_{2i}}\Big]^{w_{2i}} \Big[\frac{C_{oi}}{N_i k_1 w_{3i}}\Big]^{w_{3i}} \Big[\frac{SE(D_i)N_i}{k_2 w_{4i}}\Big]^{w_{4i}}, \\ &= \prod_{i=1}^{n} \Big[\frac{C_{oi}}{w_{1i}}\Big]^{w_{1i}} \Big[\frac{C_{hi}E(D_i)(2\gamma-1)}{2w_{2i}}\Big]^{w_{2i}} \Big[\frac{C_{oi}}{k_1 w_{3i}}\Big] \Big[\frac{SE(D_i)}{k_2 w_{4i}}\Big]^{w_{4i}} \\ &\quad \times N^{-w_{1i}+(\beta+1)w_{2i}-w_{3i}+w_{4i}}, \end{aligned} \tag{6}$$

where $\boldsymbol{w} = w_{ji}, \quad 0 < w_{ji} <$ for all $i = 1, 2, \ldots, n$ and $j = 1, 2, 3, 4$ are the weights which satisfy the following conditions (the normal and the orthogonal situations)

$$\left.\begin{array}{l} w_{1i} + w_{2i} = 1 \\ \text{and} \\ -w_{1i} + (\beta+1)w_{2i} - w_{3i} + w_{4i} = 0 \end{array}\right\}. \tag{7}$$

The problem is to find out the optimal solution of the weights w_{ji}^* for $j = 1, 2, 3, 4$, solving equation (7) as follows:

$$\left.\begin{array}{l} w_{1i} = \frac{\beta+1-w_{3i}+w_{4i}}{\beta+2} \\ w_{2i} = \frac{1+w_{3i}-w_{4i}}{\beta+2} \end{array}\right\}. \tag{8}$$

Substituting the values of w_{1i}, w_{2i} in equation (8) into equation (6) in the dual functions we obtain the following form:

$$
\begin{aligned}
g(w_{3i}, w_{4i}) =& \left[\frac{(\beta+2)C_{oi}}{\beta+1-w_{3i}+w_{4i}}\right]^{\frac{\beta+1-w_{3i}+w_{4i}}{\beta+2}} \left[\frac{(\beta+2)C_{hi}E(D_i)(2\gamma-1)}{2(1+w_{3i}-w_{4i})}\right]^{\frac{1+w_{3i}-w_{4i}}{\beta+2}} \\
& \left[\frac{C_{oi}}{k_1 w_{3i}}\right]^{w_{3i}} \left[\frac{SE(D_i)}{k_2 w_{4i}}\right]^{w_{4i}}. \qquad (9)
\end{aligned}
$$

In order to calculate w_{3i}^* and w_{4i}^* which maximize $g(w_{3i}, w_{4i})$, the logarithm for both sides in equation (9) can be applied, and then the first partial derivative of $\ln g(w_{3i}, w_{4i})$ with respect to w_{3i} and w_{4i} taken respectively and each set to zero as follows:

$$
\begin{aligned}
\frac{\partial \ln g(w_{3i}, w_{4i})}{\partial w_{3i}} =& \frac{-1}{\beta+2} + \Big[\ln(\beta+2)C_{oi} - \ln(\beta+1-w_{3i}+w_{4i})\Big] \qquad (10) \\
&+ \frac{1}{\beta+2}\Big[\ln\frac{(\beta+2)C_{hi}E(D_i)(2\gamma-1)}{2} - \ln(1+w_{3i}-w_{4i})\Big] \\
&+ \Big[\ln\frac{C_{oi}}{k_1} - \ln w_{3i}\Big] - 1 = 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \ln g(w_{3i}, w_{4i})}{\partial w_{4i}} =& \frac{-1}{\beta+2} + \Big[\ln 2C_{oi} - \ln(\beta+1-w_{3i}+w_{4i})\Big] \qquad (11) \\
&+ \frac{1}{\beta+2}\Big[\ln\frac{(\beta+2)C_{hi}E(D_i)(2\gamma-1)}{2} - \ln(1+w_{3i}-w_{4i})\Big] \\
&+ \Big[\ln\frac{SE(D_i)}{k_2} - \ln w_{4i}\Big] - 1 = 0
\end{aligned}
$$

