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The Simplest Analytical Solution of Navier-Stokes Equations

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Abstract: The nonlinear convective acceleration term in fluids performs a strong obstacle against the analytical solutions of Navier-Stokes equations up to date. The obtained solutions are valid for long wave lengths only. In this paper, the nonlinear Navier-Stokes equations are converted to the linear diffusion equations based on the concept of linear velocity operator. The simplest analytical solutions of linear Navier-Stokes equations are obtained by using Picard method for a first time for different values of wave lengths and Reynolds number. As an application, the peristaltic incompressible viscous Newtonian fluid flow in a horizontal tube is described by the continuity and linear Navier-Stokes equations. The analytical solutions are obtained in terms of stream function and fluid velocity components. Moreover, the stream function is plotted in a laminar, transit and turbulent flows for different values of parameter δ.

Keywords: Linear velocity operator concept. Linear Navier-Stokes equations. Peristaltic flow. Linear convective acceleration term Newtonian fluid. Wave lengths.

1 Introduction

There are many phenomena in physics which are described by Navier–Stokes equations [1-15]. The mathematical modeling of the weather, ocean currents, water flow in a pipes, channels and air flow around wings are described by Navier–Stokes equations. Many problems have been formulated in nonlinear partial differential equations, which face some difficulties in the way of analytical solutions [1-14]. Scientists turn to the numerical solutions according to the difficulty of the nonlinear terms in a described system of fluid flow [5]. Some scientists [8-10, 12] turn to describe the physical problems in terms of nonlinear partial differential equations for special cases of fluid and flow properties.
Recently, scientists turned to high tech programs (CFD) by using the numerical solutions for many problems in different cases of fluid and flow [6]. J. Leray proposed a backward self-similar solution of the Navier-Stokes equations as an example of the finite-time blow-up of the three-dimensional nonstationary Navier-Stokes equations [5]. A finite-difference method for solving the time-dependent Navier-Stokes equations for an incompressible fluid is introduced by Alexandre Chorin [6]. An exact solution of the three-dimensional incompressible Navier-Stokes equations with the continuity equation is produced by Gunawan Nugroho [7]. Mats et al. [8] derived a solution to the Navier–Stokes equation by considering vorticity generated at system boundaries. The transformation of the Navier-Stokes equations to the Schrödinger equation performed by application of the Riccati equation [9]. A particular class of solutions of nonlinear differential equations can be obtained by several procedures [7]. The linear partial differential equations (PDEns) are solved by the similarity parameters method [13]. The linear concentration distribution around a growing gas bubble in tissue is obtained [11,13]. The solution of Navier-Stokes equation and its application for a growth problem under the effect of magnetic field. is obtained [12]. The exact solutions of Euler equation and Navier–Stokes equation are proposed by using Lie symmetry analysis method and Bäcklund transformation respectively by using symmetry reduction method [15].
The Cole-Hopf transformation is applicable to the Navier-Stokes equation for an incompressible flow and allows reducing the Navier-Stokes equation to the Einstein-Kolmogorov equation [17].
The nonlinear Partial differential equations are transformed to the linear diffusion ones on the basis of a linear velocity operator concept [18]. The peristaltic motion of viscous fluid in different shapes of tubes and plates is obtained for long wavelength and low Reynolds number as given by [2-3,14].
In this paper, the nonlinear Navier-Stokes equations

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is converted to a linear diffusion equation based on a linear velocity operator [18]. The nonlinear convective term $(\underline{\hat{v}}.\underline{\nabla})\underline{v}$ is transformed to a linear diffusion form $-\nu\,\underline{\nabla}^2\underline{v}$ on the basis of new treatment theory [18]. The Picard method [16] can be used for solving a linear system of Navier-Stokes equations. The analytical solutions of linear Navier-Stokes equations are obtained for different values of wave lengths λ and flow patterns (laminar, transit and turbulent flow). Moreover, the peristaltic flow of an incompressible Newtonian viscous fluid in a horizontal tube is studied as an illustrated example. The continuity and linear Navier-Stokes equations represent the mathematical model of fluid flow. The analytical solution in terms of stream function and fluid velocity components are obtained for laminar, transit and turbulent flows.

