Applied Mathematics & Information Sciences
*An International Journal*



# $A$-Statistical Summability of Fourier Series and Walsh-Fourier Series

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Received: May 7, 2011; Revised Nov. 21, 2011; Accepted 16 Dec. 2011
Published online: 1 Sep. 2012

**Abstract:** In this paper we study $A$-statistical summability of conjugate Fourier series, derived Fourier series and Walsh-Fourier series. At the end of the paper it is shown that $A$-statistical summability version is stronger than $A$-summability.

**Keywords:** Statistical convergence; $A$-statistical convergence; Fourier series; Walsh-Fourier series.

## 1. Introduction

Let $K = \{k_i\}$ be an index set and let $\varphi^K$ be the characteristic sequence of $K$, i.e. $\varphi^K = (\varphi_j^K)$ with

$$\varphi_j^K = \begin{cases} 1, & \text{for } j \in K; \\ 0, & \text{otherwise.} \end{cases}$$

If $\varphi^K$ is $C_1$-summable then the limit

$$\lim_n \sum_{j=1}^{n} \varphi_j^K$$

is called the *asymptotic density* of $K$ and is denoted by $\delta(K)$. A sequence $x = (x_k)$ of real numbers is said to be *statistically convergent* to $L$ if $\delta(K_\epsilon) = 0$ for every $\epsilon > 0$, where $K_\epsilon := \{k \in N : |x_k - L| \geq \epsilon\}$ ( cf. [4], [7]). In this case $L$ is called the $st$-limit of $x$.

Let $c$ and $l_\infty$ denote the set of all convergent and bounded sequences, respectively. Let $X$ and $Y$ be two sequence spaces and $A = (a_{nk})_{n;k=1}^\infty$ be an infinite matrix of real or complex numbers. We write $Ax = (A_n(x))$ provided $A_n(x) = \sum_k a_{nk}x_k$ converges for each $n$. A sequence $x = (x_k)$ is said to be $A$-summable to $L$ if $\lim_n A_n(x) = L$. If $x = (x_k) \in X$ implies that $Ax \in Y$ , then we say that $A$ defines a matrix transformation from $X$ into $Y$ and by $(X, Y)$ we denote the class of such matrices. If $X$ and $Y$ are equipped with the limits $X$-lim and $Y$-lim, respectively, $A \in (X, Y)$ and $Y$-$\lim_n A_n(x) = X$-$\lim_k x_k$ for all $x \in X$, then we say that $A$ is a regular map from $X$ into $Y$ and in this case we write $A \in (X, Y)_{reg}$. The matrices $A \in (c, c)_{reg}$ are called *regular*.

For a non-negative regular matrix $A = (a_{nk})_{n,k=1}^\infty$, following Freedman and Sember [6], an index set $K = \{k_i\}$ is said to have $A$-*density*

$$\delta_A(K) = \lim_n A_n \varphi^K$$

if $A\varphi^K \in c$. Thus

$$\delta_A(K) = \lim_n \sum_{k \in K} a_{nk} = \lim_n \sum_i a_{n,k_i}.$$

A sequence $x$ is said to be $A$-*statistically convergent* to $L$ if $\delta_A(K_\epsilon) = 0$ for every $\epsilon > 0$. In this case we write $st_A$-$\lim x = L$, and $x_k \xrightarrow{st_A} L$. By the symbol $st_A$ we denote the set of all $A$-statistically convergent sequences. Note that $c \subset st_A$.

The following characterization is due to Kolk [9, Corollary 4] which will be needed in our results.

**Lemma 1 1** *$B = (b_{nk})_{n,k=1}^\infty \in (c, st_A \cap l_\infty)_{reg}$ if and only if*

*(i) $\| B \| = \sum_k | b_{nk} | < \infty$, and there exists $N = \{n_i\}$ so that $\delta_A(N) = 1$ and*
*(ii) $\lim_i b_{n_i k} = 0$ ($k \in N$), i.e. $st_A$-$\lim b_{nk} = 0$ for each $k$,*

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*(iii)*$\lim_i \sum_k b_{n_i k} = 1$, *i.e.* $st_A$-$\lim \sum_k b_{nk} = 1$.

