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Fixed Point Theorems in Partially Ordered Metric Spaces with Rational Expressions

N. Seshagiri Rao[1], *K. Kalyani*[2,*] *and Tekle Gemechu*[1]

[1]Department of Applied Mathematics, School of Applied Natural Sciences, Adama Science and Technology University, Post Box No.1888, Adama, Ethiopia
[2]Department of Mathematics, Vignan's Foundation for Science, Technology & Research, Vadlamudi-522213, Andhra Pradesh, India

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Abstract: The main objective of this paper is to establish some fixed point results for nonlinear contractive mappings in the context of a metric space endowed with partial order. Our results generalize, extend and unify several well known results in the literature. Some examples are illustrated to support our results.

Keywords: Partially ordered metric spaces, generalized rational contractions, fixed point, unique fixed point, well ordered set.

1 Introduction

The Banach contraction principle is one of the most versatile result in fixed point theory and approximation theory. It plays an important role in solving many existing problems in pure and applied mathematics. There is a vast literature dealing with technical extensions and generalizations of Banach contraction principle, some instances of these works are in [1–10]. Besides, this famous classical theorem gives an iteration process through which we can obtain better approximation to the fixed point. It renders a key role in solving systems of linear algebraic equations involving iteration process. Iteration procedures are using in nearly every branch of applied mathematics, convergence proof and also in estimating the process of errors, very often by an application of Banach's fixed point theorem.

In recent times, fixed points of mappings in ordered metric spaces are of great use in many branches of mathematical analysis for solving nonlinear equations. The first result in this direction was initiated by Wolk [11] and later Monjardet [12] in partially order sets. Ran and Reurings [13] studied the existence of fixed points for certain mappings in partially ordered metric spaces and applied their results to matrix equations. The results of Ran and Reurings [13] were extended by Nieto et al. [14–16] for non decreasing mappings and obtained the solutions of certain partial differential equations with periodic boundary conditions. While Agarwal et al. [17] have discussed some new results for a generalized contractions in partially ordered metric spaces. There have been a lot of generalizations and improvements of the results to obtain fixed point, common fixed point results for single valued and multivalued operators in various ordered spaces with topological properties, some of which are in [18–32, 48]. Recently, Seshagiri Rao et al. [33–40] have explored some results on fixed point, coincidence point, coupled fixed point and coupled common fixed point for the mappings in partially ordered metric spaces as well as in partially ordered b-metric spaces [41–47].

In (cf [8]), Singh, Badshah and Rathore proved the following fixed point theorem:

Theorem 11 *A mapping $T : X \to X$, defined on a complete metric space (X,d) satisfying the following condition*

$$d(Tx,Ty) \leq \alpha \frac{d(x,Tx)\,[1+d(y,Ty)]}{1+d(x,y)} + \beta[d(x,Tx)+d(y,Ty)] + \gamma[d(x,Ty)+d(y,Tx)] + \delta d(x,y), \tag{1}$$

for all distinct $x,y \in X$, where $\alpha,\beta,\gamma,\delta$ are non negative reals with $0 \leq \alpha+2(\beta+\gamma)+\delta < 1$. Then T has a unique fixed point in X.

In this paper, we generalize and extend the above Theorem 11 in a complete partially ordered metric space.

* Corresponding author e-mail: kalyani.namana@gmail.com

Also, we generalize and extend the results of [8, 14, 23, 27] and several comparable results in the literature. A few examples are given to support our results.

2 Preliminaries

We start this section with the following frequently used definitions in our study.

Definition 21 *[35] The triple $(X,d,\preceq)$ is called partially ordered metric spaces, if $(X,\preceq)$ is a partially ordered set and (X,d) is a metric space.*

Definition 22 *[35] If (X,d) is a complete metric space, then the triple $(X,d,\preceq)$ is called complete partially ordered metric spaces.*

Definition 23 *[23] A partially ordered metric space $(X,d,\preceq)$ is called ordered complete (OC) if for each convergent sequence $\{x_n\}_{n=0}^{\infty} \subset X$, the following condition holds: either*

- *if x_n is a non-increasing sequence in X such that $x_n \to x$ implies $x \preceq x_n$, for all $n \in \mathbb{N}$ that is, $x = \inf\{x_n\}$, or*
- *if x_n is a non-decreasing sequence in X such that $x_n \to x$ implies $x_n \preceq x$, for all $n \in \mathbb{N}$ that is, $x = \sup\{x_n\}$.*

Definition 24 *Let $(X,\preceq)$ be a partially ordered set and let $T : X \to X$ be a mapping. Then*

(1).elements $x,y \in X$ are comparable, if $x \preceq y$ or $y \preceq x$ holds;
(2).a non empty set X is called well ordered set, if every two elements of it are comparable;
(3).T is said to be monotone non-decreasing w.r.t. $\preceq$, if for all $x,y \in X$,

$$x \preceq y \text{ implies } Tx \preceq Ty.$$

(4).T is said to be monotone non-increasing w.r.t. $\preceq$, if for all $x,y \in X$,

$$x \preceq y \text{ implies } Tx \succeq Ty.$$

3 Main Results

3.1 Results under generalized rational type contractions

In this section, the existence and uniqueness of a fixed point of a mapping satisfying a generalized rational type contraction condition are proved in partially ordered metric space.

