Applied Mathematics & Information Sciences **2**(3) (2008), 291-307
– An International Journal

Intuitionistic Supra Gradation of Openness

A. M. Zahran[1], S. E. Abbas[2] and E. El-Sanousy[2]

[1] Department of Mathematics, Faculty of Science, Azhar University, Assuit, Eygpt

Email Address: amzahran2000@yahoo.com

[2]Department of Mathematics, Faculty of Science, Sohag University, Sohag, Eygpt

Email Addresses: sabbas73@yahoo.com; elsanowsy@yahoo.com

Received August 23, 2007; Revised January 12, 2008

In this paper, we have used the intuitionistic supra gradation of openness that was created from an intuitionistic fuzzy bitopological spaces to introduce and study the concepts of continuity, some kinds of separation axioms and compactness.

Keywords: Intuitionistic supra gradation of openness, IFP^*-continuous mapping, IFP^*-separation axioms, IFP^*-compact.

1 Introduction and Preliminaries

Kubiak [10] and Šostak [15] introduced the fundamental concept of a fuzzy topological structure, as an extension of both crisp topology and fuzzy topology [3], in the sense that not only the objects are fuzzified, but also the axiomatics. In [16,17] Šostak gave some rules and showed how such an extension can be realized. Chattopadhyay *et al.* [4] have redefined the same concept under the name gradation of openness. A general approach to the study of topological type structures on fuzzy powersets was developed in [7-11].

As a generalization of fuzzy sets, the notion of intuitionistic fuzzy sets was introduced by Atanassov [2]. By using intuitionistic fuzzy sets, Çoker and his colleague [5,6] defined the topology of intuitionistic fuzzy sets. Recently, Samanta and Mondal [14] introduced the notion of intuitionistic gradation of openness of fuzzy sets, where to each fuzzy subsets there is a definite grade of openness and there is a grade of nonopenness. Thus, the concept of intuitionistic gradation of openness is a generalization of the concept of gradation of openness and the topology of intuitionistic fuzzy sets.

In this paper, we have used the intuitionistic supra gradation of openness that was created from an intuitionistic fuzzy bitopological spaces to introduce and study the concepts of continuity, some kinds of separation axioms and compactness.

Throughout this paper, let X be a nonempty set, $I = [0,1]$, $I_0 = (0,1]$, $I_1 = [0,1)$. For $\alpha \in I$, $\underline{\alpha}(x) = \alpha$ for each $x \in X$. The set of all fuzzy subsets of X are denoted by I^X. For $x \in X$ and $t \in I_0$ a fuzzy point is defined by

$$x_t(y) = \begin{cases} t, & \text{if} \quad y = x \\ 0, & \text{if} \quad y \neq x. \end{cases}$$

$x_t \in \lambda$ iff $t \leq \lambda(x)$. We denote a fuzzy set λ which is quasi-coincident with a fuzzy set μ by $\lambda q \mu$, if there exists $x \in X$ such that $\lambda(x) + \mu(x) > 1$. Otherwise by $\lambda \overline{q} \mu$.

Definition 1.1. [1,14] An intuitionistic supra gradation of openness (ISGO, for short) on X is an ordered pair (τ, τ^*) of mappings from I^X to I such that
(ISGO1) $\tau(\lambda) + \tau^*(\lambda) \leq 1, \ \forall \lambda \in I^X$.
(ISGO2) $\tau(\underline{0}) = \tau(\underline{1}) = 1, \ \tau^*(\underline{0}) = \tau^*(\underline{1}) = 0$.
(ISGO3) $\tau(\bigvee_{i\in\Delta} \lambda_i) \geq \bigwedge_{i\in\Delta} \tau(\lambda_i)$ and $\tau^*(\bigvee_{i\in\Delta} \lambda_i) \leq \bigvee_{i\in\Delta} \tau^*(\lambda_i), \forall \lambda_i \in I^X, i \in \Delta$.

The triplet (X, τ, τ^*) is called an intuitionistic supra fuzzy topological space (isfts, for short).

An ISGO (τ, τ^*) is called an intuitionistic gradation of openness (IGO, for short) on X iff (IT) $\tau(\lambda_1 \wedge \lambda_2) \geq \tau(\lambda_1) \wedge \tau(\lambda_2)$ and $\tau^*(\lambda_1 \wedge \lambda_2) \leq \tau^*(\lambda_1) \vee \tau^*(\lambda_2), \forall \lambda_1, \lambda_2 \in I^X$.

The triplet (X, τ, τ^*) is called an intuitionistic fuzzy topological space (ifts, for short). τ and τ^* may be interpreted as gradation of opennes and gradation of nonopenness, respectively. The $(X, (\tau, \tau^*), (\nu, \nu^*))$ is called an intuitionistic fuzzy bitopological space (ifbts, for short) where (τ, τ^*) and (ν, ν^*) are IGO's on X.

Definition 1.2. [1] A map $C : I^X \times I_0 \times I_1 \to I^X$ is called an intuitionistic supra fuzzy closure operator on X if for $\lambda, \mu \in I^X$ and $r \in I_0, s \in I_1$, it satisfies the following conditions:

(C1) $C(\underline{0}, r, s) = \underline{0}$.
(C2) $\lambda \leq C(\lambda, r, s)$.
(C3) $C(\lambda, r, s) \vee C(\mu, r, s) \leq C(\lambda \vee \mu, r, s)$.
(C4) $C(\lambda, r_1, s_1) \leq C(\lambda, r_1, s_2)$ if $r_1 \leq r_2$ and $s_1 \geq s_2$.
(C5) $C(C(\lambda, r, s), r, s) = C(\lambda, r, s)$.

The pair (X, C) is called an intuitionistic supra fuzzy closure space.

The intuitionistic supra fuzzy closure space (X, C) is called the intuitionistic fuzzy closure space iff

(C) $C(\lambda, r, s) \vee C(\mu, r, s) = C(\lambda \vee \mu, r, s)$.

Theorem 1.1. [1] *Let (X, τ, τ^*) be an isfts. Then $\forall \lambda \in I^X, \ r \in I_0, s \in I_1$ we define an operator $C_{\tau,\tau^*} : I^X \times I_0 \times I_1 \to I^X$ as follows:*

$$C_{\tau,\tau^*}(\lambda, r, s) = \bigwedge \{\mu \in I^X : \lambda \leq \mu, \tau(\underline{1} - \mu) \geq r, \tau^*(\underline{1} - \mu) \leq s\}.$$

Then (X, C_{τ,τ^*}) *is an intuitionistic supra fuzzy closure space. The mapping* $I_{\tau,\tau^*} : I^X \times I_0 \times I_1 \to I^X$ *defined by*

$$I_{\tau,\tau^*}(\lambda, r, s) = \bigvee\{\mu \in I^X : \mu \leq \lambda, \tau(\mu) \geq r, \tau^*(\mu) \leq s\}$$

is an intuitionistic supra fuzzy interior space. And $I_{\tau,\tau^*}(\underline{1} - \lambda, r, s) = \underline{1} - C_{\tau,\tau^*}(\lambda, r, s)$.

Theorem 1.2. [1] *Let* (X, C) *be an intuitionistic (intuitionistic supra) fuzzy closure space. Define the mappings* $\tau_c, \tau_c^* : I^X \to I$ *on* X *by*

$$\tau_c(\lambda) = \bigvee\{r \in I_0 : C(\underline{1} - \lambda, r, s) = \underline{1} - \lambda\},$$

$$\tau_c^*(\lambda) = \bigwedge\{s \in I_1 : C(\underline{1} - \lambda, r, s) = \underline{1} - \lambda\}.$$

Then,

(1) (τ_c, τ_c^*) *is an IGO's (ISGO's) on* X*,*

(2) $C_{\tau_C,\tau_C^*} \leq C$.

Theorem 1.3. [1] *Let* $(X, (\tau_1, \tau_1^*), (\tau_2, \tau_2^*))$ *be an isfbts. We define the mappings* $C_{12}, I_{12} : I^X \times I_0 \times I_1 \to I^X$ *as follows:*

$$C_{12}(\lambda, r, s) = C_{\tau_1,\tau_1^*}(\lambda, r, s) \wedge C_{\tau_2,\tau_2^*}(\lambda, r, s),$$

$$I_{12}(\lambda, r, s) = I_{\tau_1,\tau_1^*}(\lambda, r, s) \vee I_{\tau_2,\tau_2^*}(\lambda, r, s),$$

for all $\lambda \in I^X,\ r \in I_0,\ s \in I_1$*. Then,*

(1) (X, C_{12}) *is an intuitionistic supra fuzzy closure space,*

(2) $I_{12}(\underline{1} - \lambda, r, s) = \underline{1} - C_{12}(\lambda, r, s)$.

Corollary 1.1. [1] *Let* (X, C_{12}) *be an intuitionistic supra fuzzy closure space. Then, the mappings* $\tau_{C_{12}}, \tau_{C_{12}}^* : I^X \to I$ *on* X *defined by*

$$\tau_{C_{12}}(\lambda) = \bigvee\{r \in I_0 : C_{12}(\underline{1} - \lambda, r, s) = \underline{1} - \lambda\}$$

and

$$\tau_{C_{12}}^*(\lambda) = \bigwedge\{s \in I_1 : C_{12}(\underline{1} - \lambda, r, s) = \underline{1} - \lambda\}$$

is an ISGO's on X.

Theorem 1.4. [1] *Let* $(X, (\tau_1, \tau_1^*), (\tau_2, \tau_2^*))$ *be an ifbts. Let* (X, C_{12}) *be an intuitionistic supra fuzzy closure space. Define the mappings* $\tau_{su}, \tau_{su}^* : I^X \to I$ *on* X *by*

$$\tau_{su}(\lambda) = \bigvee\{\tau_1(\lambda_1) \wedge \tau_2(\lambda_2) : \lambda = \lambda_1 \vee \lambda_2\},$$

$$\tau_{su}^*(\lambda) = \bigwedge\{\tau_1^*(\lambda_1) \vee \tau_2^*(\lambda_2) : \lambda = \lambda_1 \vee \lambda_2\},$$

where $\bigvee$ *and* $\bigwedge$ *are taken over all families* $\{\lambda_1, \lambda_2 : \lambda = \lambda_1 \vee \lambda_2\}$*. Then,*

(1) $(\tau_{su}, \tau_{su}^*) = (\tau_{c_{12}}, \tau_{c_{12}}^*)$ *is the coarsest ISGO on* X *which is finer than both of* (τ_1, τ_1^*) *and* (τ_2, τ_2^*).

