Fixed Point Results of Contractive Mappings with Altering Distance Functions in Ordered b-Metric Spaces

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Abstract: We explore the existence of a fixed point as well as the uniqueness of a mapping in an ordered b-metric space using a generalized $(\check{\psi}, \hat{\eta})$-weak contraction. In addition, some results are posed on a coincidence point and a coupled coincidence point of two mappings under the same contraction condition. These findings generalize and build on a few recent studies in the literature. At the end, we provided some examples to back up our findings.

Keywords: $(\check{\psi}, \hat{\eta})$-weak contraction, fixed points, coincidence and coupled coincidence points, ordered b-metric space.

1 Introduction

In a wide range of pure and applied mathematics problems, fixed points of mappings that satisfy contractive conditions in extended metric spaces are extremely useful. First, Ran and Reuings [31] described the existence of fixed points in this direction for certain maps in ordered metric space and exhibited matrix linear equations applications. Following that, Nieto et al. [28, 29] expanded the result of [31] to nondecreasing mappings and used their findings to obtain differential equations solutions. Agarwal et al. [4] and O'Regan et al. [30] examine the influence of generalized contractions in ordered spaces at the same time. Bhaskar and Lakshmikantham [11] first developed coupled fixed point theory for some maps, then used the results to find a unique solution to periodic boundary value problems. Following that, Lakshmikantham and Ćirić [25], which were the extensions of [11] involving monotone property to a function in the space, pioneered the idea of coupled coincidence, common fixed point results. [15,16,17,19, 21,24,26,35,36,37,38] provide additional information on coupled fixed point effects in various spaces under various contractive conditions.

A b-metric space is one of several generalizations of a standard metric space proposed by Bakhtin in his work [9], and widely used by Czerwik in his work [13,14].

Following that, a lot of progress was made in acquiring the results of fixed points to single valued as well as multi-valued operators in the space, as evidenced by [1,2, 3,5,6,7,8,10,18,20,22,23,27,32,33,34,39].

We demonstrate some fixed points results for mappings in ordered b-metric space that satisfy a generalized weak contraction in this paper. The results from [10,11,12,19,21,25,34] are expanded here as well as some examples noted to support the findings at the end of our work.

2 Preliminaries

Definition 21[14] *A b-metric is a map $\eth : \mathscr{E} \times \mathscr{E} \to [0, +\infty)$ that satisfies the properties below in $\mathscr{E}$ for all $\varepsilon, \wp, \zeta$ and some* s ≥ 1,

(a).$\eth(\varepsilon, \wp) = 0$ if and if $\varepsilon = \wp$.
(b).$\eth(\varepsilon, \wp) = \eth(\wp, \varepsilon)$.
(c).$\eth(\varepsilon, \wp) \leq s(\eth(\varepsilon, \zeta) + \eth(\zeta, \wp))$.

A b-metric space is specified as $(\mathscr{E}, \eth, s)$.

Definition 22[10,14] *In a b-metric space,*

(1).if $\eth(\varepsilon_n, \varepsilon) \to 0$ as $n \to +\infty$, $\{\varepsilon_n\}$ is said to be convergent to ε.

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(2).if $\eth(\varepsilon_n, \varepsilon_m) \to 0$ *is the same as* $n, m \to +\infty$, *then* $\{\varepsilon_n\}$ *is a Cauchy sequence.*

(3).if $(\mathscr{E}, \eth, s)$ *is a complete b-metric space, then any Cauchy sequence is convergent.*

Definition 23[14, 34, 38] *If* $\mathscr{E}$ *is a partial ordered set with respect to an ordered relation* $\preceq$ *and* $\eth$ *is a metric on it, then* $(\mathscr{E}, \eth, \preceq)$ *is a partially ordered metric space.* $(\mathscr{E}, \eth, \preceq)$ *is a complete partially ordered b-metric space, despite the fact that* $\eth$ *is complete.*

Definition 24[34, 38] *If* $\hbar(\varepsilon) \preceq \hbar(\wp)$ *for all* $\varepsilon, \wp \in \mathscr{E}$ *with* $\varepsilon \preceq \wp$, *the map is called a monotone non-decreasing.*

Definition 25[12] *Let* $\hbar, \mathscr{I} : \mathscr{A} \to \mathscr{A}$ *be two mappings, and* $\mathscr{A} \neq \emptyset \subseteq \mathscr{E}$ *be one. If* $\hbar \varepsilon = \mathscr{I} \varepsilon = \varepsilon$ $(\hbar \varepsilon = \mathscr{I} \varepsilon)$ *for* $\varepsilon \in \mathscr{A}$, *then* ε *is a common fixed point (coincidence point) of* $\hbar, \mathscr{I}$.

Definition 26[12] *If* $\hbar \mathscr{I} \varepsilon = \mathscr{I} \hbar \varepsilon$ *for all* $\varepsilon \in \mathscr{A}$, *then* $\hbar$ *and* $\mathscr{I}$ *are commuting.*

Definition 27[12, 34] *The two maps* $\hbar, \mathscr{I}$ *are compatible if* $\lim_{n \to +\infty} d(\mathscr{I} \hbar \varepsilon_n, \hbar \mathscr{I} \varepsilon_n) = 0$ *for each sequence* $\{\varepsilon_n\} \subseteq \mathscr{E}$ *so that* $\lim_{n \to +\infty} \hbar \varepsilon_n = \lim_{n \to +\infty} \mathscr{I} \varepsilon_n = \mu$, *for some* $\mu \in \mathscr{A}$.

Definition 28[12, 34] *If* $\hbar \varepsilon = \mathscr{I} \varepsilon$ *for* $\varepsilon \in \mathscr{A}$, *then* $\hbar \mathscr{I} \varepsilon = \mathscr{I} \hbar \varepsilon$, *the mappings* $\hbar$ *and* $\mathscr{I}$ *are weakly compatible.*

Definition 29[34] *If* $\hbar \varepsilon \preceq \hbar \wp$ *implies* $\mathscr{I} \varepsilon \preceq \mathscr{I} \wp$ *for any* $\varepsilon, \wp \in \mathscr{E}$, *then a map* $\mathscr{I}$ *is a monotone* $\hbar$-*non-decreasing.*

Definition 210[11] *Let* $\mathscr{I} : \mathscr{E} \times \mathscr{E} \to \mathscr{E}$ *and* $\hbar : \mathscr{E} \to \mathscr{E}$ *are two mappings,*

(a).a point $(\varepsilon, \wp) \in \mathscr{E} \times \mathscr{E}$ *is coupled coincidence point of* $\mathscr{I}$, $\hbar$ *if* $\mathscr{I}(\varepsilon, \wp) = \hbar \varepsilon$ *and,* $\mathscr{I}(\wp, \varepsilon) = \hbar \wp$. *In particular, if* $\hbar$ *is an identity map, then* $(\varepsilon, \wp)$ *is a coupled fixed point of* $\mathscr{I}$.
(b).an element $\varepsilon \in \mathscr{E}$ *is a common fixed point of* $\mathscr{I}$, $\hbar$ *if* $\mathscr{I}(\varepsilon, \varepsilon) = \hbar \varepsilon = \varepsilon$.
(c).if $\mathscr{I}(\hbar \varepsilon, \hbar \wp) = \hbar(\mathscr{I} \varepsilon, \mathscr{I} \wp)$ *for all* $\varepsilon, \wp \in \mathscr{E}$, *then* $\mathscr{I}$ *and* $\hbar$ *are commuting each other.*
(d).If any two elements in a set $\mathscr{A} \subseteq \mathscr{E}$ *are comparable, the set is well ordered.*

Definition 211 *A self map* $\breve{\psi}$ *on* $[0, +\infty)$ *that meets the conditions below is known as an altering distance function:*

(a).$\breve{\psi}$ is a non-decreasing and continuous function.
(b).$\breve{\psi}(\ell) = 0$ iff $\ell = 0$.

As seen above, the symbol $\hat{\Phi}$ represents the set of all altering distance functions.

Similarly,
$\hat{\Psi} : \{\hat{\eta} | \hat{\eta} \ \text{is a lower semi} - \text{continuous self mapping on} \ [0, +\infty) \ \text{and}, \ \hat{\eta}(\ell) = 0 \ \text{iff} \ \ell = 0\}$.

