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Approximate Solution for High Order Partial Differential Equations Subject to Integral Conditions

Ahcene Merad[*]

Departement of Mathematics, Faculty of Sciences, Larbi Ben M'hidi University, Oum El Bouaghi, 04000, Algeria

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Abstract: In this paper we apply a numerical approximate technique for solving hight order partial differential equations subject to integral conditions. Finally, we obtain the solution by using a simple and efficient algorithm Stehfest algorithm for numerical solution

Keywords: Hight order equations, Integral conditions, Laplace transform method.
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1 Introduction

We consider a high order partial differential equations

$$\frac{\partial^m u}{\partial t^m}-\frac{\partial^3 u}{\partial t\partial x^2}+\frac{\partial u}{\partial t}=f(x,t),\quad m\in\mathbb{N}^*,\quad x\in[0,1],\quad t>0, \tag{1}$$

subject to the initial conditions

$$\frac{\partial^i u(x,0)}{\partial t^i}=\varphi_i(x),\quad 0\le i\le m-1,\quad x\in[0,1], \tag{2}$$

and integral conditions

$$u(0,t)=\int_0^1 a(x)\,u(x,t)\,dx+p(t),\quad t>0, \tag{3}$$

$$u(1,t)=\int_0^1 b(x)\,u(x,t)\,dx+q(t),\quad t>0, \tag{4}$$

where x and t are the spatial and time coordinate respectively, $f,\varphi_i\,(0\le i\le m-1),a,b,p,q$ are prescribed continuous function and $u(x,t)$ is an unknown function wich is a solution of (1.1) and satisfies conditions $(1.2)-(1.4)$ at the same time.

Certain problem of modern physics and technology can be effectively describe in terms of nonlocal problems for partial differential equations. [2] has considered a one dimonsional heat equation with nonlocal (Integral) conditions. The autor has taken the Laplace transform of the problem and then used the numeriacl technique (Stehfest algorithm) for the inverse Laplace transforme to obtain the solution. We first take the Laplace transform of the equation(1.1) to reduce the problem to a second order inhomgeneous ordinary differential equation with a set of boundary conditions. After discretization, we use a numeriacl method for inverting the Laplace transform to get the solution.

2 Laplace Transform Method

Laplace transform is an efficient method for solving many differential equations and partial differential equations, the main difficulty with Laplace transform method is in inverting the Laplace domain solution into the real domain. In this section we shall apply the Laplace transform technique to find solutions of partial differential equations, we have the Laplace transform.

$$U(x;s)=\{u(x,t);t\longrightarrow s\}=\int_0^\infty u(x,t)\,e^{-st}dt, \tag{5}$$

where s is positive reel parameter. Taking the Laplace transforms on both sides of (1.1), we have

$$\frac{d^2}{dx^2}U(x;s)-\left(s^{m-1}+1\right)U(x,s)$$

$$=-\tfrac{1}{s}\left[F(x,s)+(s+1)\,\varphi_0(x)-\tfrac{d^2}{dx^2}\varphi_0(x)+\varphi_1(x)+\ldots+\varphi_{m-1}(x)\right], \tag{6}$$

[*] Corresponding author e-mail: merad_ahcene@yahoo.fr

where $F(x,s)=\{f(x,t);t\longrightarrow s\}$. Similarly, we have

$$\frac{\partial^i}{\partial t^i}U(0;s)=\varphi_i(x),\quad 0\le i\le m-1, \tag{7}$$

$$U(0;s)=\int_0^1 a(x)U(x;s)\,dx+P(x), \tag{8}$$

$$U(0;s)=\int_0^1 a(x)U(x;s)\,dx+Q(x), \tag{9}$$

Thus, considered equation is reduced in boundary value problem governed by second order inhomogeneous ordinary differential equation. We obtain a general solution of (2.2) as

$$U(x,s)=$$

$$-\frac{1}{s\sqrt{s^{m-1}+1}}\int_0^x\left[F(\tau,s)+(s+1)\varphi_0(\tau)-\frac{d^2}{dx^2}\varphi_0(\tau)+\varphi_1(\tau)+\ldots+\varphi_{m-1}(\tau)\right]$$

$$\sinh\left(\sqrt{s^{m-1}+1}\,[x-\tau]\right)d\tau+C_1(s)e^{-\sqrt{s^{m-1}+1}x}+C_2(s)e^{\sqrt{s^{m-1}+1}x}, \tag{10}$$

where $C_1$and C_2 are arbitrary functions of s. Substitution of (2.6) into $(2.4)-(2.5)$, we have