(2) $C_{12} = C_{\tau_{su},\tau_{su}^*} = C_{\tau_{c_{12}},\tau_{c_{12}}^*}$.

Definition 1.3. [13, 14] Let $f : (X, (\tau_1, \tau_1^*), (\tau_2, \tau_2^*)) \to (Y, (\nu_1, \nu_1^*), (\nu_2, \nu_2^*))$ be a mapping. Then f is called

(1) IFP-continuous iff $\tau_i(f^{-1}(\mu)) \geq \nu_i(\mu)$ and $\tau_i^*(f^{-1}(\mu)) \leq \nu_i^*(\mu)\ \forall \mu \in I^Y,\ i = 1, 2$;

(2) IFP-open iff $\tau_i(\lambda) \leq \nu_i(f(\lambda))$ and $\tau_i^*(\lambda) \geq \nu_i^*(f(\lambda))\ \forall \lambda \in I^X,\ i = 1, 2$;

(3) IFP-closed iff $\tau_i(\underline{1} - \lambda) \leq \nu_i(\underline{1} - f(\lambda))$ and $\tau_i^*(\underline{1} - \lambda) \geq \nu_i^*(\underline{1} - f(\lambda))\ \forall \lambda \in I^X,\ i = 1, 2$;

(4) IFP-weakly open iff $\tau_i(\lambda) \geq r$ and $\tau_i^*(\lambda) \leq s \Longrightarrow \nu_i(f(\lambda)) \geq r$ and $\nu_i^*(f(\lambda)) \leq s\ \forall \lambda \in I^X, i = 1, 2$;

(5) IFP-weakly closed iff $\tau_i(\underline{1} - \lambda) \geq r$ and $\tau_i^*(\underline{1} - \lambda) \leq s \Longrightarrow \nu_i(\underline{1} - f(\lambda)) \geq r$ and $\nu_i^*(\underline{1} - f(\lambda)) \leq s\ \forall \lambda \in I^X, i = 1, 2$.

2 IFP^*-Continuous Mapping

Definition 2.1. Let $f : (X, (\tau_1, \tau_1^*), (\tau_2, \tau_2^*)) \to (Y, (\nu_1, \nu_1^*), (\nu_2, \nu_2^*))$ be a mapping. Then f is called IFP^*-continuous (resp. IFP^*-open, IFP^*-closed) iff $f : (X, \tau_{su}, \tau_{su}^*) \to (Y, \nu_{su}, \nu_{su}^*)$ is IF-continuous (resp. IF-open, IF-closed).

Theorem 2.1. *Every IFP-continuous (resp. IFP-open, IFP-closed) is IFP^*-continuous (resp. IFP^*-open, IFP^*-closed).*

Proof. Let $f : (X, (\tau_1, \tau_1^*), (\tau_2, \tau_2^*)) \to (Y, (\nu_1, \nu_1^*), (\nu_2, \nu_2^*))$ be an IFP-continuous mapping and $(X, \tau_{su}, \tau_{su}^*)$, $(Y, \nu_{su}, \nu_{su}^*)$ their associated isfts. Suppose that there exists $\mu \in I^Y$ and $s_0 \in I_1$ such that

$$\tau_{su}^*(f^{-1}(\mu)) \geq s_0 \geq \nu_{su}^*(\mu).$$

There exist $\mu_1, \mu_2 \in I^Y$ with $\mu = \mu_1 \vee \mu_2$ such that $\nu_{su}^*(\mu) = \nu_1^*(\mu_1) \vee \nu_2^*(\mu_2) \leq s_0$. Then $\nu_1^*(\mu_1) \leq s_0$ and $\nu_2^*(\mu_2) \leq s_0$. By IFP-continuity, we have

$$\tau_1^*(f^{-1}(\mu_1)) \leq \nu_1^*(\mu_1) \leq s_0 \ \text{ and } \ \tau_2^*(f^{-1}(\mu_2)) \leq \nu_2^*(\mu_2) \leq s_0.$$

This implies that $\tau_1^*(f^{-1}(\mu_1)) \vee \tau_2^*(f^{-1}(\mu_2)) \leq s_0$, and so $\tau_{su}^*(f^{-1}(\mu)) \leq s_0$. It is contradiction. Hence $\tau_{su}^*(f^{-1}(\mu)) \leq \nu_{su}^*(\mu), \forall\, \mu \in I^Y$.

By the same way, we can prove $\tau_{su}(f^{-1}(\mu)) \geq \nu_{su}(\mu),\ \forall \mu \in I^Y$. So, f is IFP^*-continuous. The other parts can be proved in a similar manner. □

Example 2.1. Let $X = \{a, b, c\}$. Define $\rho_1,\ \rho_2,\ \mu_1,\ \mu_2 \in I^X$ as follows

$\rho_1(a) = 0.3$,	$\rho_1(b) = 0.5$,	$\rho_1(c) = 0.4$,
$\rho_2(a) = 0.2$,	$\rho_2(b) = 0.3$,	$\rho_2(c) = 0.5$,
$\mu_1(a) = 0.3$,	$\mu_1(b) = 0.5$,	$\mu_1(c) = 0.2$,
$\mu_2(a) = 0.5$,	$\mu_2(b) = 0.4$,	$\mu_2(c) = 0.3$.

We define $\tau_1,\ \tau_1^*,\ \tau_2,\ \tau_2^*,\ \nu_1,\ \nu_1^*,\ \nu_2,\ \nu_2^* : I^X \to I$ as follows

$$\tau_1(\rho) = \begin{cases} 1 & \text{if} \quad \rho = \underline{0},\, \underline{1} \\ 0.5 & \text{if} \quad \rho = \rho_1 \\ 0 & \text{otherwise,} \end{cases} \qquad \tau_1^*(\rho) = \begin{cases} 0 & \text{if} \quad \rho = \underline{0},\, \underline{1} \\ 0.4 & \text{if} \quad \rho = \rho_1 \\ 1 & \text{otherwise,} \end{cases}$$

$$\tau_2(\rho) = \begin{cases} 1 & \text{if} \quad \rho = \underline{0},\, \underline{1} \\ 0.6 & \text{if} \quad \rho = \rho_2 \\ 0 & \text{otherwise,} \end{cases} \qquad \tau_2^*(\rho) = \begin{cases} 0 & \text{if} \quad \rho = \underline{0},\, \underline{1} \\ 0.3 & \text{if} \quad \rho = \rho_2 \\ 1 & \text{otherwise,} \end{cases}$$

$$\nu_1(\mu) = \begin{cases} 1 & \text{if} \quad \mu = \underline{0},\, \underline{1} \\ 0.5 & \text{if} \quad \mu = \mu_1 \\ 0 & \text{otherwise,} \end{cases} \qquad \nu_1^*(\rho) = \begin{cases} 0 & \text{if} \quad \mu = \underline{0},\, \underline{1} \\ 0.4 & \text{if} \quad \mu = \mu_1 \\ 1 & \text{otherwise,} \end{cases}$$

$$\nu_2(\mu) = \begin{cases} 1 & \text{if} \quad \mu = \underline{0},\, \underline{1} \\ 0.4 & \text{if} \quad \mu = \mu_2 \\ 0 & \text{otherwise,} \end{cases} \qquad \nu_2^*(\rho) = \begin{cases} 0 & \text{if} \quad \mu = \underline{0},\, \underline{1} \\ 0.5 & \text{if} \quad \mu = \mu_2 \\ 1 & \text{otherwise.} \end{cases}$$

The mapping $f : (X, (\tau_1, \tau_1^*), (\tau_2, \tau_2^*)) \to (X, (\nu_1, \nu_1^*), (\nu_2, \nu_2^*))$ defined by $f(a) = c,\ f(b) = a,\ f(c) = b$, is IFP^*-continuous but not IFP-continuous.

Theorem 2.2. *Let $f : (X, (\tau_1, \tau_1^*), (\tau_2, \tau_2^*)) \to (Y, (\nu_1, \nu_1^*), (\nu_2, \nu_2^*))$ be a mapping. Then the following statements are equivalent:* $\forall \lambda \in I^X,\ \mu \in I^Y,\ r \in I_0, s \in I_1$

(1) *f is IFP^*-continuous.*
(2) $\tau_{su}(\underline{1} - f^{-1}(\mu)) \geq \nu_{su}(\underline{1} - \mu)$ *and* $\tau_{su}^*(\underline{1} - f^{-1}(\mu)) \leq \nu_{su}^*(\underline{1} - \mu)$.
(3) $f(C_{12}(\lambda, r, s)) \leq C_{12}(f(\lambda), r, s)$.
(4) $C_{12}(f^{-1}(\mu), r, s) \leq f^{-1}(C_{12}(\mu, r, s))$.
(5) $f^{-1}(I_{12}(\mu, r, s)) \leq I_{12}(f^{-1}(\mu), r, s)$.