Simplifying equations (10) and (11) we get

$$
\left[\frac{\beta+1-w_{3i}+w_{4i}}{1+w_{3i}-w_{4i}}\right]^{\frac{1}{\beta+2}} \left[\frac{C_{hi}E(D_i)(2\gamma-1)}{2C_{oi}}\right]^{\frac{1}{\beta+2}} \left[\frac{C_{oi}}{k_1 e w_{3i}}\right] = 1 \qquad (12)
$$

$$
\left[\frac{1+w_{3i}-w_{4i}}{\beta+1-w_{3i}+w_{4i}}\right]^{\frac{1}{\beta+2}} \left[\frac{2C_{oi}}{C_{hi}E(D_i)(2\gamma-1)}\right]^{\frac{1}{\beta+2}} \left[\frac{SE(D_i)}{2k_2 e w_{4i}}\right] = 1 \qquad (13)
$$

multiplying these equations, we obtain

$$
w_{3i}w_{4i} = \left[\frac{C_{oi}SE(D_i)}{k_1k_2e^2}\right] \qquad (14)
$$

then we obtain:

$$
f(w_{3i}) = w_{3i}^{\beta+4} + w_{3i}^{\beta+3} - Aw_{3i}^{\beta+2} + B_1w_{3i}^2 - (\beta+1)B_1Aw_{3i} - B_1A = 0 \qquad (15)
$$

$$
f(w_{4i}) = w_{4i}^{\beta+4} + (\beta+1)w_{4i}^{\beta+3} - Aw_{4i}^{\beta+2} + B_2w_{4i}^2 - B_2w_{4i} - B_2A = 0 \qquad (16)
$$

where $A = \left[\frac{C_{oi}SE(D_i)}{k_1k_2e^2}\right]$, $B_1 = \left[\frac{C_{hi}E(D_i)(2\gamma-1)}{2C_{oi}}\right]\left[\frac{C_{oi}}{k_1e}\right]^{\beta+2}$, and $B_2 = \left[\frac{2C_{oi}}{C_{hi}E(D_{)}(2\gamma-1)}\right]\left[\frac{SE(D_r)}{k_2e}\right]^{\beta+2}$. Because $f_j(0) < 0$, and $f_j(1) > 0$, $\forall j = 3,4$, there must exist roots $w_{ij} \in (0,1)$, $\jmath = 3,4$, and to calculate these roots a numerical method can be used. To clarify that any w_{ji}^*, $\jmath = 3,4$ are calculated from equations (15) and (16) maximize $g(w_{3i}^*, w_{4i}^*)$, the following conditions can be applied (negative Hessian matrix) as follows

$$
\begin{aligned}
\frac{\partial^2 g(w_{3i}, w_{4i})}{\partial w_{3i}^2} &= -\left[\frac{1}{\beta+2}\right]\left[\frac{1}{(\beta+1-w_{3i}+w_{4i})} + \frac{1}{(1+w_{3i}-w_{4i})}\right] - \frac{1}{w_{3i}} < 0 \\
\frac{\partial^2 g(w_{3i}, w_{4i})}{\partial w_{4i}^2} &= -\left[\frac{1}{\beta+2}\right]\left[\frac{1}{(\beta+1-w_{3i}+w_{4i})} + \frac{1}{(1+w_{3i}-w_{4i})}\right] - \frac{1}{w_{4i}} < 0 \\
\frac{\partial^2 g(w_{3i}, w_{4i})}{\partial w_{3i}\partial w_{4i}} &= \left[\frac{1}{\beta+2}\right]\left[\frac{1}{(\beta+1-w_{3i}+w_{4i})} + \frac{1}{(1+w_{3i}-w_{4i})}\right] > 0
\end{aligned}
$$

Hence

$$\Delta = \Big[\frac{\partial^2 g(w_{3i},w_{4i})}{\partial w_{3i}\partial w_{4i}}\Big]^2 - \frac{\partial^2 g(w_{3i},w_{4i})}{\partial w_{3i}^2}\frac{\partial^2 g(w_{3i},w_{4i})}{\partial w_{4i}^2}$$
$$= \Bigg[\Big[\frac{1}{\beta+2}\Big]\Big[\frac{1}{(\beta+1-w_{3i}+w_{4i})}+\frac{1}{(1+w_{3i}-w_{4i})}\Big]\Bigg]^2 -$$
$$\Bigg[\Bigg(\Big[\frac{1}{\beta+2}\Big]\Big[\frac{1}{(\beta+1-w_{3i}+w_{4i})}+\frac{1}{(1+w_{3i}-w_{4i})}\Big]-\frac{1}{w_{3i}}\Bigg)\Bigg(\Big[\frac{1}{\beta+2}\Big]$$
$$\Big[\frac{1}{(\beta+1-w_{3i}+w_{4i})}+\frac{1}{(1+w_{3i}-w_{4i})}\Big]-\frac{1}{w_{4i}}\Bigg)\Bigg] < 0.$$