Theorem 31 *Let $(X,d,\preceq)$ be a complete partially ordered metric space. Suppose that $T : X \to X$ be a non-decreasing, continuous self mapping satisfying*

$$d(Tx,Ty) \leq \begin{cases} \lambda d(x,y) + \eta\,[d(x,Ty)+d(y,Tx)] \\ +\mu \frac{d(x,Tx)d(x,Ty)+d(y,Tx)d(y,Ty)}{d(y,Tx)+d(x,Ty)} & , if A \neq 0 \\ 0 & , if A = 0 \end{cases} \tag{2}$$

for all distinct $x,y \in X$ with $y \preceq x$, where $A = d(y,Tx)+d(x,Ty)$ and λ,η,μ are non-negative reals such that $0 \leq \lambda+2\eta+\mu < 1$. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then T has a fixed point in X.

Proof. If $x_0 = Tx_0$, then the proof is finished. Suppose that $x_0 \prec Tx_0$. Since T is a non-decreasing mapping then by induction, we obtain that

$$x_0 \prec Tx_0 \preceq T^2x_0 \preceq \ldots \preceq T^n x_0 \preceq T^{n+1}x_0 \preceq \ldots. \tag{3}$$

Put $x_{n+1} = Tx_n$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then from $x_{n_0} = x_{n_0+1} = Tx_{n_0}$, we have x_{n_0} is a fixed point, and therefore the proof is finished. Suppose that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$. Since the points x_n and x_{n-1} are comparable for $n \in \mathbb{N}$ due to (3), we have the following two cases.
Case 1: If $A = d(x_{n-1},Tx_n)+d(x_n,Tx_{n-1}) \neq 0$, then using the contractive condition (2), we get

$$\begin{aligned} & d(x_{n+1},x_n) = d(Tx_n,Tx_{n-1}) \\ & \leq \lambda d(x_n,x_{n-1}) + \eta\,[d(x_n,Tx_{n-1})+d(x_{n-1},Tx_n)] \\ & \quad + \mu \frac{d(x_n,Tx_n)d(x_n,Tx_{n-1})+d(x_{n-1},Tx_n)d(x_{n-1},Tx_{n-1})}{d(x_{n-1},Tx_n)+d(x_n,Tx_{n-1})}, \end{aligned}$$

which implies that

$$\begin{aligned} d(x_{n+1},x_n) \leq & \lambda d(x_n,x_{n-1}) + \eta\,[d(x_n,x_n)+d(x_{n-1},x_{n+1})] \\ & + \mu \frac{d(x_n,x_{n+1})d(x_n,x_n)+d(x_{n-1},x_{n+1})d(x_{n-1},x_n)}{d(x_{n-1},x_{n+1})+d(x_n,x_n)}. \end{aligned}$$

Hence, we derived that

$$d(x_{n+1},x_n) \leq h^n d(x_1,x_0),$$

where $h = \frac{\lambda+\eta+\mu}{1-\eta} < 1$. Moreover, by the triangular inequality, for $m \geq n$

$$\begin{aligned} d(x_m,x_n) & \leq d(x_m,x_{m-1}) + d(x_{m-1},x_{m-2}) + \ldots + d(x_{n+1},x_n) \\ & \leq \frac{h^n}{1-h} d(x_1,x_0), \end{aligned}$$

as $m,n \to +\infty$, $d(x_m,x_n) \to 0$. Thus, $\{x_n\}$ is a Chachy sequence in X and by the completeness of X, there exists $z \in X$ such that $\lim_{n\to+\infty} x_n = z$. Further, the continuity of T implies that

$$\begin{aligned} Tz & = T\left(\lim_{n\to+\infty} x_n\right) \\ & = \lim_{n\to+\infty} Tx_n \\ & = \lim_{n\to+\infty} x_{n+1} \\ & = z. \end{aligned}$$

Thus, z is a fixed point of T in X.
Case 2: If $A = d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1}) = 0$, then $d(x_{n+1}, x_n) = 0$. This implies that $x_n = x_{n+1}$, a contradiction as the sequence points are comparable. Thus there exists a fixed point z of T.

We may remove the continuity criteria on T in Theorem 31 as follows:

Theorem 32*Let $(X, d, \preceq)$ be a complete partially ordered metric space. Assume that X satisfies*

$$\textit{if a nondecreasing sequence } \{x_n\} \to x \textit{ in } X, \textit{ then } x = \sup\{x_n\}. \tag{4}$$

Let $T : X \to X$ be a monotone non-decreasing mapping satisfying the contraction condition (2). If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then T has a fixed point in X.

*Proof.*We only have to check that $z = Tz$. As $\{x_n\} \subset X$ is a non-decreasing sequence such that $x_n \to z \in X$ from Theorem 31, then $z = \sup\{x_n\}$ for all $n \in \mathbb{N}$ by (4). Since T is a non-decreasing mapping, then $Tx_n \preceq Tz$ for all $n \in \mathbb{N}$ or, equivalently, $x_{n+1} \preceq Tz$ for all $n \in \mathbb{N}$. Moreover, as $x_0 \prec x_1 \preceq Tz$ and $z = \sup\{x_n\}$, we get $z \preceq Tz$.

suppose that $z \prec Tz$. Using a similar argument as that in the proof of Theorem 31 for $x_0 \preceq Tx_0$, we obtain a non-decreasing sequence $\{T^n z\}$ in X such that $\lim_{n\to+\infty} T^n z = y$ for certain $y \in X$. Again by (4), we get that $y = sup\{T^n z\}$. Moreover, from $x_0 \preceq z$, we get $x_n = T^n x_0 \preceq T^n z$ for $n \geq 1$ and $x_n \prec T^n z$ for $n \geq 1$ because $x_n \preceq z \prec Tz \preceq T^n z$ for all $n \geq 1$.