Zygmund proved some theorems on the statistical convergence of Fourier series in the first edition of his book ( [12], pp 181-188 ). Recently, $A$-statistical convergence have been used for some approximating operators in [1], [3] and [8]; and for Fourier integrals [10]. In this paper, we prove some results on statistical summability of conjugate Fourier series, derived Fourier series and Walsh-Fourier series.

## 2. Preliminaries

2.1.Let $f$ be $L$-integrable and periodic with period $2\pi$, and let the Fourier series of $f$ be

$$\frac{1}{a_0} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx). \tag{2.1}$$

Then the series conjugate to it is

$$\sum_{k=1}^{\infty} (b_k \cos kx - a_k \sin kx), \tag{2.2}$$

with partial sum $\tilde{S}_n(x)$; and the derived series is

$$\sum_{k=1}^{\infty} k (b_k \cos kx - a_k \sin kx), \tag{2.3}$$

with partial sum $S'_n(x)$. We write

$$\psi_x(t) = \psi(f,t)$$

$$\begin{cases} = f(x+t) - f(x-t) \ , & \text{for } 0 < t \leq \pi; \\ g(x) \ , & \text{for } t = 0\,; \end{cases} \tag{2.4}$$

and

$$\beta_x(t) = \frac{\psi_x(t)}{4 \sin \frac{1}{2} t}, \tag{2.5}$$

where $g(x) = f(x+0) - f(x-0)$.

We shall need the following lemma which is statistical version of the Banach Weak Convergence Theorem [2].

**Lemma 21**$st_A$-$\lim_{n\to\infty} \int_0^\pi g_n dh_x = 0$ *for all* $h_x \in BV[0,\pi]$, *if and only if* $\| g_n \| < \infty$ *for all* $n$ *and* $st_A$-$\lim_{n\to\infty} g_n = 0$ *; where* $BV[0,\pi]$ *denotes the set of all functions of bounded variations on* $[0,\pi]$.

Parallel version also holds if $[0,\pi]$ is replaced by $[0,1]$.

2.2.Let us define a sequence of functions $h_0(x), h_1(x), ......, h_n(x)$ which satisfy the following conditions:

$$h_0(x) = \begin{cases} 1 \ , & \text{for } 0 \leq x \leq \frac{1}{2}; \\ -1 \ , & \text{for } \frac{1}{2} \leq x < 1\,; \end{cases} \tag{2.6}$$

$h_0(x+1) = h_0(x)$ and $h_n(x) = h_0(2^n x)$, $n = 1,2,......$The functions $h_n(x)$ are called the $Rademacher's\ functions$.

The $Walsh\ functions$ are defined by $\phi_0(x) = 1$,

$$\phi_n(x) = h_{n_1}(x) h_{n_2}(x) .......h_{n_r}(x),\ 0 \leq x \leq 1, \tag{2.7}$$

for $n = 2^{n_1} + 2^{n_2} + ......2^{n_r}$; where the integers $n_i$ are uniquely determined by $n_{i+1} < n_i$.

Let $f$ be $L$-integrable and periodic with period $1$, and let the Walsh-Fourier series of $f$ be

$$\sum_{n=1}^{\infty} c_n \phi_n(x), \tag{2.8}$$

where

$$c_n = \int_0^1 f(x)\phi_n(x) dx, \tag{2.9}$$

are called the Walsh-Fourier coefficients of $f$.

Let us recall some basic properties of Walsh functions (see [4]).

For each fixed $x \in [0,1)$ and for all $t \in [0,1)$

(i)$\phi_n(x \dot{+} t) = \phi_n(x)\phi_n(t)$,

(ii)$\int_0^1 f(x \dot{+} t) dt = \int_0^1 f(t) dt$, and

(iii)$\int_0^1 f(t)\phi_n(x \dot{+} t) dt = \int_0^1 f(x \dot{+} t)\phi_n(t) dt$;

where $\dot{+}$ denotes the operation in the dyadic group, the set of all sequences $s = (s_n), s_n = 0,1$ for $n = 1,2,.....$, is addition modulo 2 in each coordinate.