2 Analyses

The incompressible and viscous Newtonian fluid motion is formulated by continuity and Navier-Stokes equations. The third formulation of fluid mechanics is given in section 2.1. In the same way, the Navier-Stokes equations formulated in a linear form under the effect of surface and body forces. In section 2.2, the analytical solution is obtained in terms of fluid velocity components and stream function. The discussion of analytical solution and conclusions are introduced in section 3. In section 4, the unsteady incompressible and viscous Newtonian fluid flow in a horizontal tube for different wave lengths ($\lambda \neq 0$ *and* $\delta \neq 0$) are described by linear Navier-Stokes equations. In section 5, the results and graphs are discussed in detail. Finally, in section 5, the concluded remarks are tabulated.

2.1 Third Formulation of Fluid Mechanics

The fluid state is described by Lagrange and Euler [1] as a particle and point in space, respectively. Euler described the fluid flow in the nonlinear form of Navier-Stokes equations. The nonlinear convective acceleration $(\underline{\hat{v}}.\underline{\nabla})\underline{v}$ performs a strong obstacle against the analytical solutions of Navier-Stokes equations up to date. The linear velocity operator [18] is modified in terms of the physical parameter M* as follows

$$\underbrace{\underline{\hat{v}}}_{linear\ velocity\ operator} = -M^*\underline{\nabla}, \tag{1}$$

where M^* is called Mohammadein parameter.

The new definition of total operator $\frac{D\ldots}{Dt}$ with local and linear diffusion terms in fluid mechanics becomes

$$\underbrace{\frac{D\ldots}{Dt}}_{total\ derivative} = \underbrace{\frac{\partial\ldots}{\partial t}}_{local\ derivative} - \underbrace{\nu\,\underline{\nabla}^2\ldots}_{diffusion\ derivative} \tag{2}$$

The nonlinear acceleration of fluid in the point of view of Euler has the form

$$\underbrace{\frac{D\underline{v}}{Dt}}_{total\ acceleration} = \underbrace{\frac{\partial\underline{v}}{\partial t}}_{local\ acceleration} + \underbrace{(\underline{\hat{v}}.\underline{\nabla})\underline{v}}_{convective\ acceleration} \tag{3}$$

can be converted to the linear acceleration on the basis of new treatment theory [18] in the form

$$\frac{D\underline{v}}{Dt} = \frac{\partial\underline{v}}{\partial t} - \nu\,\underline{\nabla}^2\underline{v}. \tag{4}$$

Its noted that the nonlinear convective acceleration term $(\underline{\hat{v}}.\underline{\nabla})\underline{v}$ is transformed to a linear diffusion term $-\nu\,\underline{\nabla}^2\underline{v}$ by using the linear velocity operator concept [18]. Equation (4) is called Mohammadein description of total linear acceleration with local and linear diffusion terms as in Ref. [18]. Moreover, it considered a key of third formulation of fluid mechanics for the acceleration parameter in the linear form.

2.2 Linear Navier-Stokes Equations

Consider an incompressible viscous Newtonian fluid flow under the effect of surface and body forces, which are described by continuity and nonlinear Navier-Stokes equations in the vector form

$$\underline{\nabla}.\underline{v} = 0, \tag{5}$$

$$\rho\frac{dV}{dt} = -\underline{\nabla}P + \underline{\nabla}.\tau_{ij} + \rho g\,\underline{\hat{n}}, \tag{6}$$

where $\underline{\nabla}P$ is the gradient of pressure, and τ_{ij} is the shearing stress for two different kinds of fluids (Newtonian and non-Newtonian fluids). The above equation (6) can be rewritten in the form

$$\frac{\partial V}{\partial t} + (\underline{\hat{v}}.\underline{\nabla})\underline{v} = -\frac{1}{\rho}\underline{\nabla}P + \frac{1}{\rho}\underline{\nabla}.\tau_{ij} + g\,\underline{\hat{n}}, \tag{7}$$

where

$$\underline{\nabla}.\tau_{ij} = \begin{cases} \eta\,\underline{\nabla}^2\underline{v} & for\ \text{Newtonian fluids} \\ \underline{\nabla}.\tau_{ij} & for\ \text{Non Newtonian fluids}\end{cases} \tag{8}$$