As x_n and $T^n z$ are comparable and distinct for $n \geq 1$, consider the following cases:
Case 1: If $d(T^n z, Tx_n) + d(x_n, T^{n+1} z) \neq 0$, then applying the contractive condition (2), we get

$$\begin{aligned} d(x_{n+1}, T^{n+1}z) &= d(Tx_n, T(T^n z)) \\ &\leq \lambda d(x_n, T^n z) + \eta \left[d(x_n, T^{n+1}z) + d(T^n z, x_{n+1}) \right] \\ &+ \mu \frac{d(x_n, x_{n+1}) d(x_n, T^{n+1}z) + d(T^n z, x_{n+1}) d(T^n z, T^{n+1}z)}{d(T^n z, x_{n+1}) + d(x_n, T^{n+1}z)}. \end{aligned}$$

Making $n \to +\infty$ in the above inequality, we obtain

$$d(z, y) \leq (\lambda + 2\eta) d(z, y),$$

as $\lambda + 2\eta < 1$, $d(z, y) = 0$, thus $z = y$. Particularly, $z = y = sup\{T^n z\}$ and consequently, $Tz \preceq z$, which is a contradiction. Hence, we conclude that $Tz = z$.
Case 2: If $d(T^n z, Tx_n) + d(x_n, T^{n+1} z) = 0$, then $d(x_{n+1}, T^{n+1} z) = 0$. Taking the limit as $n \to +\infty$, we get $d(z, y) = 0$. Then $z = y = sup\{T^n z\}$, which implies that $Tz \preceq z$, a contradiction. Thus $Tz = z$.

Now, we present some examples where it can be appreciated that hypotheses in Theorem 31 and Theorem 32 do not guarantee uniqueness of the fixed point. These examples appears in [14].

Example 33*Let $X = \{(1,0),(0,1)\} \subseteq \mathbb{R}^2$ with the Euclidean distance d. We consider the partial order U in X as follows:*

$$U : (u, v) \leq (r, s) \textit{ if and only if } u \leq r \textit{ and } v \leq s.$$

Let $T : X \to X$ by $T(x, y) = (x, y)$. Then T have fixed points in X.

*Proof.*It is clear that $(X, d, \leq)$ is a complete partially ordered metric space. Besides, the identity mapping $T(x, y) = (x, y)$ is trivially continuous, non-decreasing and satisfies the contraction condition

$$\begin{aligned} &d(T(u,v), T(r,s)) \leq \lambda d((u,v),(r,s)) \\ &\leq \lambda d((u,v),(r,s)) + \eta \left[d((u,v), T(r,s)) + d((r,s), T(u,v)) \right] \\ &+ \mu \frac{A_1 + A_2}{d((r,s), T(u,v)) + d((u,v), T(r,s))}, \end{aligned}$$

for all $\lambda, \eta, \mu \in [0,1)$ with $0 \leq \lambda + 2\eta + \mu < 1$, where $A_1 = d((u,v), T(u,v)) d((u,v), T(r,s))$ and $A_2 = d((r,s), T(u,v)) d((r,s), T(r,s))$. Notice that the elements of X are only comparable to themselves. Moreover, $(1,0) \leq T((1,0))$. Here all the conditions of Theorem 31 are satisfied and T has two fixed points, which are $(1,0)$ and $(0,1)$.

Example 34*Under the same assumptions in Example 33, let us consider a non-decreasing sequence $\{(x_n, y_n)\} \subseteq X$ converging to (x, y). Then necessarily, $\{(x_n, y_n)\}$ is a constant sequence and $(x_n, y_n) = (x, y)$ for all $n \in \mathbb{N}$. Also, note that the limit (x, y) is an upper bound, of course supreme for all the terms of the sequence. Hence, all the conditions of Theorem 32 are satisfied and, $(1,0)$ and $(0,1)$ are two fixed points of T in X.*

Now we give a sufficient condition for the uniqueness of the fixed point that exists in Theorem 31 and Theorem 32.

$$\text{every pair of elements has a lower bound or an upper bound.} \tag{5}$$

In [14], it is proved that the above mentioned condition is equivalent to

$$\text{for every } x, y \in X, \text{ there exists } \vartheta \in X \text{ which is comaprable to } x \text{ and } y.$$

Theorem 35*In addition to the hypotheses of Theorem 31 (or Theorem 32), condition (5) provides uniqueness of the fixed point of T in X.*

*Proof.*Suppose that there exists $y, z \in X$ are fixed points of T.

We distinguish two cases.
Case 1: If y and z are comparable and $y \neq z$. Now we have the following two subcases:

(i). If $d(z,Ty)+d(y,Tz) \neq 0$ then using the contradiction condition (2), we have

$$\begin{aligned} d(y,z) &= d(Ty,Tz) \\ &\leq \lambda d(y,z)+\eta\,[d(y,Tz)+d(z,Ty)] \\ &\quad +\mu\frac{d(y,Ty)d(y,Tz)+d(z,Ty)d(z,Tz)}{d(z,Ty)+d(y,Tz)} \\ &\leq \lambda d(y,z)+\eta\,[d(y,z)+d(z,y)] \\ &\quad +\mu\frac{d(y,y)d(y,z)+d(z,y)d(z,z)}{d(z,y)+d(y,z)}, \end{aligned}$$

which suggest that

$$\begin{aligned} d(y,z) &\leq (\lambda+2\eta)d(y,z) \\ &< d(y,z) \text{ as } \lambda+2\eta<1, \end{aligned}$$

this is a contradiction. Hence, $y=z$.
(ii). If $d(z,Ty)+d(y,Tz) = 0$, then $d(y,z) = 0$, a contradiction again. Therefore, $y=z$.
Case 2: If y and z are not comparable, then by contraction condition (2) there exists $x \in X$ comparable to y and z. Monotonicity implies that $T^n x$ is comparable to $T^n y = y$ and $T^n z = z$ for $n = 0,1,2,\ldots$.