Lemma 212[27] *Let* $\hbar : \mathscr{E} \to \mathscr{E}$ *be a mapping, and* $\mathscr{E} \neq \emptyset$. *Then* $\mathscr{M} \subseteq \mathscr{E}$ *occurs, resulting in* $\hbar \mathscr{M} = \hbar \mathscr{E}$, *where* $\hbar : \mathscr{M} \to \mathscr{E}$ *is one-to-one.*

Lemma 213[2] *Let* $\{\varepsilon_n\}$ *and* $\{\wp_n\}$ *be two sequences and b-convergent to* ε *and* $\wp$ *in a b-metric space* $(\mathscr{E}, \eth, s, \preceq)$, *where* $s > 1$. *Then*

$$\frac{1}{s^2} \eth(\varepsilon, \wp) \leq \lim_{n \to +\infty} \inf \eth(\varepsilon_n, \wp_n) \leq \lim_{n \to +\infty} \sup \eth(\varepsilon_n, \wp_n)$$
$$\leq s^2 \eth(\varepsilon, \wp).$$

In particular, if $\varepsilon = \wp$, *then* $\lim_{n \to +\infty} \eth(\varepsilon_n, \wp_n) = 0$. *In addition, for every* $\tau \in \mathscr{E}$, *we get*

$$\frac{1}{s} \eth(\varepsilon, \tau) \leq \lim_{n \to +\infty} \inf \eth(\varepsilon_n, \tau) \leq \lim_{n \to +\infty} \sup \eth(\varepsilon_n, \tau) \leq s d(\varepsilon, \tau).$$

3 Main Results

Let's get started with the theorem below.

Theorem 31 *Suppose* $(\mathscr{E}, \eth, s, \preceq)$ *is a complete partially ordered b-metric space with* $s \geq 1$. *A map* $\mathscr{I} : \mathscr{E} \to \mathscr{E}$ *is non-decreasing and continuous with respect to* $\preceq$. *If* $\varepsilon_0 \in \mathscr{E}$ *is such that* $\varepsilon_0 \preceq \mathscr{I} \varepsilon_0$ *and the following contraction condition is fulfilled, then* $\mathscr{I}$ *has a fixed point in* $\mathscr{E}$.

$$\breve{\psi}(s \eth(\mathscr{I} \varepsilon, \mathscr{I} \wp)) \leq \breve{\psi}(\mathscr{P}(\varepsilon, \wp)) - \hat{\eta}(\mathscr{P}(\varepsilon, \wp)) \quad (1)$$

for $\breve{\psi} \in \hat{\Phi}, \hat{\eta} \in \hat{\Psi}$ *and for any* $\varepsilon, \wp \in \mathscr{E}$ *so that* $\varepsilon \preceq \wp$ *and*

$$\mathscr{P}(\varepsilon, \wp) = \max\{ \frac{\eth(\wp, \mathscr{I} \wp)[1 + \eth(\varepsilon, \mathscr{I} \varepsilon)]}{1 + \eth(\varepsilon, \wp)},$$
$$\frac{\eth(\varepsilon, \mathscr{I} \varepsilon) \eth(\wp, \mathscr{I} \wp)}{1 + \eth(\varepsilon, \wp)}, \quad (2)$$
$$\frac{\eth(\varepsilon, \mathscr{I} \varepsilon) \eth(\wp, \mathscr{I} \wp)}{1 + \eth(\mathscr{I} \varepsilon, \mathscr{I} \wp)}, \eth(\varepsilon, \wp)\}.$$

Proof. For some $\varepsilon_0 \in \mathscr{E}$ with $\mathscr{I} \varepsilon_0 = \varepsilon_0$, then the result is trivial. Assuming that $\varepsilon_0 \prec \mathscr{I} \varepsilon_0$, we describe a sequence $\{\varepsilon_n\} \subset \mathscr{E}$ by $\varepsilon_{n+1} = \mathscr{I} \varepsilon_n$ for all $n \geq 0$. However, we can deduce the following as $\mathscr{I}$ is non-decreasing,

$$\varepsilon_0 \prec \mathscr{I} \varepsilon_0 = \varepsilon_1 \preceq \mathscr{I} \varepsilon_1 = \varepsilon_2 \preceq ...$$
$$\preceq \mathscr{I} \varepsilon_{n-1} = \varepsilon_n \preceq \mathscr{I} \varepsilon_n = \varepsilon_{n+1} \preceq ..... . \quad (3)$$

If $\varepsilon_{n_0} = \varepsilon_{n_0+1}$ for $n_0 \in \mathbb{N}$, then ε_{n_0} is a fixed point of $\mathscr{I}$ from (3). Otherwise, for all $n \geq 1$, $\varepsilon_n \neq \varepsilon_{n+1}$. For $n \geq 1$, let $D_n = \eth(\varepsilon_{n+1}, \varepsilon_n)$ be used. We know that for every $n \geq 1$, $\varepsilon_{n-1} \prec \varepsilon_n$ and, the equation (1) becomes

$$\breve{\psi}(D_n) = \breve{\psi}(\eth(\varepsilon_n, \varepsilon_{n+1})) = \breve{\psi}(\eth(\mathscr{I} \varepsilon_{n-1}, \mathscr{I} \varepsilon_n))$$
$$\leq \breve{\psi}(s \eth(\mathscr{I} \varepsilon_{n-1}, \mathscr{I} \varepsilon_n))$$
$$\leq \breve{\psi}(\mathscr{P}(\varepsilon_{n-1}, \varepsilon_n)) \quad (4)$$
$$- \hat{\eta}(\mathscr{P}(\varepsilon_{n-1}, \varepsilon_n)).$$

From (4), we get

$$\eth(\varepsilon_n, \varepsilon_{n+1}) = \eth(\mathscr{I} \varepsilon_{n-1}, \mathscr{I} \varepsilon_n) \leq \frac{1}{s} \mathscr{P}(\varepsilon_{n-1}, \varepsilon_n), \quad (5)$$

where

$$\mathscr{P}(\varepsilon_{n-1},\varepsilon_n) = \max\{\frac{\eth(\varepsilon_n,\mathscr{I}\varepsilon_n)[1+\eth(\varepsilon_{n-1},\mathscr{I}\varepsilon_{n-1})]}{1+\eth(\varepsilon_{n-1},\varepsilon_n)},$$

$$\frac{\eth(\varepsilon_{n-1},\mathscr{I}\varepsilon_{n-1})\,\eth(\varepsilon_n,\mathscr{I}\varepsilon_n)}{1+\eth(\varepsilon_{n-1},\varepsilon_n)},$$

$$\frac{\eth(\varepsilon_{n-1},\mathscr{I}\varepsilon_{n-1})\,\eth(\varepsilon_n,\mathscr{I}\varepsilon_n)}{1+\eth(\mathscr{I}\varepsilon_{n-1},\mathscr{I}\varepsilon_n)},\eth(\varepsilon_{n-1},\varepsilon_n)\}$$

$$\leq \max\{\eth(\varepsilon_n,\varepsilon_{n+1}),\eth(\varepsilon_{n-1},\varepsilon_n)\}$$

$$\leq \max\{D_n,D_{n-1}\}. \tag{6}$$

If $\max\{D_n,D_{n-1}\}=D_n$ for certain $n\geq 1$, equation (5) is then accompanied by

$$\eth(\varepsilon_n,\varepsilon_{n+1}) \leq \frac{1}{\jmath}\eth(\varepsilon_n,\varepsilon_{n+1}),$$

this is a contradiction. Thus, $\max\{D_n,D_{n-1}\}=D_{n-1}$ for $n\geq 1$. Hence, equation (5) becomes

$$\eth(\varepsilon_n,\varepsilon_{n+1}) \leq \frac{1}{\jmath}\eth(\varepsilon_n,\varepsilon_{n-1}),$$

Since $\frac{1}{\jmath}\in(0,1)$ then $\{\varepsilon_n\}$ is a Cauchy sequence from [1,6, 8,18]. Also, the completeness of $\mathscr{E}$ gives that $\varepsilon_n\to\mu\in\mathscr{E}$.

We may also deduce the following from $\mathscr{I}$'s continuity:

$$\mathscr{I}\mu = \mathscr{I}\big(\lim_{n\to+\infty}\varepsilon_n\big) = \lim_{n\to+\infty}\mathscr{I}\varepsilon_n = \lim_{n\to+\infty}\varepsilon_{n+1} = \mu. \tag{7}$$

As a result, $\mathscr{I}$ in $\mathscr{E}$ has a fixed point μ.

The continuity assumption on $\mathscr{I}$ is extracted from Theorem 31 and used to derive the following theorem.

Theorem 32*In Theorem 31, if $\mathscr{E}$ satisfies below condition, then $\mathscr{I}$ has a fixed point.*

If a non-decreasing sequence $\{\varepsilon_n\}\subseteq\mathscr{E}$ and $\varepsilon_n\to\sigma$ then $\varepsilon_n\leq\sigma$, for each $n\in\mathbb{N}$, i.e., $\sigma=\sup\varepsilon_n$. (8)

*Proof.*We have an increasing sequence $\{\varepsilon_n\}\subseteq\mathscr{E}$ that eventually converges to some $\sigma\in\mathscr{E}$ as a result of Theorem 31. But by the hypotheses for all n, $\varepsilon_n\preceq\sigma$, which means that $\sigma=\sup\varepsilon_n$.