Applying the new treatment theory [18] on the vector Navier-Stokes equations (7), for Newtonian fluids, then

$$\frac{\partial\underline{v}}{\partial t} = -\frac{1}{\rho}\underline{\nabla}P + 2\nu\underline{\nabla}^2\underline{v} + g\,\underline{\hat{n}}. \tag{9}$$

The Navier- Stokes equations in two dimensional cartesian coordinates has the form

$$u_x + v_y = 0\,, \tag{10}$$

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + 2\,\nu(u_{xx} + u_{yy}) + g_x, \tag{11}$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + 2\,\nu(v_{xx} + v_{yy}) + g_y, \tag{12}$$

The above linear system called linear Navier-Stokes equations and can be solved by analytical way under any physical proposed initial and boundary conditions. Moreover, the pressure field performs a real parameter for the fluid flow, and the gradient of pressure has the form

$$\underline{\nabla}P = -\rho(\underline{\hat{v}}.\underline{\nabla})\underline{v}. \tag{13}$$

The above formula of pressure gradient based on the theory [18] becomes

$$\underline{\nabla}P = -\eta\,\underline{\nabla}^2\underline{v}. \tag{14}$$

On the basis of above equation (14), the linear Navier-Stokes equation (9) for a viscous incompressible Newtonian fluid flow becomes

$$\frac{\partial \underline{v}}{\partial t} = 3\nu\underline{\nabla}^2\underline{v} + g\,\underline{\hat{n}}. \quad (15)$$

On contrary, for the non-Newtonian viscous incompressible fluid, the linear Navier-Stokes equations in the vector form become

$$\frac{\partial \underline{v}}{\partial t} = 2\nu\underline{\nabla}^2\underline{v} + \frac{1}{\rho}\underline{\nabla}.\tau_{ij} + g\,\underline{\hat{n}}. \quad (16)$$

The continuity and linear Navier-Stokes equations for Newtonian viscous incompressible fluid in a two-dimensional cartesian coordinate become

$$u_x + v_y = 0, \quad (17)$$

$$\frac{\partial u}{\partial t} = 3\,\nu(u_{xx} + u_{yy}) + g_x, \quad (18)$$

$$\frac{\partial v}{\partial t} = 3\,\nu(v_{xx} + v_{yy}) + g_y. \quad (19)$$

The stream function $\Psi(x, y, t)$ is obtained by using both relations $u = \frac{\partial \Psi}{\partial y}$ *and* $v = -\frac{\partial \Psi}{\partial x}$.

Body forces

There are two cases of body forces: First, when g_x and g_y ignored, the analytical solution by using Picard method of the above linear system (17-19) of continuity and linear Navier-Stokes equations in terms of stream function $\Psi(x, y, t)$ become

$$\Psi(x, y, t) = -\frac{A_1}{c_2} e^{3\nu({c_1}^2+{c_2}^2)t-(c_1x+c_2y)}. \quad (20)$$

Second, when g_x and g_y are considered, the solution of the same system (17-19) becomes

$$\Psi(x, y, t) = -\frac{A_1}{c_2} e^{3\nu({c_1}^2+{c_2}^2)t-(c_1x+c_2y)} - gt, \quad (21)$$

where c_1,c_2, and $A_1$are constants. The obtained analytical solution (21) in terms of stream function and fluid velocity components are satisfied by continuity and linear Navier-Stokes equations (17-19).

3 Discussion and Conclusions of Analytical Solution of Linear System of Navier-Stokes Equations

The nonlinear system of Navier-Stokes equations (7) for Newtonian fluid is transformed to the linear diffusion equations (15) on basis of New treatment theory [18]. The system of linear Navier-Stokes equations (17-19) is formulated in two dimensional cartesian coordinates, which represent the linear diffusion equations. The analytical solutions (20-21) are satisfied the continuity and linear system of Navier-Stokes equations in case of two-dimensional flow.