Proof. (1) ⇒ (2) is Obvious.
(2) ⇒ (3): For each $\lambda \in I^X, r \in I_0, s \in I_1$, we have

$$\begin{aligned} & f^{-1}(C_{12}(f(\lambda), r, s)) \\ &= f^{-1}(C_{\nu_{su}, \nu_{su}^*}(f(\lambda), r, s)) \\ &= f^{-1}[\bigwedge\{\eta \in I^Y : f(\lambda) \leq \eta, \nu_{su}(\underline{1} - \eta) \geq r, \nu_{su}^*(\underline{1} - \eta) \leq s\}] \\ &\geq \bigwedge\{f^{-1}(\eta) \in I^X : \lambda \leq f^{-1}(\eta), \tau_{su}(\underline{1} - f^{-1}(\eta)) \geq r, \tau_{su}^*(\underline{1} - f^{-1}(\eta)) \leq s\} \\ &= C_{\tau_{su}, \tau_{su}^*}(\lambda, r, s) = C_{12}(\lambda, r, s). \end{aligned}$$

Thus $f(C_{12}(\lambda, r, s)) \leq C_{12}(f(\lambda), r, s)$.
(3) ⇒ (4): For each $\mu \in I^Y,\ r \in I_0,\ s \in I_1$, put $\lambda = f^{-1}(\mu)$. From (3), we have

$$f(C_{12}(f^{-1}(\mu), r, s)) \leq C_{12}(f(f^{-1}(\mu)), r, s) \leq C_{12}(\mu, r, s),$$

which implies that

$$C_{12}(f^{-1}(\mu), r, s) \leq f^{-1}(f(C_{12}(f^{-1}(\mu), r, s))) \leq f^{-1}(C_{12}(\mu, r, s)).$$

(4) $\Rightarrow$ (5): For each $\mu \in I^Y$, $r \in I_0$, $s \in I_1$, we have

$$C_{12}(f^{-1}(\underline{1} - \mu), r, s)) \leq f^{-1}(C_{12}(\underline{1} - \mu, r, s)),$$

which implies that

$$\underline{1} - f^{-1}(C_{12}(\underline{1} - \mu, r, s)) \leq \underline{1} - C_{12}(f^{-1}(\underline{1} - \mu), r, s),$$

$$\Rightarrow \qquad f^{-1}[\underline{1} - C_{12}(\underline{1} - \mu, r, s)] \leq \underline{1} - C_{12}(f^{-1}(\underline{1} - \mu), r, s).$$

By Theorem 1.3 (2), we have

$$f^{-1}(I_{12}(\underline{1} - \mu, r, s)) \leq I_{12}(\underline{1} - f^{-1}(\underline{1} - \mu), r, s) = I_{12}(f^{-1}(\mu), r, s).$$

(5) $\Rightarrow$ (1): Suppose that there exists $\mu \in I^Y$, $r \in I$, $s \in I_1$ such that

$$\tau^*_{su}(f^{-1}(\mu)) > s \geq \nu^*_{su}(\mu) \text{ and } \tau_{su}(f^{-1}(\mu)) < r \leq \nu_{su}(\mu).$$

Then, there exist $\mu_1, \mu_2 \in I^Y$ such that $\nu^*_{su}(\mu) = \nu^*_1(\mu_1) \vee \nu^*_2(\mu_2)$, $\nu_{su}(\mu) = \nu_1(\mu_1) \wedge \nu_2(\mu_2)$, and $\mu = \mu_1 \vee \mu_2$. This implies that $\nu^*_1(\mu_1) \leq s$ and $\nu^*_2(\mu_2) \leq s$. Also, $\nu_1(\mu_1) \geq r$ and $\nu_2(\mu_2) \geq r$, then, $I_{\nu_1, \nu^*_1}(\mu_1, r, s) = \mu_1$ and $I_{\nu_2, \nu^*_2}(\mu_2, r, s) = \mu_2$. From Theorem 1.3, we have

$$I_{12}(\mu, r, s) = I_{\nu_1, \nu^*_1}(\mu_1, r, s) \vee I_{\nu_2, \nu^*_2}(\mu_2, r, s) = \mu_1 \vee \mu_2 = \mu.$$

By (5), we have

$$f^{-1}(\mu) = I_{12}(f^{-1}(\mu), r, s) = I_{\tau_{su}, \tau^*_{su}}(f^{-1}(\mu), r, s).$$

This implies that $\tau^*_{su}(f^{-1}(\mu)) \leq s$ and $\tau_{su}(f^{-1}(\mu)) \geq r$, which is a contradiction. So, $\tau^*_{su}(f^{-1}(\mu)) \leq \nu^*_{su}(\mu)$ and $\tau_{su}(f^{-1}(\mu)) \geq \nu_{su}(\mu) \forall \mu \in I^Y$. Hence, $f : (X, (\tau_1, \tau^*_1), (\tau_2, \tau^*_2)) \to (Y, (\nu_1, \nu^*_1), (\nu_2, \nu^*_2))$ is IFP^*-continuous. □

Theorem 2.3. *Let* $f : (X, (\tau_1, \tau^*_1), (\tau_2, \tau^*_2)) \to (Y, (\nu_1, \nu^*_1), (\nu_2, \nu^*_2))$ *be a mapping. Then the following statements are equivalent:* $\forall \lambda \in I^X$, $\mu \in I^Y$, $r \in I_0$, $s \in I_1$

(1) f *is* IFP^**-weakly open.*

(2) $f(I_{12}(\lambda, r, s)) \leq I_{12}(f(\lambda), r, s)$.

(3) $I_{12}(f^{-1}(\mu), r, s) \leq f^{-1}(I_{12}(\mu, r, s))$.

Proof. (1) $\Rightarrow$ (2): For each $\lambda \in I^X$ and $r \in I_0,\ s \in I_1$, Since $I_{12}(\lambda, r, s) = I_{\tau_{su},\tau^*_{su}}(\lambda, r, s) \leq \lambda$, we have

$$f(I_{\tau_{su},\tau^*_{su}}(\lambda, r, s)) \leq f(\lambda).$$

Also,

$$\tau_{su}(I_{\tau_{su},\tau^*_{su}}(\lambda, r, s)) \geq r \quad \text{and} \quad \tau^*_{su}(I_{\tau_{su},\tau^*_{su}}(\lambda, r, s) \leq s.$$

By (1),

$$\nu_{su}(f(I_{\tau_{su},\tau^*_{su}}(\lambda, r, s))) \geq r \quad \text{and} \quad \nu^*_{su}(f(I_{\tau_{su},\tau^*_{su}}(\lambda, r, s))) \leq s.$$

Hence

$$f(I_{12}(\lambda, r, s)) \leq I_{12}(f(\lambda), r, s).$$

(2) $\Rightarrow$ (3): For each $\mu \in I^Y,\ r \in I_0,\ s \in I_1$, put $\lambda = f^{-1}(\mu)$. From (2), we have

$$f(I_{12}(f^{-1}(\mu), r, s)) \leq I_{12}(f(f^{-1}(\mu)), r, s) \leq I_{12}(\mu, r, s),$$

which implies that

$$I_{12}(f^{-1}(\mu), r, s) \leq f^{-1}(f(I_{12}(f^{-1}(\mu), r, s))) \leq f^{-1}(I_{12}(\mu, r, s)).$$

(3) $\Rightarrow$ (1): For each $\lambda \in I^X$ with $\tau_{su}(\lambda) \geq r,\ \tau^*_{su}(\lambda) \leq s$ implies $I_{12}(\lambda, r, s) = \lambda$. Put $\mu = f(\lambda)$, by (3), we have

$$I_{12}(\lambda, r, s) \leq I_{12}(f^{-1}(f(\lambda)), r, s) \leq f^{-1}(I_{12}(f(\lambda), r, s)),$$

which implies that $\lambda \leq f^{-1}(I_{12}(f(\lambda), r, s))$ and so $f(\lambda) \leq I_{12}(f(\lambda), r, s)$. Then

$$\nu_{su}(f(\lambda)) \geq r \quad \text{and} \quad \nu^*_{su}(f(\lambda)) \leq s.$$

Hence,

$$f : (X, (\tau_1, \tau^*_1), (\tau_2, \tau^*_2) \to (Y, (\nu_1, \nu^*_1), (\nu_2, \nu^*_2))$$

is IFP^*-weakly open. □

Theorem 2.4. *Let $f : (X, (\tau_1, \tau^*_1), (\tau_2, \tau^*_2)) \to (Y, (\nu_1, \nu^*_1), (\nu_2, \nu^*_2))$ be a mapping. Then the following statements are equivalent:*

(1) *f is IFP^*-weakly closed.*

(2) $C_{12}(f(\lambda), r, s) \leq f(C_{12}(\lambda, r, s)),\ \forall\lambda \in I^X, r \in I_0, s \in I_1$.

Theorem 2.5. *Let $f : (X, (\tau_1, \tau^*_1), (\tau_2, \tau^*_2) \to (Y, (\nu_1, \nu^*_1), (\nu_2, \nu^*_2))$ be a bijective mapping. Then the following statements are equivalent:*

(1) *f is IFP^*-weakly closed.*

(2) $f^{-1}(C_{12}(\mu, r, s)) \leq C_{12}(f^{-1}(\mu), r, s), \forall\mu \in I^Y,\ r \in I_0,\ s \in I_1$.

Proof. (1) ⇒ (2): Put $\lambda = f^{-1}(\mu)$, from Theorem 2.4(2)

$$C_{12}(f(f^{-1}(\mu)), r, s) \leq C_{12}(\mu, r, s) \leq f(C_{12}(f^{-1}(\mu), r, s)).$$

Also, since f is onto, we have

$$f^{-1}(C_{12}(\mu, r, s)) \leq f^{-1}(f(C_{12}(f^{-1}(\mu), r, s))) = C_{12}(f^{-1}(\mu), r, s).$$

(2) ⇒ (1): Put $\mu = f(\lambda)$. Since f is injective,

$$f^{-1}(C_{12}(f(\lambda), r, s)) \leq C_{12}(f^{-1}(f(\lambda)), r, s) = C_{12}(\lambda, r, s).$$

Since f is onto,

$$C_{12}(f(\lambda), r, s) \leq f(C_{12}(\lambda, r, s)).$$

□

3 Some Types of Separation Axioms

Definition 3.1. For $i, j \in \{1, 2\}$, $i \neq j$, an ifbts $(X, (\tau_1, \tau_1^*), (\tau_2, \tau_2^*))$ is called

(1) $IFPR_0$ iff $x_t \overline{q} C_{\tau_i,\tau_i^*}(y_m, r, s)$ implies that $y_m \overline{q} C_{\tau_j,\tau_j^*}(x_t, r, s)$ for any $x_t \neq y_m$.

(2) $IFPR_1$ iff $x_t\ \overline{q} C_{\tau_i,\tau_i^*}(y_m, r, s)$ implies that there exist $\lambda, \mu \in I^X$ with $\tau_i(\lambda) \geq r$, $\tau_i^*(\lambda) \leq s$ and $\tau_j(\mu) \geq r$, $\tau_j^*(\mu) \leq s$ such that $x_t \in \lambda$, $y_m \in \mu$ and $\lambda\ \overline{q}\ \mu$.