We can now assert that σ is a fixed point of $\mathscr{I}$. Assume $\mathscr{I}\sigma\neq\sigma$ is not true. Let

$$\mathscr{P}(\varepsilon_n,\sigma) = \max\{\frac{\eth(\sigma,\mathscr{I}\sigma)[1+\eth(\varepsilon_n,\mathscr{I}\varepsilon_n)]}{1+\eth(\varepsilon_n,\sigma)},$$

$$\frac{\eth(\varepsilon_n,\mathscr{I}\varepsilon_n)\,\eth(\sigma,\mathscr{I}\sigma)}{1+\eth(\varepsilon_n,\sigma)}, \tag{9}$$

$$\frac{\eth(\varepsilon_n,\mathscr{I}\varepsilon_n)\,\eth(\sigma,\mathscr{I}\sigma)}{1+\eth(\mathscr{I}\varepsilon_n,\mathscr{I}\sigma)},\eth(\varepsilon_n,\sigma)\},$$

then taking limit as $n\to+\infty$ in the equation (9) and making use of $\lim_{n\to+\infty}\varepsilon_n=\sigma$, we get

$$\lim_{n\to+\infty}\mathscr{P}(\varepsilon_n,\sigma) = \max\{\eth(\sigma,\mathscr{I}\sigma),0\} = \eth(\sigma,\mathscr{I}\sigma), \tag{10}$$

Since, $\varepsilon_n\preceq\sigma$ for each n, then we obtain the following from equations (1) and (9)

$$\check{\psi}(\eth(\varepsilon_{n+1},\mathscr{I}\sigma)) = \check{\psi}(\eth(\mathscr{I}\varepsilon_n,\mathscr{I}\sigma)) \leq \check{\psi}(s\eth(\mathscr{I}\varepsilon_n,\mathscr{I}\sigma))$$

$$\leq \check{\psi}(\mathscr{P}(\varepsilon_n,\sigma)) - \hat{\eta}(\mathscr{P}(\varepsilon_n,\sigma)). \tag{11}$$

Take limit as $n\to+\infty$ in (11) and from equation (10) as well as the properties of $\check{\psi}$, $\hat{\eta}$, we have

$$\check{\psi}(\eth(\sigma,\mathscr{I}\sigma)) \leq \check{\psi}(\eth(\sigma,\mathscr{I}\sigma)) - \hat{\eta}(\eth(\sigma,\mathscr{I}\sigma))$$

$$< \check{\psi}(\eth(\sigma,\mathscr{I}\sigma)). \tag{12}$$

This is a contradiction to $\mathscr{I}\sigma\neq\sigma$. Hence, $\mathscr{I}\sigma=\sigma$.

In the above theorems, the fixed point is unique if $\mathscr{E}$ meets the following condition.

There is an σ in $\mathscr{E}$ that is comparable to ε and $\wp$ for each $\varepsilon,\wp\in\mathscr{E}$. (13)

Theorem 33*If $\mathscr{E}$ assumes the condition (13) in Theorem 31 & 32, then $\mathscr{I}$ has a unique fixed point in $\mathscr{E}$.*

*Proof.*Theorems 31 & 32 show that the set of fixed points of $\mathscr{I}$ is nonempty. Assume $\varepsilon^*\neq\wp^*$ are fixed points of $\mathscr{I}$ to ensure uniqueness. Following that,

$$\check{\psi}(\eth(\mathscr{I}\varepsilon^*,\mathscr{I}\wp^*)) \leq \check{\psi}(s\eth(\mathscr{I}\varepsilon^*,\mathscr{I}\wp^*))$$

$$\leq \check{\psi}(\mathscr{P}(\varepsilon^*,\wp^*)) - \hat{\eta}(\mathscr{P}(\varepsilon^*,\wp^*)) \tag{14}$$

where

$$\mathscr{P}(\varepsilon^*,\wp^*) = \max\{\frac{\eth(\wp^*,\mathscr{I}\wp^*)[1+\eth(\varepsilon^*,\mathscr{I}\varepsilon^*)]}{1+\eth(\varepsilon^*,\wp^*)},$$

$$\frac{\eth(\varepsilon^*,\mathscr{I}\varepsilon^*)\,\eth(\wp^*,\mathscr{I}\wp^*)}{1+\eth(\varepsilon^*,\wp^*)},$$

$$\frac{\eth(\varepsilon^*,\mathscr{I}\varepsilon^*)\,\eth(\wp^*,\mathscr{I}\wp^*)}{1+\eth(\mathscr{I}\varepsilon^*,\mathscr{I}\wp^*)},\eth(\varepsilon^*,\wp^*)\}. \tag{15}$$

Therefore, from equations (14) and (15), we have

$$\check{\psi}(\eth(\varepsilon^*,\wp^*)) = \check{\psi}(\eth(\mathscr{I}\varepsilon^*,\mathscr{I}\wp^*))$$

$$\leq \check{\psi}(\eth(\varepsilon^*,\wp^*)) - \hat{\eta}(\eth(\varepsilon^*,\wp^*)) \tag{16}$$

$$< \check{\psi}(\eth(\varepsilon^*,\wp^*)),$$

this contradicts to $\varepsilon^*\neq\wp^*$. Hence, $\varepsilon^*=\wp^*$.

Now, we have the below corollary from Theorems 31 to 33.

Corollary 34*Let $(\mathscr{E},\eth,\preceq)$ be a partially ordered b-metric space. Suppose the mappings $\mathscr{I},\hbar:\mathscr{E}\to\mathscr{E}$ are continuous such that*

(C_1).

$$\check{\psi}(s\eth(\mathscr{I}\varepsilon, \mathscr{I}\wp)) \leq \check{\psi}(\mathscr{P}_\hbar(\varepsilon, \wp)) - \hat{\eta}(\mathscr{P}_\hbar(\varepsilon, \wp)) \tag{17}$$

for every $\varepsilon, \wp \in \mathscr{E}$ with $\hbar\varepsilon \preceq \hbar\wp, s > 1, \check{\psi} \in \hat{\Phi}, \hat{\eta} \in \hat{\Psi}$ and, where

$$\mathscr{P}_\hbar(\varepsilon, \wp) = \max\{\frac{\eth(\hbar\wp, \mathscr{I}\wp)[1 + \eth(\hbar\varepsilon, \mathscr{I}\varepsilon)]}{1 + \eth(\hbar\varepsilon, \hbar\wp)},$$
$$\frac{\eth(\hbar\varepsilon, \mathscr{I}\varepsilon)\,\eth(\hbar\wp, \mathscr{I}\wp)}{1 + \eth(\hbar\varepsilon, \hbar\wp)},$$
$$\frac{\eth(\hbar\varepsilon, \mathscr{I}\varepsilon)\,\eth(\hbar\wp, \mathscr{I}\wp)}{1 + \eth(\mathscr{I}\varepsilon, \mathscr{I}\wp)},$$
$$\eth(\hbar\varepsilon, \hbar\wp)\}. \tag{18}$$

(C_2).$\mathscr{I}\mathscr{E} \subset \hbar\mathscr{E}$ and $\hbar\mathscr{E} \subseteq \mathscr{E}$ is complete,
(C_3).$\mathscr{I}$ is monotone $\hbar$-non-decreasing and
(C_4).$\mathscr{I}$ and $\hbar$ are compatible.

If for some $\varepsilon_0 \in \mathscr{E}$ such that $\hbar\varepsilon_0 \preceq \mathscr{I}\varepsilon_0$, then there is a coincidence point in $\mathscr{E}$ for a pair of mappings $(\mathscr{I}, \hbar)$.

*Proof.*According to lemma 212, there is a subset $\mathscr{M}$ of $\mathscr{E}$ so that $\hbar\mathscr{M} \subset \mathscr{E}$ is a complete subspace, and $\hbar : \mathscr{E} \to \mathscr{E}$ is one-to-one. Following [27]'s Corollary 2.1, there is a sequence $\{\hbar\varepsilon_n\} \subset \hbar\mathscr{M}$ for some $\varepsilon_0 \in \mathscr{M}$ so that $\hbar\varepsilon_{n+1} = \mathscr{I}\varepsilon_n = \hbar(\hbar\varepsilon_n), (n \geq 0)$ and, where $\hbar : \hbar\mathscr{M} \to \hbar\mathscr{M}$ is a mapping so that $\hbar(\hbar\varepsilon) = \mathscr{I}\varepsilon, \varepsilon \in \mathscr{M}$.