The discussion of results concluded the following points:

1. The original nonlinear Navier-Stokes equations are converted to the linear system based on the linear velocity operator concept [18].

2.The analytical solution of linear Navier-Stokes equations is obtained.

3.The parameter M*represents the kinematic viscosity ν of nanofluid state in case of Navier-Stokes equations.

4. When fluid acceleration equal to zero, the fluid velocity has a constant value in the point of view of Lagrange and Euler description. On contrary, in this treatment [18], the fluid flow velocity still existed in unsteady states, in both cases of motion and rest.

5.The fluid velocity components and stream function behave the same order of magnitude in plane (x, y) with constant difference between their values.

In the next section, the problem of unsteady incompressible and viscous Newtonian fluid flow in a horizontal tube is described by continuity, linear Navier-Stokes equations and are solved analytically as an application.

4 Unsteady Incompressible and Viscous Newtonian Fluid Flow in a Horizontal Tube for different wave lengths ($\lambda \neq 0$ and $\delta \neq 0$) described by linear Navier-Stokes equations

4.1 Introduction

In the previous efforts, scientists studied many problems of peristaltic fluid flow in several shapes under suitable boundary conditions. Most of the previous problems are described by the nonlinear Navier-Stokes equations, which are approximately solved for long wavelength $\delta = 0$ and low Reynolds number [2-15]. In the present example the proposed problem is solved analytically. Moreover, the stream function and fluid velocity components are obtained for different values of wave lengths λ and Reynolds number values.

4.2 The Physical and Mathematical Description

The peristaltic motion of fluid flow is described by many authors [2, 3, 14] in case of long wave lengths. In the follows, we consider the peristaltic flow of an incompressible Newtonian viscous fluid in a horizontal tube (see Fig. 1). The flow is caused by infinite sinusoidal wave train moving ahead with constant velocity c along the walls of the tube. The gravity force is ignored in our case. The peristaltic boundary condition has the form

$$H = a + b\sin\left(\frac{2\pi}{\lambda}(x - ct)\right), \quad (21)$$

where a is the tube half width, b is the wave amplitude, λ is the wave length and t is the time.

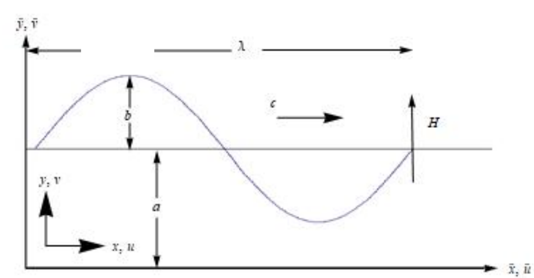


Fig. 1: Sketch of the problem.

Method of solution

The mathematical model for the fluid flow can be written in the form

Continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{22}$$

Navier-Stokes equations

$$\text{x: } \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right), \tag{23}$$

$$\text{y: } \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right), \tag{24}$$

where $\nabla P=-\rho\left(\underline{\hat{v}}.\underline{\nabla}\right)\underline{v}$.

Applying the new treatment theory [18] for the above system (22-24), in the frame $(\overline{x},\overline{y})$, then

$$\frac{\partial \overline{u}}{\partial \overline{x}}+\frac{\partial \overline{v}}{\partial \overline{y}}=0, \tag{25}$$

$$\frac{\partial \overline{u}}{\partial \overline{t}}=3\,\nu\left(\frac{\partial^2 \overline{u}}{\partial \overline{x}^2}+\frac{\partial^2 \overline{u}}{\partial \overline{y}^2}\right), \tag{26}$$

$$\frac{\partial \overline{v}}{\partial \overline{t}}=3\,\nu\left(\frac{\partial^2 \overline{v}}{\partial \overline{x}^2}+\frac{\partial^2 \overline{v}}{\partial \overline{y}^2}\right), \tag{27}$$

where $\overline{u}=\frac{\partial \overline{\psi}}{\partial \overline{y}}$ and $\overline{v}=-\frac{\partial \overline{\psi}}{\partial \overline{x}}$.