Let for $x \in [0,1)$,

$$J_k(x) = \int_0^x \phi_k(t) dt,\ k = 0,1,2,... \tag{2.10}$$

It is easy to see that $J_k(x) = 0$ for $x = 0,1$.

## 3. Main results

We prove the following theorems:

**Theorem 31***Let* $B = (b_{nk})_{n,k=1}^{\infty} \in (c, st_A \cap l_\infty)_{reg}$. *Then for every* $x \in [-\pi,\pi]$ *for which* $\beta_x(t) \in BV[0,\pi]$,

$$st_A\text{-}\lim_m \sum_{n=1}^{\infty} b_{mn} S'_n(x) = \beta_x(0+) \tag{3.1.1}$$

*if and only if*

$$st_A\text{-}\lim_m \sum_{n=1}^{\infty} b_{mn} \sin(n+\frac{1}{2})t = 0 \text{ for all } t \in [0,\pi]. \tag{3.1.2}$$

**Theorem 32***Let* $B \in (c, st_A \cap l_\infty)_{reg}$. *Then for every* $x \in [-\pi,\pi]$ *for which* $\beta_x(t) \in BV[0,\pi]$,

$$st_A\text{-}\lim_m \sum_{n=1}^{\infty} n b_{mn} \tilde{S}_n(x) = \frac{1}{\pi} g(x) \tag{3.2.1}$$

*if and only if*

$$st_A\text{-}\lim_m \sum_{n=1}^{\infty} b_{mn} \cos nt = 0 \text{ for all } t \in [0, \pi]. \quad (3.2.2)$$

**Theorem 33** *Let $B \in (c, st_A \cap l_\infty)_{reg}$. Let $z_k(x) = c_k\phi_k(x)$ for an $L$-integrable function $f \in BV[0,1)$. Then for every $x \in [0,1)$*

$$st_A\text{-}\lim_n \sum_{k=1}^{\infty} b_{nk} z_k(x) = 0 \quad (3.3.1)$$

*if and only if*

$$st_A\text{-}\lim_n \sum_{k=1}^{\infty} b_{nk} J_k(x) = 0, \quad (3.3.2)$$

*where $x$ is a point at which $f(x)$ is of bounded variation.*

## 4. Proofs

*Proof(Proof of Theorem 3.1. ).* We have

$$S_n'(x) = \frac{1}{\pi}\int_0^\pi \psi_x(t)\left(\sum_{k=1}^{n} k \sin kt\right) dt$$

$$= -\frac{1}{\pi}\int_0^\pi \psi_x(t)\frac{d}{dt}\left(\frac{\sin(n+\frac{1}{2})t}{2\sin\frac{t}{2}}\right) dt,$$

$$= I_n + \frac{2}{\pi}\int_0^\pi \sin(n+\frac{1}{2})t \, d\beta_x(t), \quad (3.1.3)$$

where

$$I_n = \frac{1}{\pi}\int_0^\pi \beta_x(t)\cos\frac{t}{2}\left(\frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}\right) dt. \quad (3.1.4)$$

Then

$$\sum_{n=1}^{\infty} b_{mn} S_n'(x) = \sum_{n=1}^{\infty} b_{mn} I_n + \frac{2}{\pi}\int_0^\pi L_m(t)\, d\beta_x(t), \quad (3.1.5)$$

where

$$L_m(t) = \sum_{n=1}^{\infty} b_{mn} \sin(n+\frac{1}{2})t. \quad (3.1.6)$$

Since $\beta_x(t)$ is of bounded variation on $[0,\pi]$ and $\beta_x(t) \to \beta_x(0+)$ as $t \to 0$, $\beta_x(t)\cos\frac{t}{2}$ has also the same property. Hence by Jordan's convergence criterion for Fourier series $I_n \to \beta_x(0+)$ as $n \to \infty$.