Thus from equation (17), we get

$$\check{\psi}(s\eth(\hbar(\hbar\varepsilon), \hbar(\hbar\wp))) \leq \check{\psi}(\mathscr{P}_\hbar(\varepsilon, \wp)) - \hat{\eta}(\mathscr{P}_\hbar(\varepsilon, \wp)), \tag{19}$$

for every $\varepsilon, \wp \in \mathscr{E}$ with $\hbar\varepsilon \preceq \hbar\wp$ and, where

$$\mathscr{P}_\hbar(\varepsilon, \wp) = \max\{\frac{\eth(\hbar\wp, \hbar(\hbar\wp))[1 + \eth(\hbar\varepsilon, \hbar(\hbar\varepsilon))]}{1 + \eth(\hbar\varepsilon, \hbar\wp)},$$
$$\frac{\eth(\hbar\varepsilon, \hbar(\hbar\varepsilon))\,\eth(\hbar\wp, \hbar(\hbar\wp))}{1 + \eth(\hbar\varepsilon, \hbar\wp)},$$
$$\frac{\eth(\hbar\varepsilon, \hbar(\hbar\varepsilon))\,\eth(\hbar\wp, \hbar(\hbar\wp))}{1 + \eth(\hbar(\hbar\varepsilon), \hbar(\hbar\wp))},$$
$$\eth(\hbar\varepsilon, \hbar\wp)\}. \tag{20}$$

We can deduce from Theorem 31 that $\{\hbar\varepsilon_n\} \subset \hbar\mathscr{M}$ is a b-Cauchy sequence that converging on $\upsilon \in \hbar\mathscr{M}$.

We get from the condition (C_4) that,

$$\lim_{n \to +\infty} \eth(\hbar(\mathscr{I}\varepsilon_n), \mathscr{I}(\hbar\varepsilon_n)) = 0.$$

We have from a b-metrics triangular inequality that

$$\eth(\mathscr{I}\upsilon, \hbar\upsilon) \leq s\eth(\mathscr{I}\upsilon, \mathscr{I}(\hbar\varepsilon_n)) + s^2\eth(\mathscr{I}(\hbar\varepsilon_n), \hbar(\mathscr{I}\varepsilon_n))$$
$$+ s^2\eth(\hbar(\mathscr{I}\varepsilon_n), \hbar\upsilon). \tag{21}$$

As $n \to +\infty$ in (21), $\eth(\mathscr{I}\upsilon, \hbar\upsilon) = 0$ this indicates that υ is a coincidence point of $\mathscr{I}, \hbar$.

The following result can get from Corollary 34 by weakening its hypotheses.

Corollary 35 If $\mathscr{E}$ satisfies the following condition in Corollary 34,

for very nondecreasing sequence $\{\hbar\varepsilon_n\} \subseteq \mathscr{E}$ so that $\hbar\varepsilon_n \to \hbar\sigma, then\hbar\varepsilon_n \leq \hbar\sigma \ (n \geq 0)$, i.e., $\hbar\sigma = \sup\hbar\varepsilon_n$. (22)

then, if $\hbar\mu \preceq \hbar(\hbar\mu)$ for some coincidence point μ, a coincidence point exists for the weakly compatible mappings $(\mathscr{I}, \hbar)$. Moreover, $(\mathscr{I}, \hbar)$ has only one common fixed point iff the set of common fixed points is well ordered.

*Proof.*A pair of maps $(\mathscr{I}, \hbar)$ has a coincidence point, according to Theorem 33 and Corollary 34.

Next, assume $(\mathscr{I}, \hbar)$ is only weakly compatible. Let $\upsilon \in \mathscr{E}$ be a point with $\upsilon = \mathscr{I}\mu = \hbar\mu$. Thence, $\mathscr{I}\upsilon = \mathscr{I}(\hbar\mu) = \hbar(\mathscr{I}\mu) = \hbar\upsilon$.

Therefore,

$$\mathscr{P}_\hbar(\mu, \upsilon) = \max\{\frac{\eth(\hbar\upsilon, \mathscr{I}\upsilon)[1 + \eth(\hbar\mu, \mathscr{I}\mu)]}{1 + \eth(\hbar\mu, \hbar\upsilon)},$$
$$\frac{\eth(\hbar\mu, \mathscr{I}\mu)\,\eth(\hbar\upsilon, \mathscr{I}\upsilon)}{1 + \eth(\hbar\mu, \hbar\upsilon)},$$
$$\frac{\eth(\hbar\mu, \mathscr{I}\mu)\,\eth(\hbar\upsilon, \mathscr{I}\upsilon)}{1 + \eth(\mathscr{I}\mu, \mathscr{I}\upsilon)}, \eth(\hbar\mu, \hbar\upsilon)\}$$
$$= \max\{0, \eth(\mathscr{I}\mu, \mathscr{I}\upsilon)\}$$
$$= \eth(\mathscr{I}\mu, \mathscr{I}\upsilon). \tag{23}$$

Thus from equation (17), we get

$$\check{\psi}(\eth(\mathscr{I}\mu, \mathscr{I}\upsilon)) \leq \check{\psi}(\mathscr{P}_\hbar(\mu, \upsilon)) - \hat{\eta}(\mathscr{P}_\hbar(\mu, \upsilon))$$
$$\leq \check{\psi}(\eth(\mathscr{I}\mu, \mathscr{I}\upsilon)) - \hat{\eta}(\eth(\mathscr{I}\mu, \mathscr{I}\upsilon)). \tag{24}$$

By the property of $\hat{\eta}$, we get $\eth(\mathscr{I}\mu, \mathscr{I}\upsilon) = 0$ implies that $\mathscr{I}\upsilon = \hbar\upsilon = \upsilon$.

Finally, we can deduce from Theorem 33 that $(\mathscr{I}, \hbar)$ only has one common fixed point iff the common fixed points of $(\mathscr{I}, \hbar)$ is well ordered.

Remark 36 Theorems *31 to 33 are the extension of Theorems 2.1,.2.2 & 2.3 of [12].*

Remark 37 Corollaries *34 & 35 are the generalizations of Corollaries 2.1 & 2.2 of [27] respectively.*

Definition 38 Consider *the partially ordered b-metric space, $(\mathscr{E}, \eth, \preceq)$. A map $\mathscr{I} : \mathscr{E} \times \mathscr{E} \to \mathscr{E}$ is known to be a generalized $(\check{\psi}, \hat{\eta})$-contractive map with regards to $\hbar : \mathscr{E} \to \mathscr{E}$, if*

$$\check{\psi}(s^k\eth(\mathscr{I}(\varepsilon, \wp), \mathscr{I}(\zeta, \mathfrak{I}))) \leq \check{\psi}(\mathscr{P}_\hbar(\varepsilon, \wp, \zeta, \mathfrak{I}))$$
$$- \hat{\eta}(\mathscr{P}_\hbar(\varepsilon, \wp, \zeta, \mathfrak{I})), \tag{25}$$

for all $\varepsilon, \wp, \zeta, \Im \in \mathscr{E}$ *with* $\hbar\varepsilon \preceq \hbar\zeta$ *and* $\hbar\wp \succeq \hbar\Im$, $k > 2$, $s > 1$, $\breve{\psi} \in \hat{\Phi}$, $\hat{\eta} \in \hat{\Psi}$ *and where*

$$\mathscr{P}_{\hbar}(\varepsilon, \wp, \zeta, \Im) = \max\{\frac{\eth(\hbar\zeta, \mathscr{I}(\zeta, \Im))[1 + \eth(\hbar\varepsilon, \mathscr{I}(\varepsilon, \wp))]}{1 + \eth(\hbar\varepsilon, \hbar\zeta)},$$
$$\frac{\eth(\hbar\varepsilon, \mathscr{I}(\varepsilon, \wp))\,\eth(\hbar\zeta, \mathscr{I}(\zeta, \Im))}{1 + \eth(\hbar\varepsilon, \hbar\zeta)},$$
$$\frac{\eth(\hbar\varepsilon, \mathscr{I}(\varepsilon, \wp))\,\eth(\hbar\zeta, \mathscr{I}(\zeta, \Im))}{1 + \eth(\mathscr{I}(\varepsilon, \wp), \mathscr{I}(\zeta, \Im))},$$
$$\eth(\hbar\varepsilon, \hbar\zeta)\}.$$