This confirms that the roots w_{3i}^* and w_{4i}^* which are calculated from equations (15) and (16) maximize the dual function $g(w_{3i},w_{4i})$. To find w_{1i}^* and w_{2i}^* substitute the value of w_{3i}^* and w_{4i}^* in expression (8).

The following relation as a result of Duffin and Peterson's theorem ([22]) of GPA can be used to find the optimal number of periods as follows:

$$\frac{C_{oi}}{N_i} = w_{1i}^* g(w_{3i}^*, w_{4i}^*),$$

and

$$\frac{C_{hi}E(D_i)N_i^{\beta+1}(2\gamma-1)}{2} = w_{2i}^* g(w_{3i}^*, w_{4i}^*).$$

Solving the above equations leads to obtaining the N_i^* as follows:

$$N_i^* = \Big[\frac{2C_{oi}(1+w_{3i}^*-w_{4i}^*)}{C_{hi}E(D_i)(2\gamma-1)(\beta+1-w_{3i}^*+w_{4i}^*)}\Big]^{\frac{1}{\beta+2}}. \tag{17}$$

If the maximum inventory level as defined earlier is $Q_{mi}^* = g(N_i)E(D_i)N_i^*$, then

$$Q_m^* = \gamma E(D_i)\Big[\frac{2C_{oi}(1+w_{3i}^*-w_{4i}^*)}{C_{hi}E(D_i)(2\gamma-1)(\beta+1-w_{3i}^*+w_{4i}^*)}\Big]^{\frac{1}{\beta+2}}. \tag{18}$$

The minimum expected total cost can be achieved by substituting N_i^* into equation (3) as follows,

$$E(TC) = \sum_{i=1}^{n}\Big[C_{pi}E(D_i) + C_{oi}\Big[\frac{C_{hi}E(D_i)(2\gamma-1)(\beta+1-w_{3i}^*+w_{4i}^*)}{2C_{oi}(1+w_{3i}^*-w_{4i}^*)}\Big]^{\frac{1}{\beta+2}} + \frac{C_{hi}E(D_i)(2\gamma-1)}{2}\Big[\frac{2C_{oi}(1+w_{3i}^*-w_{4i}^*)}{C_{hi}E(D_i)(2\gamma-1)(\beta+1-w_{3i}^*+w_{4i}^*)}\Big]^{\frac{\beta+1}{\beta+2}}\Big]. \tag{19}$$

3.2 Model II

This model considers a rational function of the number of periods, so the relational function takes the form $g(N_i) = \frac{N_i+\alpha}{N_i}$, and the expected total cost in equation (1) becomes:

$$E(TC) = \sum_{i=1}^{n}\Big[C_{pi}E(D_i) + \frac{C_{oi}}{N_i} + \frac{C_{hi}E(D_i)N_i^{\beta+1}}{2} + C_{hi}E(D_i)\alpha\Big]. \tag{20}$$

The last term of the above equation, $\sum_{i=1}^{n} C_{hi}E(D_i)\alpha$, can be seen as the cost of safety stock insurance, a cost incurred to hold an amount in excess of the expected demand as insurance against the danger of stock running out. However, equation (20) includes two terms that are not dependent on N_i. These are $\sum_{i=1}^{n} C_{pi}E(D_i)$ and $\sum_{i=1}^{n} C_{hi}E(D_i)\alpha$, and these terms can be ignored, so the expected total cost can be written as

$$E(TC) = \sum_{i=1}^{n}\Big[\frac{C_{oi}}{N_i} + \frac{C_{hi}E(D_i)N_i^{\beta+1}}{2}\Big], \tag{21}$$

subject to the constraints which are in equation (5). Applying the geometric programming approach to equations (21) and (5) we obtain