$$\begin{aligned}
&C_1(s)\left[1-\int_0^1 a(x)e^{-\sqrt{s^{m-1}+1}x}dx\right]\\
&+C_2(s)\left[1-\int_0^1 a(x)e^{\sqrt{s^{m-1}+1}x}dx\right]\\
&=-\frac{1}{s\sqrt{s^{m-1}+1}}\\
&\int_0^1\left[\left[F(\tau,s)+(s+1)\varphi_0(\tau)-\frac{d^2}{dx^2}\varphi_0(\tau)+\varphi_1(\tau)+\ldots+\varphi_{m-1}(\tau)\right]\right.\\
&\left.\times\int_\tau^1 a(x)\sinh\left(\sqrt{s^{m-1}+1}\,[x-\tau]\right)dx\right]d\tau+P(s),
\end{aligned}$$

$$\begin{aligned}
&C_1(s)\left[e^{-\sqrt{s^{m-1}+1}}-\int_0^1 b(x)e^{-\sqrt{s^{m-1}+1}x}dx\right]\\
&+C_2(s)\left[e^{\sqrt{s^{m-1}+1}}-\int_0^1 b(x)e^{-\sqrt{s^{m-1}+1}x}dx\right]\\
&=-\frac{1}{s\sqrt{s^{m-1}+1}}\\
&\int_0^1\left[\left[F(\tau,s)+(s+1)\varphi_0(\tau)-\frac{d^2}{dx^2}\varphi_0(\tau)+\varphi_1(\tau)+\ldots+\varphi_{m-1}(\tau)\right]\right.\\
&\left.\times\int_\tau^1 b(x)\sinh\left(\sqrt{s^{m-1}+1}\,[x-\tau]\right)dx\right]d\tau+Q(s),
\end{aligned}$$

where

$$\begin{pmatrix}C_1(s)\\C_2(s)\end{pmatrix}=\begin{pmatrix}a_{11}(s)&a_{12}(s)\\a_{21}(s)&a_{22}(s)\end{pmatrix}^{-1}\times\begin{pmatrix}b_1(s)\\b_2(s)\end{pmatrix}, \tag{11}$$

and

$$\begin{aligned}
a_{11}(s)&=1-\int_0^1 a(x)e^{-\sqrt{s^{m-1}+1}x}dx,\quad a_{12}(s)\\
&=1-\int_0^1 a(x)e^{\sqrt{s^{m-1}+1}x}dx,\\
a_{21}(s)&=e^{-\sqrt{s^{m-1}+1}}-\int_0^1 b(x)e^{-\sqrt{s^{m-1}+1}x}dx,\\
a_{22}(s)&=e^{\sqrt{s^{m-1}+1}}-\int_0^1 b(x)e^{-\sqrt{s^{m-1}+1}x}dx,\\
b_1(s)&=-\frac{1}{s\sqrt{s^{m-1}+1}}\\
&\int_0^1\left[\left(F(\tau,s)+(s+1)\varphi_0(\tau)-\frac{d^2}{dx^2}\varphi_0(\tau)+\varphi_1(\tau)+\ldots+\varphi_{m-1}(\tau)\right)\right.\\
&\left.\times\int_\tau^1 a(x)\sinh\left(\sqrt{s^{m-1}+1}\,[x-\tau]\right)dx\right]d\tau+P(s),\\
b_2(s)&=-\frac{1}{s\sqrt{s^{m-1}+1}}\\
&\int_0^1\left[\left(F(\tau,s)+(s+1)\varphi_0(\tau)-\frac{d^2}{dx^2}\varphi_0(\tau)+\varphi_1(\tau)+\ldots+\varphi_{m-1}(\tau)\right)\right.\\
&\left.\times\int_\tau^1 b(x)\sinh\left(\begin{matrix}\sqrt{s^{m-1}+1}\times\\[x-\tau]\end{matrix}\right)dx\right]d\tau+Q(s)\\
&+\frac{1}{s\sqrt{s^{m-1}+1}}\\
&\int_0^x\left[F(\tau,s)+(s+1)\varphi_0(\tau)-\frac{d^2}{dx^2}\varphi_0(\tau)+\varphi_1(\tau)+\ldots+\varphi_{m-1}(\tau)\right]
\end{aligned}$$

$$\times\sinh\left(\sqrt{s^{m-1}+1}\,[x-\tau]\right)d\tau. \tag{12}$$

It is possible to evaluate the integrals in (2.6) and (2.8) exactly. In general, one may have to resort to numerical integration in order to compute them, however. For example, the Gauss's formula $(25.4.30)$ given in Abramowitz and stegun [1] may be employed to calculate these integrals numerically, we have