Theorem 39 *Suppose* $(\mathscr{E}, \eth, \preceq)$ *be a complete partially ordered b-metric space. A map* $\mathscr{I} : \mathscr{E} \times \mathscr{E} \to \mathscr{E}$ *satisfies the condition* (25) *and,* $\mathscr{I}$, $\hbar$ *are continuous,* $\mathscr{I}$ *has mixed* $\hbar$-*monotone property and also commutes with* $\hbar$. *Assume, if some* $(\varepsilon_0, \wp_0) \in \mathscr{E} \times \mathscr{E}$ *so that* $\hbar\varepsilon_0 \preceq \mathscr{I}(\varepsilon_0, \wp_0)$, $\hbar\wp_0 \succeq \mathscr{I}(\wp_0, \varepsilon_0)$ *and* $\mathscr{I}(\mathscr{E} \times \mathscr{E}) \subseteq \hbar(\mathscr{E})$, *then* $\mathscr{I}$ *and* $\hbar$ *in* $\mathscr{E}$ *have a coupled coincidence point.*

Proof. From [7] of Theorem 2.2, there will be two sequences $\{\varepsilon_n\}, \{\wp_n\} \subset \mathscr{E}$ so that

$$\hbar\varepsilon_{n+1} = \mathscr{I}(\varepsilon_n, \wp_n), \quad \hbar\wp_{n+1} = \mathscr{I}(\wp_n, \varepsilon_n), n \geq 0.$$

In particular, the sequences $\{\hbar\varepsilon_n\}$, $\{\hbar\wp_n\}$ are non-decreasing and non-increasing in $\mathscr{E}$. Put $\varepsilon = \varepsilon_n, \wp = \wp_n, \zeta = \varepsilon_{n+1}, \Im = \wp_{n+1}$ in (25), we get

$$\breve{\psi}(s^k\eth(\hbar\varepsilon_{n+1}, \hbar\varepsilon_{n+2})) = \breve{\psi}(s^k\eth(\mathscr{I}(\varepsilon_n, \wp_n), \mathscr{I}(\varepsilon_{n+1}, \wp_{n+1})))$$
$$\leq \breve{\psi}(\mathscr{P}_{\hbar}(\varepsilon_n, \wp_n, \varepsilon_{n+1}, \wp_{n+1}))$$
$$- \hat{\eta}(\mathscr{P}_{\hbar}(\varepsilon_n, \wp_n, \varepsilon_{n+1}, \wp_{n+1})), \tag{26}$$

where

$$\mathscr{P}_{\hbar}(\varepsilon_n, \wp_n, \varepsilon_{n+1}, \wp_{n+1}) \leq \max\{\eth(\hbar\varepsilon_n, \hbar\varepsilon_{n+1}),$$
$$\eth(\hbar\varepsilon_{n+1}, \hbar\varepsilon_{n+2})\} \tag{27}$$

As a result of (26), we get

$$\breve{\psi}(s^k\eth(\hbar\varepsilon_{n+1}, \hbar\varepsilon_{n+2}))$$
$$\leq \breve{\psi}(\max\{\eth(\hbar\varepsilon_n, \hbar\varepsilon_{n+1}), \eth(\hbar\varepsilon_{n+1}, \hbar\varepsilon_{n+2})\}) \tag{28}$$
$$- \hat{\eta}(\max\{\eth(\hbar\varepsilon_n, \hbar\varepsilon_{n+1}), \eth(\hbar\varepsilon_{n+1}, \hbar\varepsilon_{n+2})\}).$$

Likewise by taking $\varepsilon = \wp_{n+1}, \wp = \varepsilon_{n+1}, \zeta = \varepsilon_n, \Im = \varepsilon_n$ in (25), we get

$$\breve{\psi}(s^k\eth(\hbar\wp_{n+1}, \hbar\wp_{n+2}))$$
$$\leq \breve{\psi}(\max\{\eth(\hbar\wp_n, \hbar\wp_{n+1}), \eth(\hbar\wp_{n+1}, \hbar\wp_{n+2})\}) \tag{29}$$
$$- \hat{\eta}(\max\{\eth(\hbar\wp_n, \hbar\wp_{n+1}), \eth(\hbar\wp_{n+1}, \hbar\wp_{n+2})\}).$$

We know that $\max\{\breve{\psi}(l_1), \breve{\psi}(l_2)\} = \breve{\psi}\{\max\{l_1, l_2\}\}$ for $l_1, l_2 \in [0, +\infty)$. Then we add (28) and (29) together to get,

$$\breve{\psi}(s^k\Gamma_n)$$
$$\leq \breve{\psi}(\max\{\eth(\hbar\varepsilon_n, \hbar\varepsilon_{n+1}), \eth(\hbar\varepsilon_{n+1}, \hbar\varepsilon_{n+2}),$$
$$\eth(\hbar\wp_n, \hbar\wp_{n+1}), \eth(\hbar\wp_{n+1}, \hbar\wp_{n+2})\}) \tag{30}$$
$$- \hat{\eta}(\max\{\eth(\hbar\varepsilon_n, \hbar\varepsilon_{n+1}), \eth(\hbar\varepsilon_{n+1}, \hbar\varepsilon_{n+2}),$$
$$\eth(\hbar\wp_n, \hbar\wp_{n+1}), \eth(\hbar\wp_{n+1}, \hbar\wp_{n+2})\})$$

where

$$\Gamma_n = \max\{\eth(\hbar\varepsilon_{n+1}, \hbar\varepsilon_{n+2}), \eth(\hbar\wp_{n+1}, \hbar\wp_{n+2})\}. \tag{31}$$

Let us denote,

$$\varkappa_n = \max\{\eth(\hbar\varepsilon_n, \hbar\varepsilon_{n+1}), \eth(\hbar\varepsilon_{n+1}, \hbar\varepsilon_{n+2}), \eth(\hbar\wp_n, \hbar\wp_{n+1}),$$
$$\eth(\hbar\wp_{n+1}, \hbar\wp_{n+2})\}. \tag{32}$$

Hence from equations (28)-(31), we obtain

$$s^k\Gamma_n \leq \varkappa_n. \tag{33}$$

Now to claim that

$$\Gamma_n \leq \lambda\Gamma_{n-1}, \tag{34}$$

for $n \geq 1$ and $\lambda = \frac{1}{s^k} \in [0, 1)$.

Suppose that if $\varkappa_n = \Gamma_n$ then from (33), we get $s^k\Gamma_n \leq \Gamma_n$ this leads to $\Gamma_n = 0$ since $s > 1$ and thus (34) holds.

Suppose $\varkappa_n = \max\{\eth(\hbar\varepsilon_n, \hbar\varepsilon_{n+1}), \eth(\hbar\wp_n, \hbar\wp_{n+1})\}$, i.e., $\varkappa_n = \Gamma_{n-1}$ then (33) follows (34).

Now from (33), we obtain that $\Gamma_n \leq \lambda^n\delta_0$ and hence,

$$\eth(\hbar\varepsilon_{n+1}, \hbar\varepsilon_{n+2}) \leq \lambda^n\Gamma_0 \text{ and } \eth(\hbar\wp_{n+1}, \hbar\wp_{n+2}) \leq \lambda^n\Gamma_0, \tag{35}$$

which shows that $\{\hbar\varepsilon_n\}$, $\{\hbar\wp_n\}$ in $\mathscr{E}$ are Cauchy sequences by Lemma 3.1 of [22]. Therefore, we can conclude from Theorem 2.2 of [5] that in $\mathscr{E}$, $\mathscr{I}$ and $\hbar$ have a coincidence point.

Corollary 310 *Suppose* $(\mathscr{E}, \eth, \preceq)$ *be a complete partially ordered b-metric space. A continuous map* $\mathscr{I} : \mathscr{E} \times \mathscr{E} \to \mathscr{E}$ *has mixed monotone property is satisfying the below contraction conditions for all* $\varepsilon, \wp, \zeta, \Im \in \mathscr{E}$ *such that* $\varepsilon \preceq \zeta$ *and* $\wp \succeq \Im$, $k > 2$, $s > 1$, $\breve{\psi} \in \hat{\Phi}$ *and* $\hat{\eta} \in \hat{\Psi}$:

(i).

$$\breve{\psi}(s^k\eth(\mathscr{I}(\varepsilon, \wp), \mathscr{I}(\zeta, \Im))) \leq \breve{\psi}(\mathscr{P}_{\hbar}(\varepsilon, \wp, \zeta, \Im))$$
$$- \hat{\eta}(\mathscr{P}_{\hbar}(\varepsilon, \wp, \zeta, \Im)),$$

(ii).