If there exists $n_0 \geq 1$ such that $T^{n_0}x = y$, then as y is fixed point, the sequence $\{T^n x : n \geq n_0\}$ is constant and consequently $\lim_{n\to+\infty} T^n x = y$. On the other hand, if $T^n x \neq y$ for all $n \geq 1$. Now we have the follows two subcases:
(i). If $d(T^{n-1}y,T^n x)+d(T^{n-1}x,T^n y) \neq 0$, then by (2) for $n \geq 2$, we obtain that

$$\begin{aligned} d(T^n x,y) &= d(T^n x,T^n y) \\ &\leq \lambda d(T^{n-1}x,y)+\eta\left[d(T^{n-1}x,y)+d(y,T^n x)\right] \\ &\quad +\mu\frac{d(T^{n-1}x,T^n x)d(T^{n-1}x,y)+d(y,T^n x)d(y,y)}{d(T^n x,y)+d(y,T^{n-1}x)} \\ &\leq \lambda d(T^{n-1}x,y)+\eta\left[d(T^{n-1}x,y)+d(y,T^n x)\right] \\ &\quad +\mu d(T^{n-1}x,y). \end{aligned}$$

This implies that

$$d(T^n x,y) \leq \left(\frac{\lambda+\eta+\mu}{1-\eta}\right) d(T^{n-1}x,y).$$

By induction, we obtain that

$$d(T^n x,y) \leq \left(\frac{\lambda+\eta+\mu}{1-\eta}\right)^n d(x,y).$$

As $\lambda+2\eta+\mu < 1$ and taking limit as $n \to +\infty$ in the above inequality, we get

$$\lim_{n\to+\infty} T^n x = y.$$

Using a similar argument as above, we can prove that

$$\lim_{n\to+\infty} T^n x = z.$$

Now, the uniqueness of the limit gives that $y=z$.
(ii). If $d(T^{n-1}y,T^n x)+d(T^{n-1}x,T^n y) = 0$, then by condition (2), we have $d(T^n x,y) = 0$. Therefore,

$$\lim_{n\to+\infty} T^n x = y.$$

By similar argument, we can prove that

$$\lim_{n\to+\infty} T^n x = z.$$

Now, the uniqueness of the limit gives that $y=z$. Hence, T has a unique fixed point in X.

Example 36*It is easily proved that the space $C[0,1]=\{x:[0,1]\to\mathbb{R}, continuous\}$ with the partial order given by*

$$x \leq y \text{ if and only if } x(t) \leq y(t), \text{ for } t \in [0,1],$$

and the metric given by

$$d(x,y) = \sup\{|x(t)-y(t)| : t \in [0,1]\}$$

satisfies condition (4). Moreover, as for $x,y \in [0,1]$, the function $\max(x,y)(t) = \max\{x(t),y(t)\}$ is continuous. Also $(C[0,1],\leq)$ satisfies the condition (5).

Example 37*Let $X = \{(0,0),(\frac{1}{2},0),(0,1)\}$ be a subset of $\mathbb{R}^2$ with the order $\leq$ defined as: for $(x_1,y_1),(x_2,y_2) \in X$ with $(x_1,y_1) \leq (x_2,y_2)$ if and only if $x_1 \leq x_2$ and $y_1 \leq y_2$. Let the distance $d : X\times X \to \mathbb{R}$ is defined by*

$$d((x_1,y_1),(x_2,y_2)) = \max\{|x_1-x_2|,|y_1-y_2|\}.$$

Let $T : X \to X$ be defined by $T(0,0) = (0,0)$, $T(0,1) = (\frac{1}{2},0)$ and $T(\frac{1}{2},0) = (0,0)$. Here all the conditions of Theorem 31,32 and 35 are satisfied and $(0,0)$ is the unique fixed point of T.

Note 38 *(i).If $\eta=\mu=0$ in Theorems 31, 32 and 35, then we obtain Theorems 2.1, 2.2 and 2.3 of [14].*
(ii).If $\eta=0$ in Theorems 31, 32 and 35, then we get Theorems 15, 17 and 18 of [23].
(iii).If $\lambda=\eta=0$ in Theorems 31 and 32, we obtain Theorem 20 of [23]. Also, the uniqueness of the fixed point can be proved by using Condition (5) in the hypotheses.

If in the Theorems 31 and 32, $\lambda=0$, we obtain the following fixed point theorem in complete partially ordered metric space.

Theorem 39*Let $(X,d,\preceq)$ be a complete partially ordered metric space. Suppose that a self mapping T on X be a non-decreasing, continuous and satisfying following condition*

$$d(Tx,Ty) \leq \begin{cases} \eta\,[d(x,Ty)+d(y,Tx)] \\ \quad +\mu\frac{d(x,Tx)d(x,Ty)+d(y,Tx)d(y,Ty)}{d(y,Tx)+d(x,Ty)} & , if A \neq 0 \\ 0 & , if A = 0 \end{cases} \tag{6}$$

for all $x, y \in X$ with $y \preceq x$, where $A = d(y,Tx) + d(x,Ty)$ and η, μ are non-negative reals such that $0 < 2\eta + \mu < 1$. And also suppose that either T is continuous or X satisfies condition (4). If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then T has a fixed point in X.

Theorem 310*The uniqueness of the fixed point in Theorem 39 can be proved using Condition (5).*

Theorem 311*Let $(X,d,\preceq)$ be a complete partially ordered metric space. Assume that either T is continuous or X is such that*

$$\text{if a nonincreasing sequence } \{x_n\} \to x \text{ in } X, \text{ then } x = \inf\{x_n\}.$$

Let $T : X \to X$ be a monotone non-decreasing mapping satisfying the contraction condition (2) (or (6)). If there exists $x_0 \in X$ with $x_0 \succeq Tx_0$, then T has a fixed point in X.