(3) $IFPR_2$ iff $x_t \overline{q}\ \rho = C_{\tau_i,\tau_i^*}(\rho, r, s)$ implies that there exist λ, $\mu \in I^X$ with $\tau_i(\lambda) \geq r$, $\tau_i^*(\lambda) \leq s$ and $\tau_j(\mu) \geq r$, $\tau_j^*(\mu) \leq s$ such that $x_t \in \lambda$, $\rho \leq \mu$ and $\lambda\ \overline{q} \mu$.

(4) $IFPR_3$ iff $\eta = C_{\tau_i,\tau_i^*}(\eta, r, s)\ \overline{q} \rho = C_{\tau_j,\tau_j^*}(\rho, r, s)$ implies that there exist λ, $\mu \in I^X$ with $\tau_i(\lambda) \geq r$, $\tau_i^*(\lambda) \leq s$ and $\tau_j(\mu) \geq r$, $\tau_j^*(\mu) \leq s$ such that $\eta \leq \lambda$, $\rho \leq \mu$ and $\lambda\ \overline{q}\ \mu$.

(5) $IFPT_0$ iff $x_t\ \overline{q}\ y_m$ implies that there exist $\lambda \in I^X$ such that $\tau_i(\lambda) \geq r$, $\tau_i^*(\lambda) \leq s$ and $x_t \in \lambda$, $y_m\ \overline{q}\ \mu$ or $y_m \in \lambda$, $x_t\ \overline{q}\ \mu$.

(6) $IFPT_1$ iff $x_t\ \overline{q}\ y_m$ implies that there exist $\lambda \in I^X$ such that for $i = 1$ or 2 $\tau_i(\lambda) \geq r$, $\tau_i^*(\lambda) \leq s$, $x_t \in \lambda$ and $y_m \overline{q}\ \lambda$.

(7) $IFPT_2$ iff $x_t \overline{q}\ y_m$ implies that there exist $\lambda,\ \mu \in I^X$ with $\tau_i(\lambda) \geq r,\ \tau_i^*(\lambda) \leq s$ and $\tau_j(\mu) \geq r$, $\tau_j^*(\mu) \leq s$ such that $x_t \in \lambda$, $y_m \in \mu$ and $\lambda\ \overline{q}\ \mu$.

(8) $IFPT_{2\frac{1}{2}}$ iff $x_t\ \overline{q}\ y_m$ implies that there exist $\lambda, \mu \in I^X$ with $\tau_i(\lambda) \geq r$, $\tau_i^*(\lambda) \leq s$ and $\tau_j(\mu) \geq r$, $\tau_j^*(\mu) \leq s$ such that $x_t \in \lambda$, $y_m \in \mu$ and $C_{\tau_j,\tau_j^*}(\lambda, r, s)\ \overline{q}\ C_{\tau_i,\tau_i^*}(\mu, r, s)$.

(9) $IFPT_3$ iff it is $IFPR_2$ and $IFPT_1$.

(10) $IFPT_4$ iff it is $IFPR_3$ and $IFPT_1$.

(11) IFP^*R_i iff its associated isfts $(X, \tau_{su}, \tau_{su}^*)$ is IFR_i, $i = 0, 1, 2$.

(12) IFP^*T_i iff its associated isfts $(X, \tau_{su}, \tau_{su}^*)$ is IFT_i, $i = 0, 1, 2, 2\frac{1}{2}, 3, 4$.

In this definition if $i = j$ we have the definition of IFR_0, IFR_1, IFR_2, IFR_3, IFT_0, IFT_1, IFT_2, $IFT_{2\frac{1}{2}}$, IFT_3 and IFT_4, respectively.

Theorem 3.1. *Let $(X,(\tau_1,\tau_1^*),(\tau_2,\tau_2^*))$ be an ifbts. Then we have*

(1) $IFPR_i \Rightarrow IFP^*R_i, i = 0,1,2,3.$

(2) $IFPT_i \Rightarrow IFP^*T_i, i = 0,1,2,2\frac{1}{2},3.$

(3) $IFP^*T_i \Rightarrow IFPT_i, i = 0,1.$

(4) $IFP^*R_2 \Rightarrow IFP^*R_1 \Rightarrow IFP^*R_0.$

(5) $IFP^*T_4 \Rightarrow IFP^*T_3 \Rightarrow IFP^*T_{2\frac{1}{2}} \Rightarrow IFP^*T_2 \Rightarrow IFP^*T_1 \Rightarrow IFP^*T_0.$

Proof. (1) Let $(X,(\tau_1,\tau_1^*),(\tau_2,\tau_2^*))$ be an $IFPR_0$ and let $x_t\overline{q}C_{\tau_{su},\tau_{su}^*}(y_m,r,s)$. From Theorem 1.4(2), we have $x_t\overline{q}C_{12}(y_m,r,s)$. Also, by Theorem 1.3, we have $x_t\overline{q}[C_{\tau_1,\tau_1^*}(y_m,r,s) \wedge C_{\tau_2,\tau_2^*}(y_m,r,s)]$. Then, $x_t \in \underline{1} - [C_{\tau_1,\tau_1^*}(y_m,r,s) \wedge C_{\tau_2,\tau_2^*}(y_m,r,s)] = [\underline{1} - C_{\tau_1,\tau_1^*}(y_m,r,s)] \vee [\underline{1} - C_{\tau_2,\tau_2^*}(y_m,r,s)]$, this implies that $x_t \in \underline{1} - C_{\tau_1,\tau_1^*}(y_m,r,s)$ or $x_t \in \underline{1} - C_{\tau_2,\tau_2^*}(y_m,r,s)$.

Therefore, $x_t\overline{q}C_{\tau_1,\tau_1^*}(y_m,r,s)$ or $x_t\overline{q}C_{\tau_2,\tau_2^*}(y_m,r,s)$. Since $(X,(\tau_1,\tau_1^*),(\tau_2,\tau_2^*))$ is $IFPR_0$, we have $y_m\overline{q}C_{\tau_1,\tau_1^*}(x_t,r,s)$ or $y_m\overline{q}C_{\tau_2,\tau_2^*}(x_t,r,s)$ this implies that $y_m\overline{q}[C_{\tau_1,\tau_1^*}(x_t,r,s) \wedge C_{\tau_2,\tau_2^*}(x_t,r,s)] = C_{12}(x_t,r,s) = C_{\tau_{su},\tau_{su}^*}(x_t,r,s)$, so, $(X,(\tau_1,\tau_1^*),(\tau_2,\tau_2^*))$ is IFP^*R_0.

(2) Let $(X,(\tau_1,\tau_1^*),(\tau_2,\tau_2^*))$ be an $IFPT_{2\frac{1}{2}}$ and $x_t\,\overline{q}\,y_m$. Then there exist $\lambda,\ \mu \in I^X$ with $\tau_i(\lambda) \geq r,\ \tau_i^*(\lambda) \leq s$ and $\tau_j(\mu) \geq r,\ \tau_j^*(\mu) \leq s$ for $i,j \in \{1,2\},\ i \neq j$ such that $x_t \in \lambda, y_m \in \mu$ and $C_{\tau_j,\tau_j^*}(\lambda,r,s)\overline{q}C_{\tau_i,\tau_i^*}(\mu,r,s)$. Since $C_{\tau_{su},\tau_{su}^*} \leq C_{\tau_i,\tau_i^*}$ for $i = 1,2$ we have, $C_{\tau_{su},\tau_{su}^*}(\lambda,r,s)\overline{q}\,C_{\tau_{su},\tau_{su}^*}(\mu,r,s)$. Then $(X,(\tau_1,\tau_1^*),(\tau_2,\tau_2^*))$ is $IFP^*T_{2\frac{1}{2}}$.

(3) Let $(X,(\tau_1,\tau_1^*),(\tau_2,\tau_2^*))$ be an IFP^*T_1 and $x_t\overline{q}y_m$. Then there exists $\lambda \in I^X$ such that $x_t \in \lambda,\ \tau_{su}(\lambda) \geq r,\ \tau_{su}^*(\lambda) \leq s$ and $y_m\ \overline{q}\lambda$. Since, $\tau_{su}(\lambda) \geq r,\ \tau_{su}^*(\lambda) \leq s$ there exist $\lambda_1,\ \lambda_2 \in I^X$ such that $\tau_{su}(\lambda) = \tau_1(\lambda_1) \wedge \tau_2(\lambda_2),\ \tau_{su}^*(\lambda) = \tau_1^*(\lambda_1) \vee \tau_2^*(\lambda_2)$ and $\lambda = \lambda_1 \vee \lambda_2$, then $\tau_1(\lambda_1) \geq r,\ \tau_2(\lambda_2) \geq r$ and $\tau_1^*(\lambda_1) \leq s, \tau_2^*(\lambda_2) \leq s$. And $x_t \in \lambda$ implies that $x_t \in \lambda_1$ or $x_t \in \lambda_2$. Also, $y_m\overline{q}\ \lambda$ implies that $y_m\ \overline{q}\ \lambda_1$ and $y_m\ \overline{q}\ \lambda_2$. Thus $(x_t \in \lambda_1, \tau_1(\lambda) \geq r, \tau_1^*(\lambda) \leq s$ and $y_m\ \overline{q}\ \lambda_1)$ or $(x_t \in \lambda_2,\ \tau_2(\lambda) \geq r,\ \tau_2^*(\lambda) \leq s$ and $y_m\ \overline{q}\ \lambda_2)$. Hence, $(X,(\tau_1,\tau_1^*),(\tau_2,\tau_2^*))$ is $IFPT_1$.