$$\eth(\mathscr{I}(\varepsilon, \wp), \mathscr{I}(\zeta, \Im)) \leq \frac{1}{s^k}\mathscr{P}_{\hbar}(\varepsilon, \wp, \zeta, \Im)$$
$$- \frac{1}{s^k}\hat{\eta}(\mathscr{P}_{\hbar}(\varepsilon, \wp, \zeta, \Im)).$$

where

$$\mathscr{P}_{\hbar}(\varepsilon, \wp, \zeta, \Im) = \max\{\frac{\eth(\zeta, \mathscr{I}(\zeta, \Im))[1 + \eth(\varepsilon, \mathscr{I}(\varepsilon, \wp))]}{1 + \eth(\varepsilon, \zeta)},$$
$$\frac{\eth(\varepsilon, \mathscr{I}(\varepsilon, \wp))\,\eth(\zeta, \mathscr{I}(\zeta, \Im))}{1 + \eth(\varepsilon, \zeta)},$$
$$\frac{\eth(\varepsilon, \mathscr{I}(\varepsilon, \wp))\,\eth(\zeta, \mathscr{I}(\zeta, \Im))}{1 + \eth(\mathscr{I}(\varepsilon, \wp), \mathscr{I}(\zeta, \Im))}, \eth(\varepsilon, \zeta)\}.$$

If there exists $(\varepsilon_0, \wp_0) \in \mathscr{E} \times \mathscr{E}$ *so that* $\varepsilon_0 \preceq \mathscr{I}(\varepsilon_0, \wp_0)$ *and* $\wp_0 \succeq \mathscr{I}(\wp_0, \varepsilon_0)$, *then* $\mathscr{I}$ *in* $\mathscr{E}$ *has a coupled fixed point.*

Theorem 311*The unique coupled common fixed point for* $\mathscr{I}$ *and* $\hbar$ *exists in Theorem 39, if for every* $(\varepsilon, \wp), (\Bbbk, \ell) \in \mathscr{E} \times \mathscr{E}$ *there is some* $(\Lambda, \Upsilon) \in \mathscr{E} \times \mathscr{E}$ *such that* $(\mathscr{I}(\Lambda, \Upsilon), \mathscr{I}(\Upsilon, \Lambda))$ *is comparable to* $(\mathscr{I}(\varepsilon, \wp), \mathscr{I}(\wp, \varepsilon))$ *and to* $(\mathscr{I}(\Bbbk, \mathscr{I}), \mathscr{I}(\ell, \Bbbk))$.

*Proof.*The existence of a coupled coincidence point for $\mathscr{I}$, $\hbar$ is guaranteed by the Theorem 39. Let $(\varepsilon, \wp), (\Bbbk, \ell) \in \mathscr{E} \times \mathscr{E}$ are coupled coincidence points of $\mathscr{I}$, $\hbar$. Now, we assert that $\hbar\varepsilon = \hbar\Bbbk$ and $\hbar\wp = \hbar\ell$. By hypotheses $(\mathscr{I}(\Lambda, \Upsilon), \mathscr{I}(\Upsilon, \Lambda))$ is comparable to $(\mathscr{I}(\varepsilon, \wp), \mathscr{I}(\wp, \varepsilon))$ and $(\mathscr{I}(\Bbbk, \mathscr{I}), \mathscr{I}(\ell, \Bbbk))$ for some $(\Lambda, \Upsilon) \in \mathscr{E} \times \mathscr{E}$.

Now, assume the following

$$(\mathscr{I}(\varepsilon, \wp), \mathscr{I}(\wp, \varepsilon)) \leq (\mathscr{I}(\Lambda, \Upsilon), \mathscr{I}(\Upsilon, \Lambda)) \text{ and}$$
$$(\mathscr{I}(\Bbbk, \ell), \mathscr{I}(\ell, \Bbbk)) \leq (\mathscr{I}(\Lambda, \Upsilon), \mathscr{I}(\Upsilon, \Lambda)).$$

Suppose $\Lambda_0 = \Lambda$ and $\Upsilon_0 = \Upsilon$ then there is a point $(\Lambda_1, \Upsilon_1) \in \mathscr{E} \times \mathscr{E}$ such that

$$\hbar\Lambda_1 = \mathscr{I}(\Lambda_0, \Upsilon_0), \quad \hbar\Upsilon_1 = \mathscr{I}(\Upsilon_0, \Lambda_0) \quad (n \geq 1).$$

As by applying the preceding argument repeatedly, we have the sequences $\{\hbar\Lambda_n\}$ and $\{\hbar\Upsilon_n\}$ in $\mathscr{E}$ with

$$\hbar\Lambda_{n+1} = \mathscr{I}(\Lambda_n, \Upsilon_n), \quad \hbar\Upsilon_{n+1} = \mathscr{I}(\Upsilon_n, \Lambda_n) \quad (n \geq 0).$$

Define the sequences in the same way $\{\hbar\varepsilon_n\}$, $\{\hbar\wp_n\}$ and, $\{\hbar\Bbbk_n\}$, $\{\hbar\ell_n\}$ in $\mathscr{E}$ by setting $\varepsilon_0 = \varepsilon$, $\wp_0 = \wp$ and $\Bbbk_0 = \Bbbk$, $\ell_0 = \ell$. Further, we have that

$$\begin{aligned} &\hbar\varepsilon_n \to \mathscr{I}(\varepsilon, \wp), \ \hbar\wp_n \to \mathscr{I}(\wp, \varepsilon), \\ &\hbar\Bbbk_n \to \mathscr{I}(\Bbbk, \ell), \ \hbar\ell_n \to \mathscr{I}(\ell, \Bbbk)(n \geq 1). \end{aligned} \tag{36}$$

Thus by induction, we get

$$(\hbar\varepsilon_n, \hbar\wp_n) \leq (\hbar\Lambda_n, \hbar\Upsilon_n) \text{ for every } n. \tag{37}$$

As a consequence of (25), we have

$$\begin{aligned} \breve{\psi}(\eth(\hbar\varepsilon, \hbar\Lambda_{n+1})) &\leq \breve{\psi}(s^k \eth(\hbar\varepsilon, \hbar\Lambda_{n+1})) \\ &= \breve{\psi}(s^k \eth(\mathscr{I}(\varepsilon, \wp), \mathscr{I}(\Lambda_n, \Upsilon_n))) \\ &\leq \breve{\psi}(\mathscr{P}_\hbar(\varepsilon, \wp, \Lambda_n, \Upsilon_n)) \\ &\quad - \hat{\eta}(\mathscr{P}_\hbar(\varepsilon, \wp, \Lambda_n, \Upsilon_n)), \end{aligned} \tag{38}$$

where

$$\begin{aligned} &\mathscr{P}_\hbar(\varepsilon, \wp, \Lambda_n, \Upsilon_n) \\ &= \max\Big\{ \frac{\eth(\hbar\Lambda_n, \mathscr{I}(\Lambda_n, \Upsilon_n))[1 + \eth(\hbar\varepsilon, \mathscr{I}(\varepsilon, \wp))]}{1 + \eth(\hbar\varepsilon, \hbar\Lambda_n)}, \\ &\quad \frac{\eth(\hbar\varepsilon, \mathscr{I}(\varepsilon, \wp))\, \eth(\hbar\Lambda_n, \mathscr{I}(\Lambda_n, \Upsilon_n))}{1 + \eth(\hbar\varepsilon, \hbar\Lambda_n)}, \\ &\quad \frac{\eth(\hbar\varepsilon, \mathscr{I}(\varepsilon, \wp))\, \eth(\hbar\Lambda_n, \mathscr{I}(\Lambda_n, \Upsilon_n))}{1 + \eth(\mathscr{I}(\varepsilon, \wp), \mathscr{I}(\Lambda_n, \Upsilon_n))}, \\ &\quad \eth(\hbar\varepsilon, \hbar\Lambda_n) \Big\} \\ &= \max\{0, \eth(\hbar\varepsilon, \hbar\Lambda_n)\} \\ &= \eth(\hbar\varepsilon, \hbar\Lambda_n). \end{aligned}$$

As a result of (38), we now have

$$\breve{\psi}(\eth(\hbar\varepsilon, \hbar\Lambda_{n+1})) \leq \breve{\psi}(\eth(\hbar\varepsilon, \hbar\Lambda_n)) - \hat{\eta}(\eth(\hbar\varepsilon, \hbar\Lambda_n)). \tag{39}$$

As by the similar argument, we acquire that

$$\breve{\psi}(\eth(\hbar\wp, \hbar\Upsilon_{n+1})) \leq \breve{\psi}(\eth(\hbar\wp, \hbar\Upsilon_n)) - \hat{\eta}(\eth(\hbar\wp, \hbar\Upsilon_n)). \tag{40}$$