$$G(\boldsymbol{w}) = \prod_{i=1}^{n}\Big[\frac{C_{oi}}{w_{1i}}\Big]^{w_{1i}}\Big[\frac{C_{hi}E(D_i)}{2w_{2i}}\Big]^{w_{2i}}\Big[\frac{C_{oi}}{k_1 w_{3i}}\Big]^{w_{3i}}\Big[\frac{SE(D_i)}{k_2 w_{4i}}\Big]^{w_{4i}} \times N_i^{-w_{1i}+(\beta+1)w_{2i}-w_{3i}+w_{4i}}. \tag{22}$$

where $\boldsymbol{w} = w_{ji}$ where $0 < w_{ji} < 1$ for $j = 1,2,3,4$ and $i = 1,2,3,\dots,n$,(satisfying the orthogonal and natural conditions defined earlier), substituting the values of w_{1i} and w_{2i} from (8) to equation (22), we obtain the following for the dual function:

$$g(w_{3i}, w_{4i}) = \prod_{i=1}^{n}\Big[\frac{(\beta+2)C_{oi}}{\beta+1-w_{3i}+w_{4i}}\Big]^{\frac{\beta+1-w_{3i}+w_{4i}}{\beta 2}}\Big[\frac{(\beta+2)C_{hi}E(D_i)}{2(1+w_{3i}-w_{4i})}\Big]^{\frac{1+w_{3i}-w_{4i}}{\beta+2}} \Big[\frac{C_{oi}}{k_1 w_{3i}}\Big]^{w_{3i}}\Big[\frac{SE(D_i)}{k_2 w_{4i}}\Big]^{w_{4i}}. \tag{23}$$

To calculate w_{3r}^* and w_{4r}^* that maximize $g(w_{3r}^*, w_{4r}^*)$, the logarithm of both sides in equation (23) is applied and then the first partial derivative of $\log g(w_{3i}, w_{4i})$ with respect to w_{3i} and w_{4i} is taken and set to zero as follows:

$$f(w_{3i}) = w_{3i}^{\beta+4} + w_{3i}^{\beta+3} - Aw_{3i}^{\beta+2} + B_3 w_{3i}^2 - (\beta+1)B_3 w_{3i} - B_3 A = 0. \tag{24}$$

$$f(w_{4i}) = w_{4i}^{\beta+4} + (\beta+1)w_{4i}^{\beta+3} - Aw_{4i}^{\beta+2} + B_4 w_{4i}^2 - B_4 w_{4i} - B_4 A = 0. \tag{25}$$

A as defined earlier, $B_3 = \Big[\frac{C_{hi}E(D_i)}{2C_{oi}}\Big]\Big[\frac{C_{oi}}{k_1 e}\Big]^{\beta+2}$, and $B_4 = \Big[\frac{2C_{oi}}{C_{hi}E(D_i)}\Big]\Big[\frac{SE(D_i)}{k_2 e}\Big]^{\beta+2}$. As we see in model I, $f_j(0) < 0$ and $f_j(1) > 0$ for $j = 3,4$, which means there are roots $w_j \in (0,1)$ for $j = 3,4$. To check that w_{3r}^* and w_{4r}^* maximize $g(w_{3i}^*, w_{4i}^*)$, the second derivative with respect to w_{3r} and w_{4r} is applied to obtain the Hessian matrix as follows:

$$\Delta = \Big[\frac{\partial^2 g(w_{3i}, w_{4i})}{\partial w_{3i}\partial w_{4i}}\Big]^2 - \frac{\partial^2 g(w_{3i}, w_{4i})}{\partial w_{3i}^2}\frac{\partial^2 g(w_{3i}, w_{4i})}{\partial w_{4i}^2} < 0,$$

This confirms that the roots w_{3i}^* and w_{4i}^* from equation (24) and (25) maximize the dual function $g(w_{3i}^*, w_{4i}^*)$. The results of the Duffin and Peterson theorem ([22]) of GPA is adopted to find N_r^* as follows:

$$N_r^* = \Big[\frac{2C_{or}(1+w_{3i}-w_{4i})}{C_{hi}E(D_i)(\beta+1-w_{3i}+w_{4i})}\Big]^{\frac{1}{\beta+2}}. \tag{26}$$

The maximum inventory level Q_{mr} is

$$Q_{mi}^* = E(D_i)\Big[\frac{2C_{oi}(1+w_{3i}^*-w_{4i}^*)}{C_{hi}E(D_i)(\beta+1-w_{3i}^*+w_{4i}^*)}\Big]^{\frac{1}{\beta+2}} + E(D_i)\alpha. \tag{27}$$