*Proof.*The scheme of the proof is similar to the procedure followed in the proof of the previous Theorems 31 and 32.

Theorem 312*Condition (5) provides uniqueness of the fixed point of T in the hypotheses of Theorem 311.*

3.2 Results under Singh, Badshah and Rathore contractions

We start this section with the following definition.

Definition 313*Let $(X,d,\preceq)$ be a partially ordered metric space. A self-mapping T on X is called an almost Singh, Badshah and Rathore contraction if it satisfies the following condition:*

$$\begin{aligned} d(Tx,Ty) \leq\ & \alpha \frac{d(x,Tx)\,[1+d(y,Ty)]}{1+d(x,y)} \\ & + \beta[d(x,Tx)+d(y,Ty)] \\ & + \gamma[d(x,Ty)+d(y,Tx)] + \delta d(x,y) \\ & + L\ \min\{d(x,Ty),d(y,Tx),d(x,Tx),d(y,Ty)\}, \end{aligned} \tag{7}$$

for all distinct $x, y \in X$ with $x \preceq y$, where $L \geq 0$ and there exist $\alpha, \beta, \gamma, \delta \in [0,1)$ such that $0 \leq \alpha + 2(\beta+\gamma) + \delta < 1$.

In the sequel, we prove the following theorem which is a version of Theorem 11 in the context of partially ordered metric spaces.

Theorem 314*Let $(X,d,\preceq)$ be a complete partially ordered metric space. Suppose that a self-mapping T is an almost Singh, Badshah and Rathore contraction, continuous and non-decreasing. Suppose there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then T has a fixed point in X.*

*Proof.*Let $x_0 \in X$ and set $x_{n+1} = Tx_n$. If $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then T has a fixed point. In particular, x_{n_0} is a fixed point of T. We assume that $x_n \neq x_{n+1}$ for all n. Since $x_0 \preceq Tx_0$, then

$$x_0 \preceq x_1 \preceq \ldots \preceq x_n \preceq x_{n+1} \preceq \ldots. \tag{8}$$

Now,

$$\begin{aligned} d(x_{n+1},x_n) &= d(Tx_n,Tx_{n-1}) \\ &\leq \alpha \frac{d(x_n,Tx_n)\,[1+d(x_{n-1},Tx_{n-1})]}{1+d(x_n,x_{n-1})} \\ &\quad + \beta[d(x_n,Tx_n)+d(x_{n-1},Tx_{n-1})] \\ &\quad + \gamma[d(x_n,Tx_{n-1})+d(x_{n-1},Tx_n)] + \delta d(x_n,x_{n-1}) \\ &\quad + L\ \min\{d(x_n,Tx_{n-1}),d(x_{n-1},Tx_n),d(x_n,Tx_n), \\ &\qquad d(x_{n-1},Tx_{n-1})\}, \end{aligned}$$

which implies that

$$\begin{aligned} d(x_{n+1},x_n) &= \left(\frac{\beta+\gamma+\delta}{1-\alpha-\beta-\gamma}\right) d(x_n,x_{n-1}) \leq \ldots.. \\ &\leq \left(\frac{\beta+\gamma+\delta}{1-\alpha-\beta-\gamma}\right)^n d(x_1,x_0). \end{aligned}$$

From the triangular inequality for $m \geq n$, we have

$$\begin{aligned} d(x_n,x_m) &= d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \ldots + d(x_{m-1},x_m) \\ &\leq \left(k^n + k^{n+1} + \ldots + k^{m-1}\right) d(x_0,Tx_0) \\ &\leq \frac{k^n}{1-k} d(x_1,x_0), \end{aligned} \tag{9}$$

where $k = \frac{\beta+\gamma+\delta}{1-\alpha-\beta-\gamma}$. Letting $m,n \to +\infty$ in the above inequality (9), we get $d(x_n,x_m) = 0$. Thus, the sequence $\{x_n\}$ is a Cauchy sequence. Since X is complete, then there exists a point $z \in X$ such that $x_n \to z$. Furthermore, the continuity of T in X implies that

$$Tz = T\left(\lim_{n\to+\infty} x_n\right) = \lim_{n\to+\infty} Tx_n = \lim_{n\to+\infty} x_{n+1} = z.$$

Therefore, z is fixed point of T in X.

Theorem 315*Let $(X,d,\preceq)$ be a complete partially ordered metric space. Assume that X satisfies*

$$\text{if a nondecreasing sequence } \{x_n\} \to x \text{ in } X, \text{ then } x = \sup\{x_n\}. \tag{10}$$

Let $T : X \to X$ be a monotone non-decreasing mapping satisfying the contraction condition (7). If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then T has a fixed point in X.

*Proof.*The proof follows Theorem 32.

Now, we give the examples for Theorem 314.

Example 316*Let $X = \{(2,0),(0,2)\} \subseteq \mathbb{R}^2$ with the Euclidean distance d. We consider the partial order in X as follows:*

$$(x_1,y_1) \leq (x_2,y_2) \text{ if and only if } x_1 \leq x_2 \text{ and } y_1 \leq y_2.$$