Hence from (39) and (40), we have

$$\begin{aligned} &\breve{\psi}(\max\{\eth(\hbar\varepsilon, \hbar\Lambda_{n+1}), \eth(\hbar\wp, \hbar\Upsilon_{n+1})\}) \\ &\leq \breve{\psi}(\max\{\eth(\hbar\varepsilon, \hbar\Lambda_n), \eth(\hbar\wp, \hbar\Upsilon_n)\}) \\ &\quad - \hat{\eta}(\max\{\eth(\hbar\varepsilon, \hbar\Lambda_n), \eth(\hbar\wp, \hbar\Upsilon_n)\}) \\ &< \breve{\psi}(\max\{\eth(\hbar\varepsilon, \hbar\Lambda_n), \eth(\hbar\wp, \hbar\Upsilon_n)\}). \end{aligned} \tag{41}$$

Thus, the property of $\breve{\psi}$ implies,

$$\begin{aligned} &\max\{\eth(\hbar\varepsilon, \hbar\Lambda_{n+1}), \eth(\hbar\wp, \hbar\Upsilon_{n+1})\} \\ &< \max\{\eth(\hbar\varepsilon, \hbar\Lambda_n), \eth(\hbar\wp, \hbar\Upsilon_n)\}. \end{aligned}$$

Hence, $\max\{\eth(\hbar\varepsilon, \hbar\Lambda_n), \eth(\hbar\wp, \hbar\Upsilon_n)\}$ is a decreasing sequence of positive reals and bounded below and by a result, we have

$$\lim_{n \to +\infty} \max\{\eth(\hbar\varepsilon, \hbar\Lambda_n), \eth(\hbar\wp, \hbar\Upsilon_n)\} = \Gamma, \ \Gamma \geq 0.$$

Therefore as $n \to +\infty$ in equation (41), we get

$$\breve{\psi}(\Gamma) \leq \breve{\psi}(\Gamma) - \hat{\eta}(\Gamma), \tag{42}$$

from which we have $\hat{\eta}(\Gamma) = 0$, implies that $\Gamma = 0$. Therefore,

$$\lim_{n \to +\infty} \max\{\eth(\hbar\varepsilon, \hbar\Lambda_n), \eth(\hbar\wp, \hbar\Upsilon_n)\} = 0.$$

Hence, we have that,

$$\lim_{n \to +\infty} \eth(\hbar\varepsilon, \hbar\Lambda_n) = 0 \text{ and } \lim_{n \to +\infty} \eth(\hbar\wp, \hbar\Upsilon_n) = 0. \tag{43}$$

By the similar argument as above, we obtain

$$\lim_{n \to +\infty} \eth(\hbar\Bbbk, \hbar\Lambda_n) = 0 \text{ and } \lim_{n \to +\infty} \eth(\hbar\mathscr{I}, \hbar\Upsilon_n) = 0. \tag{44}$$

Therefore from (43) and (44), we get $\hbar\varepsilon = \hbar\Bbbk$ and $\hbar\wp = \hbar\mathscr{I}$. Since $\hbar\varepsilon = \mathscr{I}(\varepsilon, \wp)$ and $\hbar\wp = \mathscr{I}(\wp, \varepsilon)$ and, the commutativity property of $\mathscr{I}$, $\hbar$ implies that

$$\begin{aligned} &\hbar(\hbar\varepsilon) = \hbar(\mathscr{I}(\varepsilon, \wp)) = \mathscr{I}(\hbar\varepsilon, \hbar\wp) \text{ and} \\ &\hbar(\hbar\wp) = \hbar(\mathscr{I}(\wp, \varepsilon)) = \mathscr{I}(\hbar\wp, \hbar\varepsilon). \end{aligned} \tag{45}$$

If $\hbar\varepsilon = \Lambda^*$ and $\hbar\wp = \Upsilon^*$ then from (45), we get

$$\hbar(\Lambda) = \mathscr{I}(\Lambda^*, \Upsilon^*) \text{ and } \hbar(\Upsilon^*) = \mathscr{I}(\Upsilon^*, \Lambda^*), \tag{46}$$

which exhibits that (Λ^*, Υ^*) is a coupled coincidence point of $\mathscr{I}$, $\hbar$. Hence, $\hbar(\Lambda^*) = \hbar\Bbbk$ and $\hbar(\Upsilon^*) = \hbar\mathscr{I}$ which in turn gives that $\hbar(\Lambda) = \Lambda^*$ and $\hbar(\Upsilon^*) = \Upsilon^*$. Therefore

from (46), (Λ^*, Υ^*) is a coupled common fixed point of $\mathcal{I}$, $\hbar$.

Let $(\Lambda_1^*, \Upsilon_1^*)$ be another coupled common fixed point of $\mathcal{I}$, $\hbar$. Thence, $\Lambda_1^* = \hbar \Lambda_1^* = \mathcal{I}(\Lambda_1^*, \Upsilon_1^*)$ and $\Upsilon_1^* = \hbar \Upsilon_1^* = \mathcal{I}(\Upsilon_1^*, \Lambda_1^*)$. But $(\Lambda_1^*, \Upsilon_1^*)$ is a coupled common fixed point of $\mathcal{I}$, $\hbar$ then, $\hbar \Lambda_1^* = \hbar \varepsilon = \Lambda$ and $\hbar \Upsilon_1^* = \hbar \wp = \Upsilon^*$. Therefore, $\Lambda_1^* = \hbar \Lambda_1^* = \hbar \Lambda = \Lambda$ and $\Upsilon_1^* = \hbar \Upsilon_1^* = \hbar \Upsilon^* = \Upsilon^*$. Hence the uniqueness.

Theorem 312 *In Theorem 311, if $\hbar \varepsilon_0 \preceq \hbar \wp_0$ or $\hbar \varepsilon_0 \succeq \hbar \wp_0$, then an unique common fixed point of $\mathcal{I}$, $\hbar$ can be found.*

Proof. Assume $(\varepsilon, \wp) \in \mathcal{E}$ is a unique coupled common fixed point of $\mathcal{I}$, $\hbar$. Then, to demonstrate that $\varepsilon = \wp$. Suppose that $\hbar \varepsilon_0 \preceq \hbar \wp_0$ then we get by induction, $\hbar \varepsilon_n \preceq \hbar \wp_n$, $n \geq 0$. From Lemma 2 of [23], we have

$$
\begin{aligned}
\breve{\psi}(s^{k-2}\eth(\varepsilon, \wp)) &= \breve{\psi}(s^k \frac{1}{s^2}\eth(\varepsilon, \wp)) \\
&\leq \lim_{n \to +\infty} \sup \breve{\psi}(s^k \eth(\varepsilon_{n+1}, \wp_{n+1})) \\
&= \lim_{n \to +\infty} \sup \breve{\psi}(s^k \eth(\mathcal{I}(\varepsilon_n, \wp_n), \mathcal{I}(\wp_n, \varepsilon_n))) \\
&\leq \lim_{n \to +\infty} \sup \breve{\psi}(\mathscr{P}_\hbar(\varepsilon_n, \wp_n, \wp_n, \varepsilon_n)) \\
&\quad - \lim_{n \to +\infty} \inf \hat{\eta}(\mathscr{P}_\hbar(\varepsilon_n, \wp_n, \wp_n, \varepsilon_n)) \\
&\leq \breve{\psi}(\eth(\varepsilon, \wp)) \\
&\quad - \lim_{n \to +\infty} \inf \hat{\eta}(\mathscr{P}_\hbar(\varepsilon_n, \wp_n, \wp_n, \varepsilon_n)) \\
&< \breve{\psi}(\eth(\varepsilon, \wp)),
\end{aligned}
$$

a contradiction. Hence, $\varepsilon = \wp$.

The result can also see in the case of $\hbar \varepsilon_0 \succeq \hbar \wp_0$.

Remark 313 *While $s = 1$ and the result of [21], the condition*

$$
\begin{aligned}
\breve{\psi}(\eth(\mathcal{I}(\varepsilon, \wp), \mathcal{I}(\eth, \Im))) \\
\leq \breve{\psi}(\max\{\eth(\hbar \varepsilon, \hbar \eth), \eth(\hbar \wp, \hbar \Im)\}) \\
- \hat{\eta}(\max\{\eth(\hbar \varepsilon, \hbar \eth), \eth(\hbar \wp, \hbar \Im)\})
\end{aligned}
$$

is equivalent to,

$$
\eth(\mathcal{I}(\varepsilon, \wp), \mathcal{I}(\eth, \Im)) \leq \varphi(\max\{\eth(\hbar \varepsilon, \hbar \eth), \eth(\hbar \wp, \hbar \Im)\}),
$$

here $\breve{\psi} \in \breve{\psi}$, $\hat{\eta} \in \hat{\eta}$ and φ is a continuous self map on $[0, +\infty)$ with $\varphi(y) < y$ for every $y > 0$ with $\varphi(y) = 0$ iff $y = 0$. Hence the results found here are generalized and extended the results of [11, 18, 25, 26, 27] and a number of comparable results.