The nondimensional parameters in terms of dimensional ones have the form

$$\overline{x}=\lambda x,\ \overline{y}=ay,\ \overline{u}=cu,\ \overline{v}=c\delta v,\ \delta=\frac{a}{\lambda},\ \overline{t}=\frac{\lambda}{c}t,$$

$$\overline{\psi}=a\,c\psi\,,\ e=\frac{b}{a}$$

$$h=\frac{H}{a} \tag{28}$$

The above equations (25-27) by using the above transformations (28) in frame (x, y) introduces a linear partial differential equation in terms of stream function ψ in the form

$$R_e\,\delta\,\psi_t=3(\delta^2\psi_{xx}+\psi_{yy}). \tag{29}$$

The analytical solution by using Picard method [16] of above linear partial differential equation (29) has the form

$$\psi(x,y,t)=A_1 e^{\frac{3}{Re\,\delta}(c_1^2\,\delta^2+c_2^2)t\,-(c_1x+c_2y)}. \tag{30}$$

under the effect of initial and boundary conditions

$$\psi(x,y,0)=f(x,y)=e^{-(c_1\,x+\,c_2\,y)}$$

$$\psi(0,y,t)=2,\qquad \psi(L_1,y,t)=1, \tag{31}$$

$$\psi(x,0,t)=3,\qquad \psi(x,h_1,t)=1,$$

where c_1, c_2 and A_1 are constants can be estimated from the boundary conditions (31) as follow:

$$c_1=\frac{1}{L_1}\ln 2,\quad c_2=\frac{1}{h_1}\ln 3,\ A_1=1,$$

$$h_1=1+e\sin\left(2\pi(x-t)\right), L_1=3 \tag{32}$$

4.3 Discussion of Results

The peristaltic flow of an incompressible Newtonian fluid in a horizontal tube is studied as an application of linear Navier-stokes equations. The nonlinear system of Navier-Stokes equations (22-24) is transformed to a linear system of Navier-Stokes equations (25-27) on the basis of New treatment theory [18]. The system of linear partial differential equations (25-27) is transformed to the non-dimensional linear equation (29). The analytical solution is obtained by Picard method [16] in terms of stream function $\boldsymbol{\psi}$. The stream function (30) is obtained graphically for three different values of wave lengths $\boldsymbol{\lambda}$ as a function of the physical parameters.

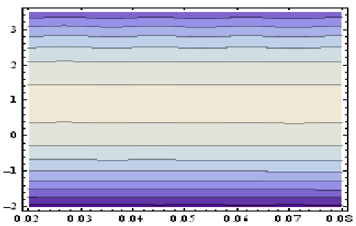
Here we are going to display, few but valuably different cases of flow patterns at $e=0.01$, Re $=5$, $L_1=3$ and $h_1=1+0.01\sin\left(2\pi(x-t)\right)$, as shown in Figs. 2-4 such that each group of alphabetically lettered figures are put in one row so that all parameters are fixed except one parameter (t or δ).

In Figs. 2a-g, the streamlines are plotted for $\delta=0.1$ for time intervals $t=0.0001, 1, 3, 7.5, 9, 16$ and 25 respectively.

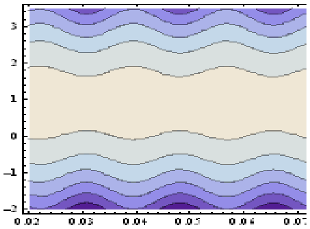
In Fig. 2a, the streamlines are straight and uniform at time$t=0.0001$. This means that, the streamlines are laminar flow. By increasing the time $t=1$, in Fig. 2b, the streamlines are transitional flow at $t=3$. In Fig. 2c, the streamlines transformed to a turbulent case at $t=7.5$. Moreover, the trapped bolus appears for large value of time where the formation of internally circulating bolus of fluid by the closed streamlines is known as trapping. By increasing the time, the amplitude of wave becomes much more widely spaced at time $t=\ 9, 16$ and 25.