(4) and (5) obvious from the definition. Other parts are similarly proved. □

Lemma 3.1. *Let $(X,(\tau_1,\tau_1^*),(\tau_2,\tau_2^*))$ be an ifbts. Then*

(1) *If (X,τ_1,τ_1^*) or (X,τ_2,τ_2^*) is IFT_i, then $(X,(\tau_1,\tau_1^*),(\tau_2,\tau_2^*))$ is $IFP^*T_i, i = 0,1,2,2.5,3.$*

(2) *If (X,τ_1,τ_1^*) or $(X,\tau_2,\tau_2^*))$ is IFR_i, then $(X,(\tau_1,\tau_1^*),(\tau_2,\tau_2^*))$ is $IFP^*R_i,\ i = 0,1,2.$*

Proof. (2) Let (X,τ_1,τ_1^*) or (X,τ_2,τ_2^*) be an IFR_0. For any two fuzzy points $x_t \neq y_m$ such that $x_t\ \overline{q}\ C_{\tau_{su},\tau_{su}^*}(y_m,r,s)$ this implies that $x_t\ \overline{q}\ [C_{\tau_1,\tau_1^*}(y_m,r,s) \wedge C_{\tau_2,\tau_2^*}(y_m,r,s)]$ implies $x_t\ \overline{q}\ C_{\tau_1,\tau_1^*}(y_m,r,s)$ or $x_t\ \overline{q}\ C_{\tau_2,\tau_2^*}(y_m,r,s)$. Then $y_m\ \overline{q}\ C_{\tau_1,\tau_1^*}(x_t,r,s)$ or $y_m\ \overline{q}\ C_{\tau_2,\tau_2^*}(x_t,r,s)$ this implies that $y_m\ \overline{q}\ [C_{\tau_1,\tau_1^*}(x_t,r,s) \wedge C_{\tau_2,\tau_2^*}(x_t,r,s)] = C_{12}(x_t,r,s) = C_{\tau_{su},\tau_{su}^*}(x_t,r,s)$. This implies that $(X,(\tau_1,\tau_1^*),(\tau_2,\tau_2^*))$ is IFP^*R_0. □

Example 3.1. Let $X = \{a, b\}$. Define $\tau_i,\ \tau_i^* : I^X \to I$, $i \in \{1, 2, 3, \dots, 12\}$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if} \quad \lambda = \underline{0}, \underline{1} \\ 0.5 & \text{if} \quad \lambda \in \{a_\alpha \vee b_{0.5}, a_{0.5} \vee b_\alpha\}, \qquad \alpha \in (0,1) - \{0.5\} \\ 0.5 & \text{if} \quad \lambda = \underline{\alpha}, \qquad \alpha \in (0,1) \\ 0 & \text{otherwise}, \end{cases}$$

$$\tau_1^*(\lambda) = \begin{cases} 0 & \text{if} \quad \lambda = \underline{0}, \underline{1} \\ 0.5 & \text{if} \quad \lambda \in \{a_\alpha \vee b_{0.5}, a_{0.5} \vee b_\alpha\}, \qquad \alpha \in (0,1) - \{0.5\} \\ 0.5 & \text{if} \quad \lambda = \underline{\alpha}, \qquad \alpha \in (0,1) \\ 1 & \text{otherwise}, \end{cases}$$

$$\tau_2(\lambda) = \begin{cases} 1 & \text{if} \quad \lambda = \underline{0}, \underline{1} \\ 0.5 & \text{if} \quad \lambda = \underline{0.4} \\ 0 & \text{otherwise}, \end{cases} \qquad \tau_2^*(\lambda) = \begin{cases} 0 & \text{if} \quad \lambda = \underline{0},\ \underline{1} \\ 0.5 & \text{if} \quad \lambda = \underline{0.4},\ \underline{0.5} \\ 1 & \text{otherwise}, \end{cases}$$

$$\tau_3(\lambda) = \begin{cases} 1 & \text{if} \quad \lambda = \underline{0}, \underline{1} \\ 0.5 & \text{if} \quad \underline{0} \neq \lambda < \underline{0.5} \\ 0.4 & \text{if} \quad \underline{0.5} < \lambda \neq \underline{1} \\ 0 & \text{otherwise}, \end{cases} \qquad \tau_3^*(\lambda) = \begin{cases} 0 & \text{if} \quad \lambda = \underline{0}, \underline{1} \\ 0.5 & \text{if} \quad \underline{0} \neq \lambda < \underline{0.5} \\ 0.6 & \text{if} \quad \underline{0.5} < \lambda \neq \underline{1} \\ 0.6 & \text{if} \quad \lambda = \underline{0.5} \\ 1 & \text{otherwise}, \end{cases}$$

$$\tau_4(\lambda) = \begin{cases} 1 & \text{if} \quad \lambda = \underline{0},\ \underline{1} \\ 0.4 & \text{if} \quad \lambda = \underline{0.5} \\ 0 & \text{otherwise}, \end{cases} \qquad \tau_4^*(\lambda) = \begin{cases} 0 & \text{if} \quad \lambda = \underline{0},\ \underline{1} \\ 0.6 & \text{if} \quad \lambda = \underline{0.5} \\ 1 & \text{otherwise}, \end{cases}$$

$$\tau_5(\lambda) = \begin{cases} 1 & \text{if} \quad \lambda = \underline{0},\ \underline{1} \\ 0.5 & \text{if} \quad \lambda \in \{a_1, b_1, a_{0.4}, b_{0.4}\} \\ 0.6 & \text{if} \quad \lambda \in \{\underline{0.4}, a_{0.4} \vee b_1, a_1 \vee b_{0.4}\} \\ 0 & \text{otherwise}, \end{cases}$$

$$\tau_5^*(\lambda) = \begin{cases} 0 & \text{if} \quad \lambda = \underline{0},\ \underline{1} \\ 0.5 & \text{if} \quad \lambda \in \{a_1, b_1, a_{0.4}, b_{0.4}, a_{0.6}, b_{0.6}, a_{0.4} \vee b_{0.6}, a_{0.6} \vee b_{0.4}\} \\ 0.4 & \text{if} \quad \lambda \in \{\underline{0.4}, \underline{0.6}, a_{0.4} \vee b_1, a_1 \vee b_{0.4}, a_{0.6} \vee b_1, a_1 \vee b_{0.6}\} \\ 1 & \text{otherwise}, \end{cases}$$

$$\tau_6(\lambda) = \begin{cases} 1 & \text{if} \quad \lambda = \underline{0},\ \underline{1} \\ 0.6 & \text{if} \quad \lambda = \underline{0.7} \\ 0 & \text{otherwise,} \end{cases} \qquad \tau_6^*(\lambda) = \begin{cases} 0 & \text{if} \quad \lambda = \underline{0},\ \underline{1} \\ 0.4 & \text{if} \quad \lambda = \underline{0.7} \\ 0.6 & \text{if} \quad \lambda = \underline{0.3} \\ 1 & \text{otherwise,} \end{cases}$$

$$\tau_7(\lambda) = \begin{cases} 1 & \text{if} \quad \lambda = \underline{0},\ \underline{1} \\ 0.5 & \text{if} \quad \lambda \in \{a_\alpha, b_\alpha\}, \qquad \alpha \in (0,1) \\ 0.6 & \text{if} \quad \lambda = \underline{\alpha}, \qquad \alpha \in (0,1) \\ 0 & \text{otherwise,} \end{cases}$$

$$\tau_7^*(\lambda) = \begin{cases} 0 & \text{if} \quad \lambda = \underline{0},\ \underline{1} \\ 0.5 & \text{if} \quad \lambda \in \{a_\alpha, b_\alpha, a_\alpha \vee b_1, a_1 \vee b_\alpha\}, \qquad \alpha \in (0,1) \\ 0.4 & \text{if} \quad \lambda = \underline{\alpha}, \qquad \alpha \in (0,1) \\ 1 & \text{otherwise,} \end{cases}$$

$$\tau_8(\lambda) = \begin{cases} 1 & \text{if} \quad \lambda = \underline{0},\ \underline{1} \\ 0.5 & \text{if} \quad \lambda = \underline{0.6} \\ 0 & \text{otherwise,} \end{cases} \qquad \tau_8^*(\lambda) = \begin{cases} 0 & \text{if} \quad \lambda = \underline{0},\ \underline{1} \\ 0.5 & \text{if} \quad \lambda = \underline{0.6} \\ 0.6 & \text{if} \quad \lambda = \underline{0.4} \\ 1 & \text{otherwise,} \end{cases}$$

$$\tau_9(\lambda) = \begin{cases} 1 & \text{if} \quad \lambda = \underline{0},\ \underline{1} \\ 0.5 & \text{if} \quad \underline{0} \neq \lambda < \underline{1} \\ 0 & \text{otherwise,} \end{cases} \qquad \tau_9^*(\lambda) = \begin{cases} 0 & \text{if} \quad \lambda = \underline{0},\ \underline{1} \\ 0.5 & \text{if} \quad \underline{0} \neq \lambda < \underline{1} \\ 1 & \text{otherwise,} \end{cases}$$

$$\tau_{10}(\lambda) = \begin{cases} 1 & \text{if} \quad \lambda = \underline{0},\ \underline{1} \\ 0.4 & \text{if} \quad \underline{0.5} < \lambda < \underline{1} \\ 0 & \text{otherwise,} \end{cases} \qquad \tau_{10}^*(\lambda) = \begin{cases} 0 & \text{if} \quad \lambda = \underline{0},\ \underline{1} \\ 0.6 & \text{if} \quad \underline{0.5} < \lambda < \underline{1} \\ 1 & \text{otherwise,} \end{cases}$$

$$\tau_{11}(\lambda) = \begin{cases} 1 & \text{if} \quad \lambda = \underline{0},\ \underline{1} \\ 0.5 & \text{if} \quad \lambda \in \{\underline{\alpha}, a_\alpha \vee b_1, a_1 \vee b_\alpha\}, \qquad \alpha \in (0,1) \\ 0 & \text{otherwise,} \end{cases}$$

$$\tau_{11}^*(\lambda) = \begin{cases} 0 & \text{if} \quad \lambda = \underline{0},\ \underline{1} \\ 0.5 & \text{if} \quad \lambda \in \{\alpha, 1\}, \{1, \alpha\}, \qquad \alpha \in (0,1) \\ 0.4 & \text{if} \quad \lambda = \underline{\alpha}, \qquad \alpha \in (0,1) \\ 1 & \text{otherwise,} \end{cases}$$

$$\tau_{12}(\lambda)=\begin{cases}1 & \text{if} \quad \lambda=\underline{0},\underline{1}\\ 0.5 & \text{if} \quad \lambda=\underline{0.3}\\ 0 & \text{otherwise,}\end{cases} \qquad \tau^*_{12}(\lambda)=\begin{cases}0 & \text{if} \quad \lambda=\underline{0},\underline{1}\\ 0.5 & \text{if} \quad \lambda=\underline{0.3}\\ 0.6 & \text{if} \quad \lambda=\underline{0.4}\\ 1 & \text{otherwise.}\end{cases}$$