Thus, $(X,d,\leq)$ is a complete partially ordered metric space. The mapping $T(x,y) = (x,y)$ is continuous, *non-decreasing and the condition*

$$\begin{aligned} d(T(x_1,y_1),T(x_2,y_2)) &\le \delta d((x_1,y_1),(x_2,y_2)) \\ &\le \alpha \frac{d((x_1,y_1),T(x_1,y_1))\,[1+d((x_2,y_2),T(x_2,y_2))]}{1+d((x_1,y_1),(x_2,y_2))} \\ &+\beta\,[d((x_1,y_1),T(x_1,y_1))+d((x_2,y_2),T(x_2,y_2))] \\ &+\gamma[d((x_1,y_1),T(x_2,y_2))+d((x_2,y_2),T(x_1,y_1))] \\ &+\delta d((x_1,y_1),(x_2,y_2)) \\ &+L\min\{d((x_1,y_1),T(x_2,y_2)),d((x_2,y_2),T(x_1,y_1)), \\ &\quad d((x_1,y_1),T(x_1,y_1)),d((x_2,y_2),T(x_2,y_2))\}, \end{aligned}$$

holds for any $\alpha,\beta,\gamma,\delta \in [0,1)$ *with* $0 \le \alpha+2(\beta+\gamma)+\delta < 1$ *and for any* $L \ge 0$. *Notice that the elements of* X *are only comparable to themselves and no different elements are comparable. Moreover,* $(0,2) \le T((0,2))$. *Here all the conditions of Theorem 314 are satisfied,* $(2,0)$ *and* $(0,2)$ *are the fixed points of* T.

Example 317*Let* $X = \{(x,-x), x \in \mathbb{R}\}$ *with usual order and* d *be the Euclidean distance. The identity map has an infinite number of fixed points in* X. *Note that two different points in* X_2 *are not comparable.*

Theorem 318*In addition to the hypotheses of Theorem 314 (or Theorem 315), condition (5) provides uniqueness of the fixed point of* T *in* X.

*Proof.*The proof follows Theorem 35.

Now, we illustrate an example for Theorem 318.

Example 319*Let us define a metric* $d : X \times X \to R$ *on* $X = [0,1]$ *as*

$$d(x,y) = |x-y|.$$

And also define a self-mapping T *on* X *by*

$$Tx = \frac{x^3}{10}.$$

Then T *has a unique fixed point in* X.

*Proof.*The mapping T is continuous and non-decreasing and, let $x_0 = 0$ then $x_0 \le Tx_0$. Note that any two different points are comparable in X. Take $\delta = \frac{1}{3}$. Then for any $\alpha,\beta,\gamma \in [0,1)$ with $0 \le \alpha+2(\beta+\gamma)+\delta < 1$, we have the result. Let us examine in detail. Without loss of generality, we assume that $y \preceq x$. Also note that $0 \le d(x,Tx) \le \frac{9}{10}$, $0 \le d(y,Ty) \le \frac{9}{10}$, $0 \le d(x,Ty) \le \frac{9}{10}$ and $0 \le d(y,Tx) \le 1$.

Now, consider the following

$$\begin{aligned} d(Tx,Ty) = \frac{1}{10}|x^3-y^3| &= \frac{1}{10}|(x-y)(x^2+xy+y^2)| \\ &\le \frac{1}{3}|x-y| = \frac{1}{3}\, d(x,y), \end{aligned}$$

that is,

$$\begin{aligned} d(Tx,Ty) \le \alpha &\frac{d(x,Tx)\,[1+d(y,Ty)]}{1+d(x,y)} \\ &+\beta\,[d(x,Tx)+d(y,Ty)] \\ &+\gamma[d(x,Ty)+d(y,Tx)]+\frac{1}{3}\,d(x,y) \\ &+L\,\min\{d(x,Ty),d(y,Tx),d(x,Tx),d(y,Ty)\}, \end{aligned}$$

holds for any $L \ge 0$ and any $\alpha,\beta,\gamma \in [0,1)$ with $\alpha+2(\beta+\gamma)+\delta < 1$. Thus all conditions of Theorem 314 and Condition (315) are satisfied in X. Therefore, $0 \in X$ is the unique fixed point of T.

Definition 320*Let* $(X,d,\preceq)$ *be a partially ordered metric space. A self-mapping* T *on* X *is called Singh, Badshah and Rathore contraction if it satisfies the following condition:*

$$\begin{aligned} d(Tx,Ty) \le \alpha \frac{d(x,Tx)\,[1+d(y,Ty)]}{1+d(x,y)} &+\beta[d(x,Tx)+d(y,Ty)] \\ +\gamma[d(x,Ty)+d(y,Tx)]&+\delta d(x,y), \end{aligned} \tag{11}$$

for all distinct $x,y \in X$ *with* $x \preceq y$*, there exist* $\alpha,\beta,\gamma,\delta \in [0,1)$ *such that* $0 \le \alpha+2(\beta+\gamma)+\delta < 1$.

Corollary 321*Let* $(X,d,\preceq)$ *be a complete partially ordered metric space. suppose that a self-mapping* T *is Singh, Badshah and Rathore contraction, continuous and non-decreasing. Suppose there exists* $x_0 \in X$ *with* $x_0 \preceq Tx_0$*, then* T *has a fixed point in* X.

*Proof.*Set $L = 0$ in Theorem 314.

Besides, if X satisfies the condition (4), then a map T has a fixed point and also, if X satisfies condition (5), then one obtains uniqueness of the fixed point.

Theorem 322*Let* $(X,d,\preceq)$ *be a complete partially ordered metric space. Assume that either* T *is continuous or* X *is such that*

$$\text{if a nonincreasing sequence } \{x_n\} \to x \text{ in } X, \text{ then } x = \inf\{x_n\}.$$

Let $T : X \to X$ *be a monotone nondecreasing mapping satisfying the contraction condition (7) (or (11)). If there exists* $x_0 \in X$ *with* $x_0 \succeq Tx_0$*, then* T *has a fixed point in* X.

*Proof.*The scheme of the proof is similar to the procedure followed in the proof of the previous theorems.

We present an example where Theorem 314 (or Corollary 321) can be applied and this example cannot be treated by the main theorem of Singh, Badshah and Rathore (cf [8]) in complete metric space.