In Fig. 3, the streamlines are plotted for $\delta=0.5$ at various values of time $t=0.0001, 1, 3, 7.5, 9, 16$ and 25 respectively As shown in the Figs. 3a-b, the streamlines are straight and uniform at time $t=0.0001$ and $t=1$ i.e. the streamlines are laminar flow. At time $t=3$, the streamlines look like a laminar flow as shown in Fig. 3c. But by increasing the time at $t=7.5$, the streamlines are beginning transformed to transitional fluid flow as in Fig. 3d. In Figs. 3e-g, the streamlines are transitional fluid flow at $t=9, 16$ and 25. The fluid flow in this case transformed to a turbulent flow at time $t=35$. Also, it is observed that, by increasing the time, the amplitude of the wave becomes much more widely spaced.

In Fig. 4, the streamlines are plotted for $\delta=0.9$ at various values of time $t=0.0001, 1, 3, 7.5, 9, 16$ and 25 respectively. The fluid flow represents a laminar flow as in Figs 4a-e for time $t=0.0001$ to $t=9$. On contrary, when $t=16$ the fluid flow is transitional flow as in Figs. 4f-g. Moreover, the streamlines are beginning transformed to a turbulent fluid flow at time $t=64$.



(2-a)



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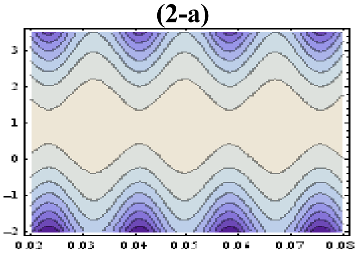

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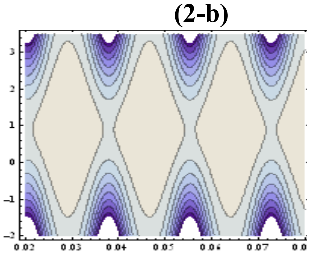

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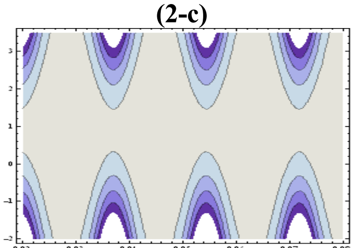

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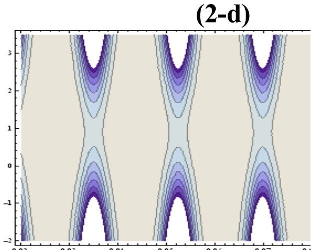
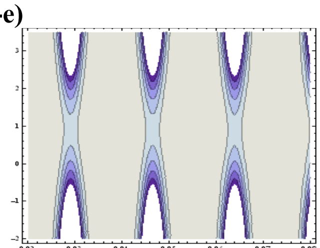
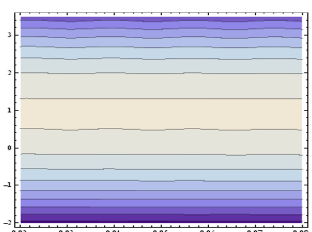


(2-f)

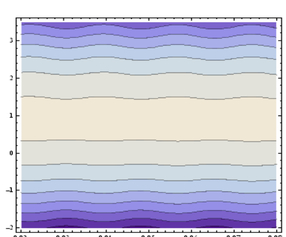


(2-g)

Fig. 2: The streamlines for $\delta = 0.1$ at various times.



(3-a)



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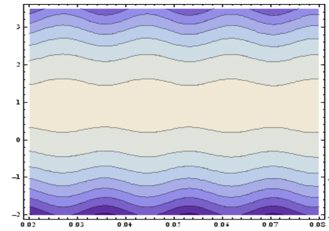

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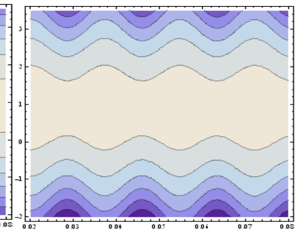

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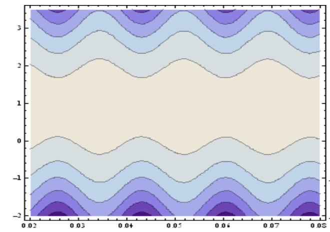