The minimum expected total cost can be found by replacing the value of N_i^* in equation (20) as follows:

$$E(TC) = \sum_{i=1}^{n}\Big[C_{pi}E(D_i) + C_{oi}\Big[\frac{C_{hi}E(D_i)(\beta+1-w_{3i}^*+w_{4i}^*)}{2C_{oi}(1+w_{3i}^*-w_{4i}^*)}\Big]^{\frac{1}{\beta+2}} + \frac{C_{hr}E(D_i)}{2}\Big[\frac{2C_{oi}(1+w_{3i}^*-w_{4i}^*)}{C_{hi}E(D_i)(\beta+1-w_{3i}^*+w_{4i}^*)}\Big]^{\frac{\beta+1}{\beta+2}} + C_{hi}E(D_i)\alpha\Big]. \tag{28}$$

4 Special Case

Let $\beta = 0, i = 1 \Rightarrow C_{hi}(N_i) = C_h =$ constant and $k_1, k_2 \to \infty$ so $w_{3i}^*, w_{4i}^* = 0$ and $w_{1i}^* = w_{2i}^* = \frac{1}{2}$. Assume that $\gamma = 1, \quad \alpha = 0$, this will lead to a probabilistic single-item inventory model, where all the cost components are constant and without any constraints. This leads to the classical inventory model of [4]. Therefore, N_i^*, Q_{mi}^*, and min$E(TC)$ for model I and model II become:

$$N^* = \sqrt{\frac{2C_o}{C_h E(D)}},$$

$$Q_m^* = E(D)\sqrt{\frac{2C_o}{C_h E(D)}}.$$

$$\min E(TC) = C_p E(D) + \sqrt{C_{or} C_h E(D)}.$$

Harris's results ([23]) can be obtained in the case of a deterministic inventory model without constraints if $E(D) = D$, $\gamma = 1$, and $\alpha = 0$.

5 An illustrative example

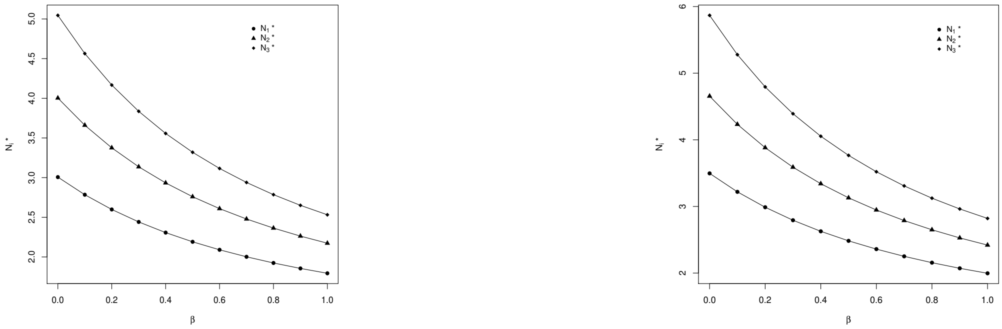
The following table represents the inventory parameters of the probabilistic inventory model for 3 items. The holding cost is a continuous increasing function and the lead time is equal to zero as we assumed earlier in the model assumptions. Let the expected order cost limitation be $k_1 = 1500$, the storage space limitation $S = 80m^2$, $k_2 = 6000$, $\gamma = 1.2$, and $\alpha = 3$.

parameter	item 1	item 2	item 3
$E(D)$	34	23	16
C_{hr}	0.40	0.42	0.44
C_{or}	130	160	180
C_{pr}	80	100	120

The roots w_{3r}^* and w_{4r}^* calculated in equations (15), and (16) for model I, and equations (24), and (25) for model II, so N_i^* in model I can be calculated in equation (17), and in equation (26) for model II. Similarly, the optimal maximum inventory levels for models I and II are calculated from equations (18), and (27) respectively. Finally, min$E(TC)$ which is equal to the sum of the min$E(TC)$ for each item is calculated from equations (19); and (28) for models I and II respectively. All these results are presented in Table (1).