(1) For $0 < r \leq 0.5,\ 0.5 \leq s < 1,\ (X,(\tau_1,\tau_1^*),(\tau_2,\tau_2^*))$ is IFP^*R_1, but it is neither $IFPR_1$ nor $IFPR_0$.
(2) For $0 < r \leq 0.4,\ 0.6 \leq s < 1,\ (X,(\tau_3,\tau_3^*),(\tau_4,\tau_4^*))$ is IFP^*R_2, but it is not $IFPR_2$.
(3) For $0 < r \leq 0.4,\ 0.6 \leq s < 1,\ (X,(\tau_5,\tau_5^*),(\tau_6,\tau_6^*))$ is IFP^*T_2, but it is not $IFPT_2$.
(4) For $0 < r \leq 0.4,\ 0.6 \leq s < 1,\ (X,(\tau_7,\tau_7^*),(\tau_8,\tau_8^*))$ is $IFP^*T_{2\frac{1}{2}}$, but it is not $IFPT_{2\frac{1}{2}}$.
(5) For $0 < r \leq 0.4,\ 0.6 \leq s < 1,\ (X,(\tau_9,\tau_9^*),(\tau_{10},\tau_{10}^*))$ is IFP^*T_3, but it is not $IFPT_3$.
(6) For $0 < r \leq 0.4,\ 0.6 \leq s < 1,\ (X,(\tau_{11},\tau_{11}^*),(\tau_{12},\tau_{12}^*))$ is IFP^*R_0, but it is not IFP^*R_1.

Lemma 3.2. [12] *Let $(X,(\tau_1,\tau_1^*),(\tau_2,\tau_2^*))$ be an ifbts. For $r \in I_0,\ s \in I_1$, we have*

(1) *For $\lambda \in I^X$ with $\tau_{su}(\lambda) \geq r,\ \tau^*_{su}(\lambda) \leq s,\ \lambda q \mu$ iff $\lambda q C_{12}(\mu,r,s),\ \mu \in I^X$.*

(2) *$x_t q C_{12}(\lambda,r,s)$ iff $\lambda q \mu$ for all $\mu \in I^X$ with $\tau_{su}(\mu) \geq r,\ \tau^*_{su}(\mu) \leq s$ and $x_t \in \mu$.*

Theorem 3.2. *Let $(X,(\tau_1,\tau_1^*),(\tau_2,\tau_2^*))$ be an ifbts. Then, $\forall \lambda \in I^X,\ r \in I_0, s \in I_1$, the following statements are equivalent:*

(1) *$(X,(\tau_1,\tau_1^*),(\tau_2,\tau_2^*))$ is IFP^*R_0.*

(2) *$C_{12}(x_t,r,s) \leq \lambda$ with $\tau_{su}(\lambda) \geq r,\ \tau^*_{su}(\lambda) \leq s,\ x_t \in \lambda$.*

(3) *If $x_t\ \overline{q}\ \lambda = C_{\tau_{su},\tau^*_{su}}(\lambda,r,s)$, there exists $\mu \in I^X$ with $\tau_{su}(\mu) \geq r,\ \tau^*_{su}(\mu) \leq s$ such that $x_t\ \overline{q}\ \mu$ and $\lambda \leq \mu$.*

(4) *If $x_t\ \overline{q}\ \lambda = C_{\tau_{su},\tau^*_{su}}(\lambda,r,s)$ then, $C_{\tau_{su},\tau^*_{su}}(x_t,r,s)\ \overline{q}\ \lambda = C_{\tau_{su},\tau^*_{su}}(\lambda,r,s)$.*

Proof. (1) $\Rightarrow$ (2): Let $y_m\ q\ C_{12}(x_t,r,s)$. By Theorem 1.3, we have $y_m\ q\ C_{\tau_{su},\tau^*_{su}}(x_t,r,s)$. Using (1), we obtain $x_t\ q\ C_{\tau_{su},\tau^*_{su}}(y_m,r,s)$, i.e. $x_t\ q\ C_{12}(y_m,r,s)$. Using Lemma 3.2(2), we find that $y_m\ q\ \mu\ \forall \mu \in I^X$ with $\tau_{su}(\mu) \geq r,\ \tau^*_{su}(\mu) \leq s$ and $x_t \in \mu$. Then, we have $C_{12}(x_t,r,s) \leq \mu$.

(2) $\Rightarrow$ (1): If $y_m\ \overline{q}\ C_{\tau_{su},\tau^*_{su}}(x_t,r,s)$, we have $y_m \in \underline{1} - C_{\tau_{su},\tau^*_{su}}(x_t,r,s)$. By (2) and the fact $\tau_{su}(\underline{1} - C_{\tau_{su},\tau^*_{su}}(x_t,r,s)) \geq r, \quad \tau^*_{su}(\underline{1} - C_{\tau_{su},\tau^*_{su}}(x_t,r,s)) \leq s$, we get

$$C_{12}(y_m,r,s) \leq \underline{1} - C_{\tau_{su},\tau^*_{su}}(x_t,r,s) \leq \underline{1} - x_t.$$

Thus, $x_t\ \overline{q}\ C_{12}(y_m,r,s) = C_{\tau_{su},\tau^*_{su}}(y_m,r,s)$. Hence, $(X,(\tau_1,\tau_1^*),(\tau_2,\tau_2^*))$ is IFP^*R_0.

(1) $\Rightarrow$ (3): Let $x_t \; \overline{q} \; \lambda = C_{\tau_{su},\tau^*_{su}}(\lambda, r, s)$. Since $C_{\tau_{su},\tau^*_{su}}(y_m, r, s) \leq C_{\tau_{su},\tau^*_{su}}(\lambda, r, s)$, $\forall y_m \in \lambda$, we have $x_t \; \overline{q} \; C_{\tau_{su},\tau^*_{su}}(y_m, r, s)$. By (1), we have $y_m \; \overline{q} \; C_{\tau_{su},\tau^*_{su}}(x_t, r, s)$. Using Lemma 3.2(2), $\forall \, y_m \; \overline{q} \; C_{\tau_{su},\tau^*_{su}}(x_t, r, s)$, there exists $\eta \in I^X$ such that $x_t \; \overline{q} \; \eta$, $\tau_{su}(\eta) \geq r$, $\tau^*_{su}(\eta) \leq s$ and $y_m \in \eta$. Let $\mu = \bigvee_{y_m \in \lambda} \{\eta : x_t \overline{q} \eta, y_m \in \eta\}$. From the definition of ISGO, we have $\tau_{su}(\mu) \geq r$, $\tau^*_{su}(\mu) \leq s$. Then, $x_t \; \overline{q} \; \mu, \lambda \leq \mu$, $\tau_{su}(\mu) \geq r$, $\tau^*_{su}(\mu) \leq s$.

(3) $\Rightarrow$ (4): Let $x_t \; \overline{q} \; \lambda = C_{\tau_{su},\tau^*_{su}}(\lambda, r, s)$. By (3), there exists $\mu \in I^X$ such that $x_t \; \overline{q} \; \mu$, $\lambda \leq \mu$ with $\tau_{su}(\mu) \geq r$, $\tau^*_{su}(\mu) \leq s$. Since $x_t \; \overline{q} \; \mu$, it follows that $x_t \in \underline{1} - \mu$, which implies that

$$C_{\tau_{su},\tau^*_{su}}(x_t, r, s) \leq C_{\tau_{su},\tau^*_{su}}(\underline{1} - \mu, r, s) = \underline{1} - \mu \leq \underline{1} - \lambda.$$

Hence, $C_{\tau_{su},\tau^*_{su}}(x_t, r, s) \; \overline{q} \; \lambda = C_{\tau_{su},\tau^*_{su}}(\lambda, r, s)$.

(4) $\Rightarrow$ (1): Let $x_t \; \overline{q} \; C_{\tau_{su},\tau^*_{su}}(y_m, r, s)$. By (4), we have $C_{\tau_{su},\tau^*_{su}}(x_t, r, s) \; \overline{q} \; C_{\tau_{su},\tau^*_{su}}(y_m, r, s)$ and since $y_m \leq C_{\tau_{su},\tau^*_{su}}(y_m, r, s)$, $y_m \; \overline{q} \; C_{\tau_{su},\tau^*_{su}}(x_t, r, s)$. Hence $(X, (\tau_1, \tau^*_1), (\tau_2, \tau^*_2))$ is IFP^*R_0. □

Theorem 3.3. *An ifbts $(X, (\tau_1, \tau^*_1), (\tau_2, \tau^*_2))$ is an IFP^*R_1 iff $x_t \; \overline{q} \; C_{12}(y_m, r, s)$, there exist $\lambda_i \in I^X$ for $i = 1, 2$ such that $(\underline{1} - \lambda_1) \; \overline{q} \; (\underline{1} - \lambda_2)$ and $C_{12}(x_t, r, s) \leq \lambda_2$, $C_{12}(y_m, r, s) \leq \lambda_1$, $\tau_{su}(\lambda_i) \geq r$, $\tau^*_{su}(\lambda_i) \leq s$.*

Proof. $(\Rightarrow)$ Let $x_t \; \overline{q} \; C_{12}(y_m, r, s) = C_{\tau_{su},\tau^*_{su}}(y_m, r, s)$. By IFP^*R_1, there exist $\lambda_i \in I^X$ for $i = 1, 2$ with $\lambda_1 \; \overline{q} \; \lambda_2$ such that

$$x_t \in \lambda_1, \; y_m \in \lambda_2 \;\text{ and }\; \tau_{su}(\lambda_i) \geq r, \; \tau^*_{su}(\lambda_i) \leq s.$$

Since $(X, (\tau_1, \tau^*_1), (\tau_2, \tau^*_2))$ is an IFP^*R_1 implies that $(X, (\tau_1, \tau^*_1), (\tau_2, \tau^*_2))$ is an IFP^*R_0, by Theorem 3.2(4), $x_t \; \overline{q} \; (\underline{1} - \lambda_1)$ with $\tau_{su}(\lambda_1) \geq r$, $\tau^*_{su}(\lambda_1) \leq s$ implies $C_{12}(x_t, r, s) \leq \underline{1} - \lambda_1 \leq \lambda_2$. Similarly, $C_{12}(y_m, r, s) \leq \underline{1} - \lambda_2 \leq \lambda_1$.