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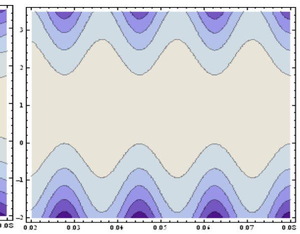

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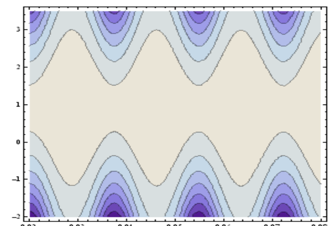
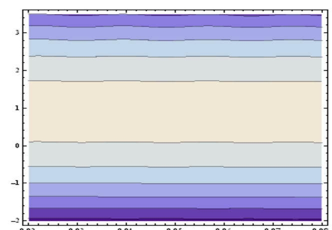
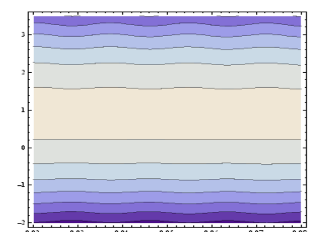


(3-g)

Fig. 3: The streamlines for $\delta = 0.5$ at various times.



(4-a)



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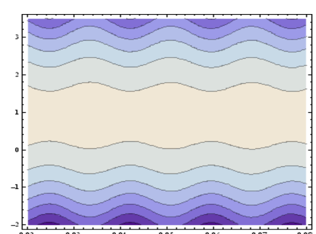

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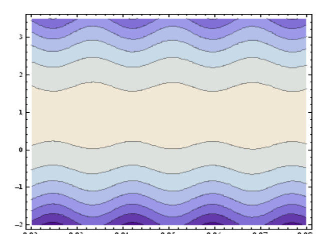

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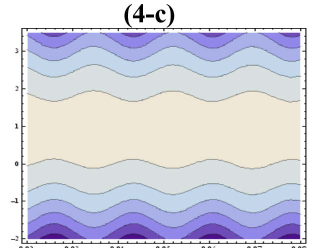

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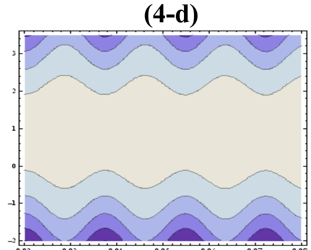

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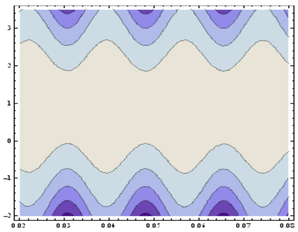

(4-g)
Fig. 4. The streamlines for $\delta = 0.9$ at various times.

5 Conclusions

The peristaltic flow of an incompressible and Newtonian viscous fluid in a horizontal tube is studied as application of linear Navier-Stokes equations. The linear system of Navier-Stokes equations (25-27) is obtained based on New treatment theory [18]. The stream function ψ and fluid velocity components u and v are obtained as an analytical solution of equation (30). The discussion of results and figures concluded the following remarks:

1. The peristaltic motion of Newtonian fluid flow in horizontal tube is obtained.
2. The analytical solution in terms of stream function and velocity components is obtained for laminar, transit and turbulent flows.
3. The stream function and fluid velocity components are obtained for different values of wave lengths λ and Reynolds number Re.
4. The time of transformation of flow patterns stages (laminar, transit and turbulent) is proportional directly with the different values of parameter δ; which $t(\delta = 0.1) < t(\delta = 0.5) < t(\delta = 0.9)$.Moreover, the radii $a(\delta = 0.1) < a(\delta = 0.5) <$ a (δ =0.9).
5. The boundary layer is wholly laminar and when the thickness of the boundary layer increases with distance from the leading edge as more and more fluid is slowed down by the viscous boundary, becomes unstable and breaks into turbulent boundary layer over a transition region and finally tends to separate layers.
6. The fluid velocity components are similar to the stream function in plane (x, y) with small difference between them in calculation values.
7. The fluid flow takes more time to transformed from laminar to turbulent flow when parameter δ increases.
8. The equation (4) represents the third formulation of fluid mechanics in a linear acceleration diffusion form.
9. The Navier-Stokes equations can be solved in the different cases of fluid and flow as a future prospect.