Example 323*Let* $X = \{(0,1),(1,0),(1,1)\}$ *and consider the partial order relation on* X *by* $R = \{(x,x) : x \in X\}$. *Notice that elements in* X *are only comparable to themselves. Besides,* (X,d) *is a complete metric space,* *where d is an Euclidean distance. Also, $(X,\leq)$ is a partially ordered set.*

Let $T: X \to X$ be defined by

$$T(0,1)=(1,0),\ \ T(1,0)=(0,1),\ \ T(1,1)=(1,1).$$

Thus, T is trivially continuous and non-decreasing and satisfy the condition (7) (or condition (1)) of Theorem 314 (or Corollary 321), since elements of X are only comparable to themselves. Moreover, $(1,1)\leq T(1,1)=(1,1)$ and, by Theorem 314 (or Corollary 321), T has fixed point $(1,1)$.

On the other hand, for $x=(0,1)$, $y=(1,0)$ in X, we have

$$d(Tx,Ty)=\sqrt{2},\ d(x,Ty)=0,\ d(y,Tx)=0,\ d(x,Tx)=\sqrt{2},$$
$$d(y,Ty)=\sqrt{2},$$

and the contractive condition of the main theorem of Singh, Badshah and Rathore (cf [8]) is not satisfied because

$$\begin{aligned}
d(Tx,Ty)=\sqrt{2}\leq\alpha&\frac{d(x,Tx)\left[1+d(y,Ty)\right]}{1+d(x,y)}\\
&+\beta[d(x,Tx)+d(y,Ty)]\\
&+\gamma[d(x,Ty)+d(y,Tx)]+\delta d(x,y)\\
\leq\alpha.&\frac{\sqrt{2}\left[1+\sqrt{2}\right]}{1+\sqrt{2}}+\beta.2\sqrt{2}+\gamma.0+\delta.\sqrt{2}\\
=(\alpha&+2\beta+\delta).\sqrt{2},
\end{aligned}\tag{12}$$

and thus, $\alpha+2\beta+\delta\geq 1$. Consequently, this example can not treated by the main theorem of Singh, Badshah and Rathore (cf [8]).

Moreover, notice that in this example we have the uniqueness of fixed point and $(X,\leq)$ does not satisfy condition (5). This proves that condition (5) is not necessary condition for the uniqueness of the fixed point.

Now, in the next theorem we establish a fixed point of a self mapping T by assuming only the continuity of some iteration of T.

Theorem 324*Let $(X,d,\preceq)$ be a complete partially ordered metric space. Suppose that a self-mapping T is non-decreasing and an almost Singh, Badshah and Rathore contraction. Suppose there exists $x_0\in X$ with $x_0\preceq Tx_0$. If the operator T^p is continuous for some positive integer p, then T has a fixed point in X.*

*Proof.*From Theorem 314, we construct a non-decreasing sequence $\{x_n\}$ in X such that $x_n\to z$, for some $z\in X$. Also, its subsequence $x_{n_k}(n_k=kr)$ converges to the same point z. Therefore,

$$T^pz=T^p\left(\lim_{n\to+\infty}x_{n_k}\right)=\lim_{n\to+\infty}x_{n_{k+1}}=z.$$

Thus, z is a fixed point of T^p.

Next to prove that z is a fixed point of T. Let m be the smallest positive integer such that $T^mz=z$ but $T^qz\neq z$ $(q=1,2,3,...,m-1)$. If $m>1$, then

$$\begin{aligned}
d(Tz,z)&=d(Tz,T^mz)\\
&\leq\alpha\frac{d(z,Tz)\left[1+d(T^{m-1}z,T^mz)\right]}{1+d(z,T^{m-1}z)}\\
&\quad+\beta[d(z,Tz)+d(T^{m-1}z,T^mz)]\\
&\quad+\gamma[d(z,T^mz)+d(T^{m-1}z,Tz)]+\delta d(z,T^{m-1}z)\\
&\quad+L\ \min\{d(z,T^mz),d(T^{m-1}z,Tz),d(z,T^mz),\\
&\qquad d(T^{m-1}z,T^mz)\},
\end{aligned}$$

which implies that

$$d(z,Tz)\leq\left(\frac{\beta+\gamma+\delta}{1-\alpha-\beta-\gamma}\right)d(z,T^{m-1}z).$$

Regarding (314), we have

$$\begin{aligned}
&d(z,T^{m-1}z)=d(T^mz,T^{m-1}z)\\
&\leq\alpha\frac{d(T^{m-1}z,T^mz)\left[1+d(T^{m-2}z,T^{m-1}z)\right]}{1+d(T^{m-2}z,T^{m-1}z)}\\
&\quad+\beta[d(T^{m-1}z,T^mz)+d(T^{m-2}z,T^{m-1}z)]\\
&\quad+\gamma[d(T^{m-1}z,T^{m-1}z)+d(T^{m-2}z,T^mz)]\\
&\quad+\delta d(T^{m-1}z,T^{m-2}z)\\
&\quad+L\ \ \min\{d(T^{m-1}z,T^mz),d(T^{m-2}z,T^{m-1}z),\\
&\qquad d(T^{m-1}z,T^{m-1}z),d(T^{m-2}z,T^mz)\}.
\end{aligned}$$

Inductively, we get

$$d(z,T^{m-1}z)=d(T^mz,T^{m-1}z)\leq kd(T^{m-1}z,T^{m-2}z)\leq...$$
$$.......\leq k^{m-1}d(Tz,z),$$

where $k=\frac{\beta+\gamma+\delta}{1-\alpha-\beta-\gamma}$. Notice that $k<1$. Therefore

$$d(Tz,z)\leq k^md(Tz,z)<d(Tz,z),$$

a contradiction. Hence $Tz=z$.