Now, depending on the type of metric, some examples are shown.

Example 314 *Let $\mathcal{E} = \{e_{11}, e_{22}, e_{33}, e_{44}, e_{55}, e_{66}\}$ and $\eth : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ be a metric defined by*

$$
\begin{aligned}
&\eth(\varepsilon, \wp) = \eth(\wp, \varepsilon) = 0, \\
&\quad if\ \varepsilon = \wp = \{e_{11}, e_{22}, e_{33}, e_{44}, e_{55}, e_{66}\}\ and\ \varepsilon = \wp, \\
&\eth(\varepsilon, \wp) = \eth(\wp, \varepsilon) = 3, \\
&\quad if\ \varepsilon = \wp = \{e_{11}, e_{22}, e_{33}, e_{44}, e_{55}\}\ and\ \varepsilon \neq \wp, \\
&\eth(\varepsilon, \wp) = \eth(\wp, \varepsilon) = 12, \\
&\quad if\ \varepsilon = \{e_{11}, e_{22}, e_{33}, e_{44}\}\ and\ \wp = e_{66}, \\
&\eth(\varepsilon, \wp) = \eth(\wp, \varepsilon) = 20,\ if\ \varepsilon = e_{55}\ and\ \wp = e_{66}, \\
&\quad with\ usual\ order\ \leq.
\end{aligned}
$$

A self-map $\mathcal{I}$ on $\mathcal{E}$ defined by $\mathcal{I}e_{11} = \mathcal{I}e_{22} = \mathcal{I}e_{33} = \mathcal{I}e_{44} = \mathcal{I}e_{55} = 1, \mathcal{I}e_{66} = 2$ has a fixed point with $\breve{\psi}(y) = \frac{y}{2}, \hat{\eta}(y) = \frac{y}{4}$ where $y \in [0, +\infty)$.

Proof. When $s = 2$, $(\mathcal{E}, \eth, \leq)$ is a complete partially ordered b-metric space. Let $\varepsilon, \wp \in \mathcal{E}$ such that $\varepsilon < \wp$ then we'll look at the cases below.
Case 1. If $\varepsilon, \wp \in \{e_{11}, e_{22}, e_{33}, e_{44}, e_{55}\}$ then $\eth(\mathcal{I}\varepsilon, \mathcal{I}\wp) = \eth(e_{11}, e_{11}) = 0$. Hence,

$$
\breve{\psi}(2\eth(\mathcal{I}\varepsilon, \mathcal{I}\wp)) = 0 \leq \breve{\psi}(\mathscr{P}(\varepsilon, \wp)) - \hat{\eta}(\mathscr{P}(\varepsilon, \wp)).
$$

Case 2. If $\varepsilon \in \{e_{11}, e_{22}, e_{33}, e_{44}, e_{55}\}$ and $\wp = e_{66}$, then $\eth(\mathcal{I}\varepsilon, \mathcal{I}\wp) = \eth(e_{11}, e_{22}) = 3$, $\mathscr{P}(e_{66}, e_{55}) = 20$ and $\mathscr{P}(\varepsilon, e_{66}) = 12$, for $\varepsilon \in \{e_{11}, e_{22}, e_{33}, e_{44}\}$. Hence,

$$
\breve{\psi}(2\eth(\mathcal{I}\varepsilon, \mathcal{I}\wp)) \leq \frac{\mathscr{P}(\varepsilon, \wp)}{4} = \breve{\psi}(\mathscr{P}(\varepsilon, \wp)) - \hat{\eta}(\mathscr{P}(\varepsilon, \wp)).
$$

As a result, all of the conditions of Theorem 31 are met, and $\mathcal{I}$ has a fixed point.

Example 315 *Let us define a metric $\eth$ with usual order $\leq$ by*

$$
\eth(\varepsilon, \wp) = \begin{cases} 0 & ,\ if\ \varepsilon = \wp \\ 1 & ,\ if\ \varepsilon \neq \wp \in \{0, 1\} \\ |\varepsilon - \wp| & ,\ if\ \varepsilon, \wp \in \{0, \frac{1}{2n}, \frac{1}{2m} : n \neq m \geq 1\} \\ 6 & ,\ otherwise. \end{cases}
$$

where $\mathcal{E} = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ..., \frac{1}{n}, ...\}$. A self-map $\mathcal{I}$ on $\mathcal{E}$ by $\mathcal{I}0 = 0, \mathcal{I}\frac{1}{n} = \frac{1}{12n}(n \geq 1)$ has a fixed point with $\breve{\psi}(y) = y$, $\hat{\eta}(y) = \frac{4y}{5}$ for $y \in [0, +\infty)$.

Proof. $\eth$ is clearly discontinuous, and $(\mathcal{E}, \eth, \leq)$ is a complete partially ordered b-metric space for $s = \frac{12}{5}$. Now we'll look at the cases for $\varepsilon, \wp \in \mathcal{E}$ with $\varepsilon < \wp$.
Case 1. Suppose $\varepsilon = 0$ and $\wp = \frac{1}{n}$ $(n > 0)$, then $\eth(\mathcal{I}\varepsilon, \mathcal{I}\wp) = \eth(0, \frac{1}{12n}) = \frac{1}{12n}$ and $\mathscr{P}(\varepsilon, \wp) = \frac{1}{n}$ and $\mathscr{P}(\varepsilon, \wp) = \{1, 6\}$. Thus,

$$
\begin{aligned}
\breve{\psi}\left(\frac{12}{5}\eth(\mathcal{I}\varepsilon, \mathcal{I}\wp)\right) &\leq \frac{\mathscr{P}(\varepsilon, \wp)}{5} \\
&= \breve{\psi}(\mathscr{P}(\varepsilon, \wp)) - \hat{\eta}(\mathscr{P}(\varepsilon, \wp)).
\end{aligned}
$$

Case 2. Let $\varepsilon = \frac{1}{m}$ and $\wp = \frac{1}{n}$ where $m > n \geq 1$, thence

$$\eth(\mathscr{I}\varepsilon, \mathscr{I}\wp) = \eth(\frac{1}{12m}, \frac{1}{12n}) \text{ and}$$

$$\mathscr{P}(\varepsilon, \wp) \geq \frac{1}{n} - \frac{1}{m} \text{ or } \mathscr{P}(\varepsilon, \wp) = 6.$$

Thus,

$$\check{\psi}\left(\frac{12}{5}\eth(\mathscr{I}\varepsilon, \mathscr{I}\wp)\right) \leq \frac{\mathscr{P}(\varepsilon, \wp)}{5}$$
$$= \check{\psi}(\mathscr{P}(\varepsilon, \wp)) - \hat{\eta}(\mathscr{P}(\varepsilon, \wp)).$$

Hence, we have the conclusion from Theorem 31 as all assumptions are fulfilled.

Example 316 *Define a metric* $\eth : \mathscr{E} \times \mathscr{E} \to \mathscr{E}$, *where* $\mathscr{E} = \{\tilde{\ell}/\tilde{\ell} : [a_1, a_2] \to [a_1, a_2] \text{ continuous}\}$ *by*

$$\eth(\tilde{\ell}_1, \tilde{\ell}_2) = \sup_{y \in [a_1, a_2]} \{|\tilde{\ell}_1(y) - \tilde{\ell}_2(y)|^2\}$$

for any $\tilde{\ell}_1, \tilde{\ell}_2 \in \mathscr{E}$, $0 \leq a_1 < a_2$ *with* $\tilde{\ell}_1 \preceq \tilde{\ell}_2$ *implies* $a_1 \leq \tilde{\ell}_1(y) \leq \tilde{\ell}_2(y) \leq a_2, y \in [a_1, a_2]$. *A self-map* $\mathscr{I}$ *on* $\mathscr{E}$ *defined by* $\mathscr{I}\tilde{\ell} = \frac{\tilde{\ell}}{5}, \tilde{\ell} \in \mathscr{E}$ *has a unique fixed point with* $\check{\psi}(y) = y$, $\hat{\eta}(y) = \frac{y}{3}$, *for any* $y \in [0, +\infty]$.

Proof. As $\min(\tilde{\ell}_1, \tilde{\ell}_2)(y) = \min\{\tilde{\ell}_1(y), \tilde{\ell}_2(y)\}$ is continuous and all other assumptions of Theorem 33 are fulfilled for $s = 2$. Therefore, $0 \in \mathscr{E}$ is an unique fixed point $\mathscr{I}$.

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Competing interests

The authors declare that they have no competing interests.

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