Conflict of interest

The authors have no conflicts of interest to disclose.

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References

[1] Landau, L. D. and E. M. Lifshitz,. Fluid Mechanics., **6** (1987).

[2] N. T. M. Eldabe, M. O. Shaker, and Maha S. A, Peristaltic Flow of MHD Jeffrey Fluid Through Porous Medium in a Vertical Channel with Heat and Mass Transfer with Radiation. Journal of Nanofluids., **7(3)**, 595-602 (2018).

[3] Abo-Eldahab, E. M. The effects of temperature-dependent fluid properties on free convective flow along a semi-infinite vertical plate by the presence of radiation. Heat and Mass Transfer., **41(2)**, 163-169 (2004).

[4] Abd-Alla, A. M. Generation of waves in an infinite micropolar elastic solid body under initial stress. Astrophysics and Space Science (ISSN 0004-640X).,**178(2)**, 175-183 (1991).

[5] Hisashi Okamoto. Exact solutions of the Navier-Stokes equations via Leray's scheme.Japan Journal of Industrial and Applied Mathematics., **14**,169169 (1997).

[6] Alexandre Chorin. Numerical Solution of the Navier–Stokes Equations. Mathematics of Computation., **22**, 104 (1968).

[7] Gunawan Nugroho_, Ahmed M.S. Ali, Zainal A. Abdul Karim. A class of exact solutions to the three-dimensional incompressible Navier_Stokes equations. Applied Mathematics Letters., **23**, 1388_1396 (2010).

[8] Mats D Lyberg1 and Henrik Tryggeson. An analytical solution of the Navier–Stokes equation for internal flows. 2007CC J of Physics A: Mathematical and Theoretical., **40**, 24 (2007).

[9] V. Christianto, F. Smarandache, An exact mapping from Navier_Stokes equation to Schrödinger equation via Riccati equation, Progress in Physics., **1**, (2008).

[10] G. Nugroho, A.M.S. Ali, Z.A. Abdul Karim. Toward a new simple analytical formulation of Navier_Stokes equations. International Journal of Mechanical Systems Science and Engineering 1 (2) (2009).

[11] Mohammadein, S. A., KG Mohamed. Concentration distribution around a growing gas bubble in tissue. Mathematical biosciences., **225 (1)**, 11-17(2010).

[12] Mohammadein, S. A., R. A. GadEl-Rab, A. A Megahed, A. Mosbah. Solution of Navier-Stokes' equation and its application for a growth problem under the effect of magnetic field. Faculty of Education, Ain Shams University (2011).

[13] Mohammadein, S. A., The concentration distribution around a growing gas bubble in a bio tissue under the effect of suction process. Mathematical biosciences., **253**, 88-93(2014).

[14] Mohammadein, S. A. and A. K. Abu-Nab. Growth of Vapour Bubble Flow inside a Symmetric Vertical Cylindrical Tube. Fluid Mechanics., **2(2)**, 28-32 (2016).

[15] Mingshuo Liu, Xinyue Li and Qiulan Zhao Exact solutions to Euler equation and Navier–Stokes equation. Zeitschrift f¨ur angewandte Mathematik und Physik ZAMP, Z. Angew. Math. Phys., **70**, 43(2019).

[16] Ahmad, H.; Khan, T.A.; Stanimirovic, P.S.; Ahmad, I. Modified variational iteration technique for the numerical solution of fifth order KVD type equations. J. Appl. Comput. Mech. (2020).

[17] Vladimir V. KULISH* & José L. LAGE. Exact solutions to the Naiver-Stokes equation for an incompressible flow from the interpretation of the Schrödinger wave function. School of Mechanical & Aerospace Engineering, Nanyang Technological University, Singapore 639798, Source arXiv, Jan 2013.

[18] Mohammadein, S. A. New Treatment of Fluid Mechanics with heat and Mass Transfer: Theory of Diffusion. Submitted to Appl. Math & Info Sci Lett. J