Since $B \in (c, st_A \cap l_\infty)_{reg}$, by the conditions of Lemma 1.1, we get

$$st_A\text{-}\lim_m \sum_{n=1}^{\infty} b_{mn} I_n = \beta_x(0+). \quad (3.1.7)$$

Now, it is enough to show that (3.1.2) holds if and only if

$$st_A\text{-}\lim_m \int_0^\pi L_m(t)\, d\beta_x(t) = 0. \quad (3.1.8)$$

Hence, by Lemma 2.1, it follows that (2.1.8) holds if and only if

$$\| L_m(t) \| \le M \text{ for all } m \text{ and for all } t \in [0,\pi], \quad (3.1.9)$$

and (3.1.2) holds. Since (3.1.9) is satisfied by Lemma 1.1$(i)$, it follows that (3.1.8) holds if and only if (3.1.2) holds. Hence the result follows immediately.

*Proof(Proof of Theorem 3.2).* We have

We have

$$\tilde{S}_n(x) = \frac{1}{\pi}\int_0^\pi \psi_x(t)\sin nt\, dt,$$

$$= \frac{g(x)}{n\pi} + \frac{1}{n\pi}\int_0^\pi \cos nt\, d\psi_x(t).$$

Therefore

$$\sum_{n=1}^{\infty} n b_{mn}\tilde{S}_n(x) = \frac{g(x)}{\pi}\sum_{n=1}^{\infty} b_{mn} + \frac{1}{\pi}\int_0^\pi K_m(t)\, d\psi_x(t), \quad (3.2.3)$$

where

$$K_m(t) = \sum_{n=1}^{\infty} b_{mn}\cos nt.$$

Now, taking $st_A$-lim on both sides of (3.2.3) and using Lemmas 1.1 and 2.1 as in the proof of Theorem 3.1, we get the required result.

*Proof(Proof of Theorem 3.3).* We have

$$z_k(x) = c_k\phi_k(x) = \int_0^1 f(t)\phi_k(t)\phi_k(x)dt,$$

$$= \int_0^1 f(t)\phi_k(x\dot{+}t)dt = \int_0^1 f(x\dot{+}t)\phi_k(t)dt,$$

where $x\dot{+}t$ belongs to the set $\Omega$ of dyadic rationals in $[0,1)$, in particular each element of $\Omega$ has the form $p/2^n$ for some non-negative integers $p$ and $n$, $0 \le p < 2^n$. Now, on integration by parts, we obtain

$$z_k(x) = [f(x\dot{+}t)J_k(t)]_0^1 - \int_0^1 J_k(t)df(x\dot{+}t),$$

$$= -\int_0^1 J_k(t)df(x\dot{+}t), \text{ since } J_k(x) = 0 \text{ for } x = 0, 1.$$

Hence

$$\sum_{k=1}^{\infty} b_{nk} z_k(x) = -\int_0^1 D_n(t)\, dh_x(t), \quad (3.3.3)$$

where

$$D_n(t)=\sum_{k=1}^{\infty} b_{nk}J_k(t), \tag{3.3.4}$$

and $h_x(t)=f(x\dot{+}t)$. Write, for any $t\in R$, $g_n=(D_n(t))$.

Since $B\in(c,st_A\cap l_\infty)_{reg}$, it follows by Lemma 1.1 that $\| g_n \|<\infty$ for all $n$, and $g_n \xrightarrow{st_A} 0$. Hence by Lemma 2.1,

$$\int_0^1 D_n(t)\, dh_x(t) \xrightarrow{st_A} 0.$$

Now, taking $st_A$-$\lim$ in (3.3.3) and (3.3.4) and using Lemma 2.1, we get the desired result.

*Remark.* Theorem 3.3 generalizes the result of Mursaleen [11] and the following example shows that this theorem is stronger than that of [11].

**Example.** Let $A$ be a $(C,1)$ matrix, $B=I$ (identity matrix). Let us define a sequence $u=(u_k)$ by

$$u_k:=\begin{cases}1, & \text{if } k=m^2, m\in N;\\ 0, & \text{otherwise.}\end{cases}$$

Then $u$ is $A$-statistically convergent to $0$ but not $A$-summable. Now write $\hat{z}_k(x)=(1+u_k)z_k(x)$ and $\hat{J}_k(x)=(1+u_k)J_k(x)$, it is easy to see that Theorem 3.3 holds if we replace $z_k(x)$ by $\hat{z}_k(x)$ and $J_k(x)$ by $\hat{J}_k(x)$ but Theorem 3.1 of [11] does not hold.

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