$$\begin{aligned}
&\int_0^1\begin{pmatrix}a(x)\\b(x)\end{pmatrix}e^{\pm\sqrt{s^{m-1}+1}x}dx\\
&\simeq\frac{1}{2}\sum_{i=1}^N w_i\begin{pmatrix}a\left(\frac{1}{2}(x_i+1)\right)\\b\left(\frac{1}{2}(x_i+1)\right)\end{pmatrix}e^{\pm\sqrt{s^{m-1}+1}\left(\frac{1}{2}(x_i+1)\right)},\\
&\int_0^x\left[F(\tau,s)+(s+1)\varphi_0(\tau)-\frac{d^2}{dx^2}\varphi_0(\tau)+\varphi_1(\tau)+\ldots+\varphi_{m-1}(\tau)\right]\\
&\times\sinh\left(\sqrt{s^{m-1}+1}\,[x-\tau]\right)d\tau\\
&\simeq\frac{x}{2}\sum_{i=1}^N w_i\left[F\left(\frac{x}{2}(x_i+1),s\right)+(s+1)\varphi_0\left(\frac{x}{2}(x_i+1)\right)\right.\\
&\left.-\frac{d^2}{d\tau}\varphi_0\left(\frac{x}{2}(x_i+1)\right)+\varphi_1\left(\frac{x}{2}(x_i+1)\right)+\ldots+\varphi_{m-1}\left(\frac{x}{2}(x_i+1)\right)\right]\\
&\times\sinh\left(\sqrt{s^{m-1}+1}\left(x-\frac{x}{2}(x_i+1)\right)\right),
\end{aligned}$$

where $\overline{\tau} = \frac{x}{2}(x_i+1)$.

$$\int_0^1 \left[\left(F(\tau,s)+(s+1)\varphi_0(\tau)-\frac{d^2}{dx^2}\varphi_0(\tau)+\varphi_1(\tau)+\ldots+\varphi_{m-1}(\tau)\right)\right.$$
$$\left.\times\int_\tau^1\begin{pmatrix}a(x)\\b(x)\end{pmatrix}\sinh\left(\sqrt{s^{m-1}+1}(x-\tau)\right)dx\right]d\tau$$
$$\simeq \frac{1}{2}\sum_{i=1}^{N} w_i\left[F\left(\frac{1}{2}(x_i+1),s\right)+(s+1)\varphi_0\left(\frac{1}{2}(x_i+1)\right)\right.$$
$$\left.-\frac{d^2}{d\overline{\tau}}\varphi_0\left(\frac{1}{2}(x_i+1)\right)+\varphi_1\left(\frac{1}{2}(x_i+1)\right)+\ldots+\varphi_{m-1}\left(\frac{1}{2}(x_i+1)\right)\right]$$
$$\times\left(\frac{1-\frac{1}{2}(x_i+1)}{2}\right)\times\sum_{j=1}^{N} w_j\begin{pmatrix}a\left(\frac{1-\frac{1}{2}(x_i+1)}{2}\right)x_j+\frac{1-\frac{1}{2}(x_i+1)}{2}\\b\left(\frac{1-\frac{1}{2}(x_i+1)}{2}\right)x_j+\frac{1-\frac{1}{2}(x_i+1)}{2}\end{pmatrix}$$
$$\times\sinh\left(\sqrt{s^{m-1}+1}\left(\left(\frac{1-\frac{1}{2}(x_i+1)}{2}\right)x_j+\frac{1+\frac{1}{2}(x_i+1)}{2}-\frac{1}{2}(x_i+1)\right)\right),$$

where x_i and w_i are the abscissa and weights, defined as

$$x_i : i^{th} \text{ zero of } P_n(x), \qquad \omega_i = 2/\left(1-x_i^2\right)\left[P_n'(x)\right]^2.$$

Their tabulated values can be found in [1] for different values of N.

2.1 Numerical inversion of Laplace transform

Sometimes, an analytical inversion of a Laplace domain solution is difficult to obtain; therefore a numerical inversion method must be used. A nice comparison of four frequently used numerical Laplace inversion algorithms is given by Hassan Hassanzadeh, Mehran Pooladi-Darvish [16]. In this work we use the Stehfest's algorithm [26] that is easy to implement. This numerical technique was first introduced by Graver [14] and its algorithm then offered by [26].Stehfest's algorithm approximates the time domain solution as

$$u(x,t) \approx \frac{\ln 2}{t}\sum_{n=1}^{2m}\beta_n U\left(x;\frac{n\ln 2}{t}\right), \tag{13}$$

where, m is the positive integer,

$$\beta_n = (-1)^{n+m}\sum_{k=\left[\frac{n+1}{2}\right]}^{\min(n,m)}\frac{k^m(2k)!}{(m-k)!k!(k-1)!(n-k)!(2k-n)!}, \tag{14}$$

and $[q]$ denotes the integer part of the real number q.

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Ahcene Merad get this M.Sc and Ph.D degree from Jijel University-Algeria. He is an assistant professor at Oum El Bouaghi University-Algeria. His area Of research is nonlocal PDE, numerical solution of linear PDE and fractional Partial dierential equations.