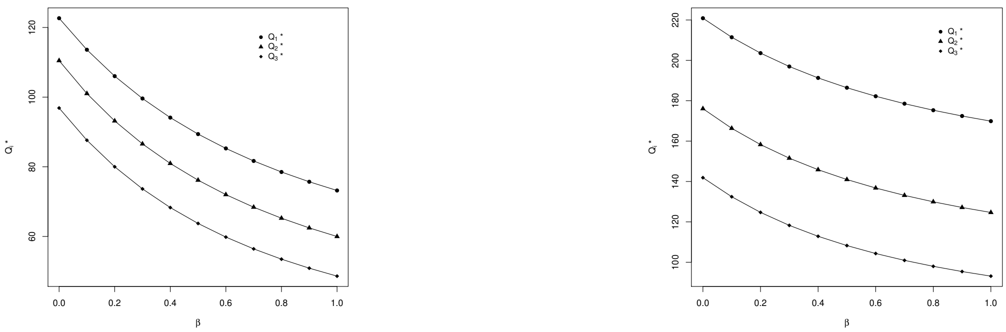
Table 1: The optimal solution for model I and model II.

β	N_1^*	N_2^*	N_3^*	Q_1^*	Q_2^*	Q_3^*	min$E(TC)$
Model I							
0.00	3.01	4.00	5.05	122.66	110.48	96.87	7058.89
0.10	2.78	3.66	4.56	113.61	101.02	87.63	7069.84
0.20	2.60	3.38	4.17	106.03	93.17	80.01	7080.62
0.30	2.44	3.14	3.84	99.61	86.56	73.65	7091.19
0.40	2.31	2.93	3.56	94.12	80.96	68.29	7101.51
0.50	2.19	2.76	3.32	89.39	76.16	63.73	7111.55
0.60	2.09	2.61	3.12	85.28	72.01	59.82	7121.30
0.70	2.00	2.48	2.94	81.68	68.40	56.43	7130.74
0.80	1.92	2.36	2.79	78.51	65.24	53.48	7139.87
0.90	1.86	2.26	2.65	75.70	62.45	50.89	7148.69
1.00	1.79	2.17	2.53	73.20	59.98	48.61	7157.19
Model II							
0.00	3.50	4.66	5.87	173.21	155.82	136.33	7042.23
0.10	3.22	4.23	5.28	163.88	145.50	125.90	7052.30
0.20	2.99	3.88	4.79	156.03	136.86	117.23	7062.28
0.30	2.79	3.59	4.39	149.37	129.55	109.94	7072.12
0.40	2.63	3.34	4.05	143.65	123.31	103.75	7081.79
0.50	2.48	3.13	3.77	138.71	117.94	98.45	7091.27
0.60	2.36	2.95	3.52	134.42	113.28	93.87	7100.52
0.70	2.25	2.79	3.31	130.67	109.22	89.89	7109.54
0.80	2.16	2.65	3.12	127.36	105.64	86.41	7118.31
0.90	2.07	2.53	2.96	124.44	102.48	83.34	7126.81
1.00	2.00	2.42	2.82	121.85	99.67	80.62	7135.06



(a) Model I (b) Model 2

Fig. 1: The optimal number of periods for different values of β.



(a) Model I (b) Model 2

Fig. 2: The optimal maximum inventory level for different values of β.

The results show a decrease in N_i^* and Q_{mi}^* as the values of β increases. This is clear from Figure (1) and Figure (2), whereas value of min$E(TC)$ increases as the value of β increases as shown in Figure (3). Comparing the min$E(TC)$ of model I with model II, we can see that the minimum expected total cost in model II is less than in model I. This means that model II is better than model I because it achieves the goal which is the lowest expected total cost. Furthermore, when the value of β is equal to zero, this means models I and II become crisp models (without varying holding costs), the values of min$E(TC)$ of crisp model I and crisp model II are 7058.89 and 7042.23 respectively. This confirms that model II is better than model I. However, increasing the values of α and γ will increase min$E(TC)$ for both models.

6 Conclusion

This paper has investigated which relational function form can lead to minimum expected total costs for a periodic-review probabilistic inventory model under two constraints, where the relational function can be either constant or a rational function, and the constraints are the expected ordering cost and the limit of the storage space. The holding cost is a continuously increasing function of the number of periods. The geometric programming approach is considered for finding the optimal solutions of N_i^*, Q_{mi}^* and min$E(TC)$ for the i^{th} item in the two probabilistic inventory models. The classical inventory model of [4] is derived as a special case. The results show that the rational function form achieved min$E(TC)$ for all values of β compared to the constant form.

(a) Model I (b) Model 2

Fig. 3: The minimum annual expected total cost for different values of β.

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Conflict of interest

The authors declare that they have no conflict of interest.

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