$(\Leftarrow)$ Straightforward. □

Theorem 3.4. *Let $(X, (\tau_1, \tau^*_1), (\tau_2, \tau^*_2))$ be an ifbts. Then, $\forall \, r \in I_0, s \in I_1$, the following statements are equivalent:*

(1) *$(X, (\tau_1, \tau^*_1), (\tau_2, \tau^*_2))$ is IFP^*R_2.*

(2) *If $x_t \in \lambda$ with $\tau_{su}(\lambda) \geq r$, $\tau^*_{su}(\lambda) \leq s$, there exist $\mu_1 \in I^X$ with $\tau_{su}(\mu_1) \geq r$, $\tau^*_{su}(\mu_1) \leq s$ such that $x_t \in \mu_1 \leq C_{\tau_{su},\tau^*_{su}}(\mu_1, r, s) \leq \lambda$.*

(3) *If $x_t \; \overline{q} \; \lambda$ with $\tau_{su}(\underline{1} - \lambda) \geq r$, $\tau^*_{su}(\underline{1} - \lambda) \leq s$, there exists $\mu_i \in I^X$ with $\tau_{su}(\mu_i) \geq r$, $\tau^*_{su}(\mu_i) \leq s$, $i = 1, 2$ such that $x_t \in \mu_1$, $\lambda \leq \mu_2$ and $C_{\tau_{su},\tau^*_{su}}(\mu_1, r, s) \; \overline{q} \; C_{\tau_{su},\tau^*_{su}}(\mu_2, r, s)$.*

Proof. (1) $\Rightarrow$ (2): Let $x_t \in \lambda$ with $\tau_{su}(\lambda) \geq r,\ \tau^*_{su}(\lambda) \leq s$. Then $x_t\ \overline{q}\ (\underline{1} - \lambda)$. Since $(X, (\tau_1, \tau^*_1), (\tau_2, \tau^*_2))$ is IFP^*R_2, there exists $\mu_i \in I^X$ with $\tau_{su}(\mu_i) \geq r,\ \tau^*_{su}(\mu_i) \leq s$ for $i = 1, 2$ such that $x_t \in \mu_1,\ \underline{1} - \lambda \leq \mu_2$ and $\mu_1\ \overline{q}\ \mu_2$, which implies $x_t \in \mu_1 \leq \underline{1} - \mu_2 \leq \lambda$.

(2) $\Rightarrow$ (3): Let $x_t\ \overline{q}\ \lambda$ with $\tau_{su}(\underline{1} - \lambda) \geq r,\ \tau^*_{su}(\underline{1} - \lambda) \leq s$. Then $x_t \in \underline{1} - \lambda$. By (2), there exists $\mu \in I^X$ with $\tau_{su}(\mu) \geq r,\ \tau^*_{su}(\mu) \leq s$ such that

$$x_t \in \mu \leq C_{\tau_{su}, \tau^*_{su}}(\mu, r, s) \leq \underline{1} - \lambda.$$

Since $\tau_{su}(\mu) \geq r,\ \tau^*_{su}(\mu) \leq s$ and $x_t \in \mu$. Again by (2), there exists $\mu_1 \in I^X$ with $\tau_{su}(\mu_1) \geq r,\ \tau^*_{su}(\mu_1) \leq s$ such that

$$x_t \in \mu_1 \leq C_{\tau_{su}, \tau^*_{su}}(\mu_1, r, s) \leq \mu \leq C_{\tau_{su}, \tau^*_{su}}(\mu, r, s) \leq \underline{1} - \lambda,$$

which implies that

$$\lambda \leq (\underline{1} - C_{\tau_{su}, \tau^*_{su}}(\mu, r, s)) = I_{\tau_{su}, \tau^*_{su}}(\underline{1} - \mu, r, s) \leq \underline{1} - \mu.$$

Put $\mu_2 = I_{\tau_{su}, \tau^*_{su}}(\underline{1} - \mu, r, s)$. Then,

$$C_{\tau_{su}, \tau^*_{su}}(\mu_2, r, s) \leq \underline{1} - \mu \leq \underline{1} - C_{\tau_{su}, \tau^*_{su}}(\mu_1, r, s),$$

that is, $C_{\tau_{su}, \tau^*_{su}}(\mu_1, r, s)\ \overline{q}\ C_{\tau_{su}, \tau^*_{su}}(\mu_2, r, s)$.

(3) $\Rightarrow$ (1): It is trivial. □

4 IFP^*-Compactness

Definition 4.1. Let (X, τ, τ^*) be an ifts and $\mu \in I^X,\ r \in I_0,\ s \in I_1$. Then

(1) The family $\{\eta_j : \tau(\eta_j) \geq r, \tau^*(\eta_j) \leq s, j \in J\}$ is called (τ, τ^*)-cover of μ iff for each $x_t \in \mu$ there exists $j_0 \in J$ such that $x_t \in \eta_{j_0}$.

(2) μ is C-set iff every (τ, τ^*)-cover of μ have a finite subcover.

(3) (X, τ, τ^*) is IF -compact iff $\forall \lambda \in I^X$ such that $\tau(\underline{1} - \lambda) \geq r,\ \tau^*(\underline{1} - \lambda) \leq s$ is C-set.

(4) An ifbts $(X, (\tau_1, \tau^*_1), (\tau_2, \tau^*_2))$ is called IFP^*-compact iff its associated isfts $(X, \tau_{su}, \tau^*_{su})$ is IF-compact.

Theorem 4.1. *Let $(X, (\tau_1, \tau^*_1), (\tau_2, \tau^*_2))$ be an ifbts. If (X, τ_1, τ^*_1) or (X, τ_2, τ^*_2) is IF-compact, then $(X, (\tau_1, \tau^*_1), (\tau_2, \tau^*_2))$ is IFP^*-compact.*

Proof. Suppose that (X, τ_1, τ^*_1) is IF-compact, and $\lambda \in I^X$ such that $\tau_{su}(\underline{1} - \lambda) \geq r$, $\tau^*_{su}(\underline{1} - \lambda) \leq s,\ r \in I_0, s \in I_1$ and $\{\eta_j : \tau_{su}(\eta_j) \geq r,\ \tau^*_{su}(\eta_j) \leq s,\ j \in J\}$ be (τ_{su}, τ^*_{su})-cover of λ. Since $\tau_{su}(\underline{1} - \lambda) \geq r,\ \tau^*_{su}(\underline{1} - \lambda) \leq s$, we can write

$$\lambda = \lambda_1 \wedge \lambda_2, \quad \tau_i(\underline{1} - \lambda_i) \geq r, \quad \tau^*_i(\underline{1} - \lambda_i) \leq s, \quad (i = 1, 2).$$

Then, for every $x_t \in \lambda$, there exists $\eta_{j_0} \in I^X$ with $\tau_{su}(\eta_{j_0}) \geq r,\ \tau^*_{su}(\eta_{j_0}) \leq s$ such that $x_t \in \eta_{j_0} = \eta^{(1)} \vee \eta^{(2)}$, for some $\eta^{(i)} \in I^X$ with $\tau_{su}(\eta^{(i)}) \geq r,\ \tau^*_{su}(\eta^{(i)}) \leq s,\ (i = 1, 2)$. Then, $x_t \in \eta^{(1)}$ or $x_t \in \eta^{(2)}$. Now, the family $\{\eta_i^{(1)} : \tau_1(\eta_i^{(1)}) \geq r, \tau^*_1(\eta_i^{(1)}) \leq s, i \in \Delta\}$ is (τ_1, τ^*_1)-cover of λ_1 or $\{\eta_i^{(2)} : \tau_2(\eta_i^{(2)}) \geq r,\ \tau^*_2(\eta_i^{(2)}) \leq s, i \in \Delta\}$ is (τ_2, τ^*_2)-cover of λ_2. If (X, τ_1, τ^*_1) is IF-compact, then λ_1 is C-set i.e., there exists finite subset Δ_0 of Δ such that $\lambda \leq \lambda_1 \leq \bigvee_{i\in\Delta_0} \eta_i^{(1)}$. Hence, λ is C-set. Consequently $(X, (\tau_1, \tau^*_1), (\tau_2, \tau^*_2))$ is IFP^*-compact. Similarly, if (X, τ_2, τ^*_2) is IF-compact, then $(X, (\tau_1, \tau^*_1), (\tau_2, \tau^*_2))$ is IFP^*-compact. □

Theorem 4.2. *Let $(X, (\tau_1, \tau^*_1), (\tau_2, \tau^*_2))$ be an IFP^*T_2, $x_t \in Pt(X)$, $\lambda, \mu \in I^X$, $r \in I_0$, $s \in I_1$. Then*

(1) *If λ is C-set such that $x_t\overline{q}\lambda$, then there exist $\eta_i \in I^X$ with $\tau_{su}(\eta_i) \geq r,\ \tau^*_{su}(\eta_i) \leq s$, $(i = 1, 2)$ such that $x_t \in \eta_1$, $\lambda \leq \eta_2$ and $\eta_1\ \overline{q}\ \eta_2$.*

(2) *If λ, μ are C-sets such that $\lambda\ \overline{q}\ \mu$, then there exist $\rho_i \in I^X,\ \tau_{su}(\rho_i) \geq r,\ \tau^*_{su}(\rho_i) \leq s,\ (i = 1, 2)$ such that $\lambda \leq \rho_1$, $\mu \leq \rho_2$ and $\rho_1\ \overline{q}\ \rho_2$.*