Corollary 325*Let $(X,d,\preceq)$ be a complete partially ordered metric space. Suppose that a self-mapping T is non-decreasing and a Singh, Badshah and Rathore contraction. Suppose there exists $x_0\in X$ with $x_0\preceq Tx_0$. If the operator T^p is continuous for some positive integer p, then T has a fixed point in X.*

*Proof.*Set $L=0$ in Theorem 324.

Theorem 326*Let $(X,d,\preceq)$ be a complete partially ordered metric space and let T be a non-decreasing self mapping defined on X. Suppose that for some positive integer m, self* *mapping T satisfies the following condition*

$$d(T^m x, T^m y) \leq \alpha \frac{d(x, T^m x)\,[1+d(y, T^m y)]}{1+d(x,y)} + \beta[d(x,T^m x)+d(y,T^m y)] + \gamma[d(x,T^m y)+d(y,T^m x)] + \delta d(x,y) + L\min\{d(x,Ty), d(y,Tx), d(x,Tx), d(y,Ty)\}, \quad (13)$$

for all distinct $x,y \in X$ with $x \preceq y$, where $L \geq 0$ and $\alpha,\beta,\gamma,\delta \in [0,1)$ with $0 \leq \alpha+2(\beta+\gamma)+\delta < 1$. Suppose there exists $x_0 \in X$ with $x_0 \preceq T^m x_0$. If T^m is continuous, then T has a fixed point in X.

*Proof.*The proof follows from Theorem 314 and Theorem 324.

Corollary 327*Let $(X,d,\preceq)$ be a complete partially ordered metric space and let T be a non-decreasing self mapping defined on X. Suppose that for some positive integer m, self mapping T satisfies the following condition:*

$$d(T^m x, T^m y) \leq \alpha \frac{d(x, T^m x)\,[1+d(y, T^m y)]}{1+d(x,y)} + \beta[d(x,T^m x)+d(y,T^m y)] + \gamma[d(x,T^m y)+d(y,T^m x)] + \delta d(x,y), \quad (14)$$

for all $x,y \in X$ with $x \preceq y$, where $\alpha,\beta,\gamma,\delta \in [0,1)$ with $0 \leq \alpha+2(\beta+\gamma)+\delta < 1$. Suppose there exists $x_0 \in X$ with $x_0 \preceq T^m x_0$. If T^m is continuous, then T has a fixed point in X.

*Proof.*Set $L = 0$ in Theorem 326.

Now, we give the following example.

Example 328*Let $X = [0,1]$ with the usual metric and usual order $\leq$. Define an operator $T : X \to X$ as follows:*

$$Tx = \begin{cases} 0 & ,if\ x \in [0, \frac{1}{7}], \\ \frac{1}{7} & ,if\ x \in (\frac{1}{7}, 1]. \end{cases}$$

It can be easily seen that T is discontinuous and does not satisfy (7) for any $\alpha,\beta,\gamma,\delta \in [0,1)$ with $0 \leq \alpha+2(\beta+\gamma)+\delta < 1$ when $x = \frac{1}{7}, y = 1$. Now $T^2(x) = 0$ for all $x \in [0,1]$. It can be verified that T^2 satisfies the conditions of Theorem 326 and 0 is a unique fixed point of T^2.

Remark 329*In [14], instead of condition (4), the authors use the following weaker condition:*

if a nondecreasing (nonincreasing) sequence $\{x_n\} \to x$ in X, then $x_n \preceq x$ ($x \preceq x_n$), for all $n \in \mathbb{N}$. (15)

we have not been able to prove Theorem 32, 314 and its consequences using (15).

Some other consequences of the main Theorem 314 for the self mapping involving in the integral type contractions are as follows.

Corollary 330*Let $(X,d,\preceq)$ be a T-orbitally complete partially ordered metric space. Suppose that $T : X \to X$ be a non-decreasing, continuous mapping such that*

$$\int_0^{d(Tx,Ty)} ds \leq \alpha \int_0^{\frac{d(x,Tx)[1+d(y,Ty)]}{1+d(x,y)}} ds + \beta \int_0^{d(x,Tx)+d(y,Ty)} ds + \gamma \int_0^{d(x,Ty)+d(y,Ty)} ds + \delta \int_0^{d(x,y)} ds + L \int_0^{\min\{d(x,Ty),d(y,Tx),d(x,Tx),d(y,Ty)\}} ds, \quad (16)$$

for all distinct $x,y \in X$ with $x \preceq y$ and there exist $\alpha,\beta,\gamma,\delta \in [0,1)$ such that $0 \leq \alpha+2(\beta+\gamma)+\delta < 1$, where $L \geq 0$. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then T has at least one fixed point in X.

Similarly, the following results is the consequence of Corollary321.

Corollary 331*Let T be a continuous, non-decreasing self-mapping defined on a complete partially ordered metric space $(X,d,\preceq)$. Suppose that T satisfies the following condition*

$$\int_0^{d(Tx,Ty)} ds \leq \alpha \int_0^{\frac{d(x,Tx)[1+d(y,Ty)]}{1+d(x,y)}} ds + \beta \int_0^{d(x,Tx)+d(y,Ty)} ds + \gamma \int_0^{d(x,Ty)+d(y,Ty)} ds + \delta \int_0^{d(x,y)} ds, \quad (17)$$

for all distinct $x,y \in X$ with $x \preceq y$ and for $\alpha,\beta,\gamma,\delta \in [0,1)$ with $0 < \alpha+2(\beta+\gamma)+\delta < 1$. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then T has a fixed point in X.

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