(3) *If λ is C-set, then $C_{\tau_{su},\tau^*_{su}}(\lambda, r, s) = \lambda$.*

Proof. (1): Since $x_t\ \overline{q}\ \lambda$, then $x_t\ \overline{q}\ y_m\ \forall\ y_m \in \lambda$. By IFP^*T_2 of $(X, (\tau_1, \tau^*_1), (\tau_2, \tau^*_2))$ there exist $\eta_1, \upsilon \in I^X$ with $\tau_{su}(\eta_1) \geq r,\ \tau^*_{su}(\eta_1) \leq s,\ \tau_{su}(\upsilon) \geq r,\ \tau^*_{su}(\upsilon) \leq s$ such that $x_t \in \eta_1$, $y_m \in \upsilon$ and $\eta_1\ \overline{q}\ \upsilon$. Then the family $\{\upsilon_i : \tau_{su}(\upsilon_i) \geq r, \tau^*_{su}(\upsilon_i) \leq s, i \in \Delta\}$ is (τ_{su}, τ^*_{su})-cover of λ. Since λ is C-set, there exists a finite subset Δ_0 of Δ_0 such that $\lambda \leq \bigvee_{i\in\Delta_0} \upsilon_i$. Put $\eta_2 = \bigvee_{i\in\Delta_0} \upsilon_i$. Then

$$\tau_{su}(\eta_2) = \tau_{su}(\bigvee_{i\in\Delta_0} \upsilon_i) \geq \bigwedge_{i\in\Delta_0} \tau_{su}(\upsilon_i) \geq r,$$

$$\tau^*_{su}(\eta_2) = \tau^*_{su}(\bigvee_{i\in\Delta_0} \upsilon_i) \leq \bigvee_{i\in\Delta_0} \tau^*_{su}(\upsilon_i) \leq s.$$

Since $\eta_1\ \overline{q}\ \upsilon_i$, $i \in \Delta_0$, then $\eta_1 \leq \underline{1} - \upsilon_i$, which implies that

$$\eta_1 \leq \bigwedge_{i\in\Delta_0} (\underline{1} - \upsilon_i) = \underline{1} - \bigvee_{i\in\Delta_0} \upsilon_i = \underline{1} - \eta_2.$$

Then, $\eta_1\ \overline{q}\ \eta_2$.

(2): Let $x_t \in \mu$ and $\lambda\ \overline{q}\ \mu$, then $x_t\ \overline{q}\ \lambda$. By (1) there exist $\sigma,\ \rho_2 \in I^X$ with $\tau_{su}(\sigma) \geq r$, $\tau^*_{su}(\sigma) \leq s,\ \tau_{su}(\rho_2) \geq r,\ \tau^*_{su}(\rho_2) \leq s$ such that $x_t \in \sigma,\ \lambda \leq \rho_2$ and $\sigma\ \overline{q}\ \rho_2$. Then the family $\{\sigma_i : \tau_{su}(\sigma_i) \geq r, \tau^*_{su}(\sigma_i) \leq s,\ i \in \Delta\}$ is (τ_{su}, τ^*_{su})-cover of μ, so there exists a finite subset Δ_0 of Δ such that $\mu \leq \bigvee_{i\in\Delta_0} \sigma_i$. Put $\rho_1 = \bigvee_{i\in\Delta_0} \rho_i$, then $\tau_{su}(\rho_1) \geq r$ and $\tau^*_{su}(\rho_1) \leq s$. Since $\rho_2\ \overline{q}\ \sigma_i$, $i \in \Delta_0$ we have $\rho_2\ \overline{q}\ \rho_1$.

(3): Let $x_t \in \underline{1} - \lambda$, then $x_t\ \overline{q}\ \lambda$. Since λ is C-set, then by (2), there exist $\eta_1, \eta_2 \in I^X$ with $\tau_{su}(\eta_i) \geq r,\ \tau^*_{su}(\eta_i) \leq s\ \ (i = 1, 2)$ such that $x_t \in \eta_1,\ \lambda \leq \eta_2$ and $\eta_1\ \overline{q}\ \eta_2$. This implies that $x_t \in \eta_1 \leq \underline{1} - \eta_2 \leq \underline{1} - \lambda$. Thus, $\underline{1} - \lambda = \bigvee\{\eta_1 : x_t \in \underline{1} - \lambda\}$. So, $\tau_{su}(\underline{1} - \lambda) \geq r, \tau^*_{su}(\underline{1} - \lambda) \leq s$. Hence, $C_{\tau_{su},\tau^*_{su}}(\lambda, r, s) = \lambda$. □

Theorem 4.3. *Let $(X, (\tau_1, \tau_1^*), (\tau_2, \tau_2^*))$ be an IFP^*-compact. Then*

$$IFP^*T_2 \Longleftrightarrow IFP^*T_4.$$

Proof. $(\Rightarrow)$: Since $(X, (\tau_1, \tau_1^*), (\tau_2, \tau_2^*))$ is IFP^*T_2 it is clear that it is IFP^*T_1. We only need to prove that $(X, (\tau_1, \tau_1^*), (\tau_2, \tau_2^*))$ is IFP^*R_3,

Let $\lambda_1 = C_{\tau_{su}, \tau_{su}^*}(\lambda_1, r, s)\overline{q}\lambda_2 = C_{\tau_{su}, \tau_{su}^*}(\lambda_2, r, s)$. Then, $\tau_{su}(\underline{1} - \lambda_i) \geq r,\ \tau_{su}^*(\underline{1} - \lambda_i) \leq s,\ (i = 1, 2)$. Since $(X, (\tau_1, \tau_1^*), (\tau_2, \tau_2^*))$ is IFP^*-compact, λ_1 and λ_2 are C-sets. Since $\lambda_1\ \overline{q}\ \lambda_2$, by Theorem 4.2(2), there exist $\rho_1, \rho_2 \in I^X$, such that $\lambda_2 \leq \rho_2$ and $\rho_1\ \overline{q}\ \rho_2$. Thus, $(X, (\tau_1, \tau_1^*), (\tau_2, \tau_2^*))$ is IFP^*R_3. Hence, $(X, (\tau_1, \tau_1^*), (\tau_2, \tau_2^*))$ is IFP^*T_4.

$(\Leftarrow)$: See Theorem 3.1(5). □

Theorem 4.4. *Let $f : (X, (\tau_1, \tau_1^*), (\tau_2, \tau_2^*)) \to (Y, (\nu_1, \nu_1^*), (\nu_2, \nu_2^*))$ be an IFP^*-continuous and $\mu \in I^X$ is C-set. Then $f(\mu)$ is C-set in Y.*

Proof. Let $\{\eta_i : i \in J\}$ be (ν_{su}, ν_{su}^*)-cover of $f(\mu)$. Then, $f(\mu) \leq \bigvee_{i \in J} \eta_i$, $\nu_{su}(\eta_i) \geq r$, $\nu_{su}^*(\eta_i) \leq s$. By IFP^*-continuity of f we have

$$\tau_{su}(f^{-1}(\eta_i)) \geq \nu_{su}(\eta_i) \geq r, \quad \tau_{su}^*(f^{-1}(\eta_i)) \leq \nu_{su}^*(\eta_i) \leq s.$$

Also,

$$\mu \leq f^{-1}(f(\mu)) \leq f^{-1}(\bigvee_{i \in J} \eta_i) = \bigvee_{i \in J} f^{-1}(\eta_i).$$

Then, the family $\{f^{-1}(\eta_i) : i \in J\}$ is (τ_{su}, τ_{su}^*)-cover of μ.

But μ is C-set, there exist a finite subset J_0 of J such that $\mu \leq \bigvee_{i \in J_0} f^{-1}(\eta_i)$, which implies that

$$f(\mu) \leq f(\bigvee_{i \in J_0} f^{-1}(\eta_i)) = \bigvee_{i \in J_0} f(f^{-1}(\eta_i)) \leq \bigvee_{i \in J_0} \eta_i.$$

Hence, $f(\mu)$ is C-set in Y. □

Corollary 4.1. *The IFP^*-continuous image of an IFP^*-compact is IFP^*-compact.*

References

[1] S. E. Abbas, Intuitionistic supra fuzzy topological spaces, *Chaos, Solitons and Fractals* **21** (2004), 1205–1214.

[2] K. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems* **20** (1986), 87–96.

[3] C. L. Chang, Fuzzy topological spaces, *J. Math. Anal. Appl.* **24** (1968), 182–190.

[4] K. C. Chattopadhyay, R. N. Hazra and S. K. Samanta, Gradation of openness: fuzzy topology, *Fuzzy Sets and Systems* **49** (1992), 237–242.

[5] D. Çoker, An introduction to intuitionistic fuzzy topological spaces, *Fuzzy Sets and Systems* **88** (1997), 81–98.

[6] D. Çoker and M. Demirsi, An introduction to intuitionistic fuzzy topological spaces in Šostak sense, *Busefal* **67** (1996), 67–76.

[7] U. Höhle, Upper semicontinuous fuzzy sets and applications, *J. Math. Anal. Appl.* **78** (1980), 659–673.

[8] U. Höhle and A. Šostak, General theory of fuzzy topological spaces, *Fuzzy Sets and Systems* **73** (1995), 131–149.

[9] U. Höhle and A. Šostak, Axiomatic foundations of fixed-basis fuzzy topology, In: *The Handbooks of Fuzzy Sets Series*, Kluwer Academic Publishers, Dordrecht, 1999, 3: Chapter 3.

[10] T. Kubiak, *On Fuzzy Topologies*, Ph. D. Thesis, A. Mickiewicz, Poznan, 1985.

[11] T. Kubiak and A. Šostak, Lower set-valued fuzzy topologies, *Questions Math.* **20** (1997), 423–429.

[12] A. A. Ramadan, S. E. Abbas and A. A. Abd El-latif, On fuzzy topological spaces in Šostak's sense, *Commun. Korean Math. Soc.* **21** (2006), 497–514.

[13] A. A. Ramadan, Y. C. Kim and S. E. Abbas, Compactness in intuitionistic gradation of openness, *The Journal of Fuzzy Mathematics* **13** (2005), 581–600.

[14] S. K. Samanta and T. K. Mondal, On intuitionistic gradation of openness, *Fuzzy Sets and Systems* **131** (2002), 323–336.

[15] A. P. Šostak, On fuzzy topological structure, *Supp. Rend. Circ. Matem. Palerms Ser.* **II** (1985), 89–103.

[16] A. P. Šostak, Two decades of fuzzy topology: basic ideas, notion and results, *Russ. Math. Serveys* **44** (1989), 125–186.

[17] A. P. Šostak, Basic structures of fuzzy topology, *J. Math. Sci.* **78** (1996), 622–701.