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A Modified Runge-Kutta-Nyström Method by using Phase Lag Properties for the Numerical Solution of Orbital Problems

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Abstract: In this paper, a new modified Runge-Kutta-Nyström method of third algebraic order is developed. The new modified RKN method has phase-lag and amplification error of order infinity, also the first derivative of the phase lag is of order infinity. Numerical results indicate that the new method presented in this paper, is much more efficient than other methods of the same algebraic order, for the numerical integration of orbital problems.

Keywords: Runge-Kutta-Nyström methods, Orbital problems, Phase-fitted, Amplification-fitted, derivatives, initial value problems, oscillating solution

1. Introduction

Much research has been done on the second order periodic initial and boundary value problems with oscillating and/or periodic solutions the recent years (see [13] - [28] and references therein).

Last decades many researchers developed optimized methods based on the phase lag properties ([1] - [10]). Moreover, the very last years a new methodology has been developed which is basing the optimization of a method to the nullification of the phase lag and its derivative ([4],[6], [7],[8],[9],[10]). This new methodology was applied on the multistep methods [4].

In the present paper, an effort is being made to combine for first time in literature the nullification of phase lag, amplification error and phase lag's derivative. The new modified Runge-Kutta-Nyström that is being constructed, contains three additional variable coefficients (in comparison with the classical RKN method), which depend on $z = wh$, where w is the dominant frequency of the problem and h is the step length of integration. In order to evaluate the above coefficients,the new method combines the nullification of phase-lag, amplification factor and phase-lag's derivative.

The new modified RKN method that obtained, will be used for the numerical solution of some well-known orbital problems.

2. The modified Runge-Kutta-Nyström method

In this section we present the general form of the new modified method,which can be used for the numerical integration of second order ordinary differential equations with the following form

$$\frac{d^2y(t)}{dt^2} = f(t, y(t)) \tag{1}$$

The general form of the new modified RKN method is given below

$$y_n = y_{n-1} + hy'_{n-1} + h^2 \sum_{i=1}^{m} b_i f(t_{n-1} + c_i h, f_i),$$

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$$y_n = y'_{n-1}G + h\sum_{i=1}^{m} b'_i f(t_{n-1}+c_i h, f_i), \quad (2)$$

where

$$f_i = y_{n-1} + hc_i y'_{n-1} + h^2 \sum_{j=1}^{i-1} \alpha_{ij} f(t_{n-1}+c_j h, f_j) \quad (3)$$

$$i = 1,\dots,m$$

As it is obvious, for $G = 1$, the classical Runge Kutta Nyström method is obtained. In the present paper and based on the requirement of the development of the new method, value G is a variable and depends on z (which is the product of the frequency w and the step-size h). In section (4) we will present a development of a three-stage modified Runge-Kutta-Nyström method of third algebraic order.

3. Phase-lag analysis of the modified Runge-Kutta-Nyström method

For the development of the new modified method we compare the exact and the numerical solution of the following test equation

$$\frac{d^2y(t)}{dt^2} = (iw)^2 y(t) \Longrightarrow y''(t) = -w^2 y(t), \quad w \in R \quad (4)$$

In the test equation (4) we apply the modified RKN method (2) and we are led to the numerical solution

$$\begin{bmatrix} y_n \\ hy'_n \end{bmatrix} = D^n \begin{bmatrix} y_0 \\ hy'_0 \end{bmatrix}, \quad D = \begin{bmatrix} A(z^2) & B(z^2) \\ \dot{A}(z^2) & \dot{B}(z^2) \end{bmatrix}, \quad (5)$$

where $z = wh$ and $A, B, \dot{A}, \dot{B}$ are polynomials in z^2, completely determined by the parameters of the method (2).

The eigenvalues of the amplification matrix $D(z^2)$ are the roots of the characteristic equation

$$r^2 - tr(D(z^2))r + det(D(z^2)) = 0 \quad (6)$$

In phase analysis one compares the phases of $exp(iz)$ with the phases of the roots of the characteristic equation (6). The following definition is originally formulated by van der Houwen and Sommeijer [1].

Definition* 1(Phase-lag). Apply the RKN method (2) to the general method (4). Then we define the phase-lag $\Phi(z) = z - arccos(tr(D)/2\sqrt{det(D)})$. If $\Phi(z) = O(z^{q+1})$, then the RKN method is said to have phase-lag order q. In addition, the quantity $a(z) = 1 - \sqrt{det(D)}$ is called amplification error.*

where $z = wh$. From definition 1 it follows that

$$\Phi(z) = z - arccos\left(\frac{R(z^2)}{2\sqrt{Q(z^2)}}\right),$$
$$a(z) = 1 - \sqrt{Q(z^2)}. \quad (7)$$

If at a point z, $a(z) = 0$, then the Runge Kutta Nyström method has zero dissipation at this point.

According to the definition 1 we have the following theorem.

0			
1/2	1/8		
1	0	1/2	
	1/6	2/6	0
	1/6	b'_2	b'_3

Table 1 third-stage explicit Runge-Kutta-Nystöm method

Theorem 1 If we have phase-lag of order infinity and at a point z, $\alpha(z) = 0$ then,

$$\left.\begin{array}{l} z - arccos\left(\frac{R(z^2)}{2\sqrt{Q(z^2)}}\right) = 0 \\ 1 - \sqrt{Q(z^2)} = 0 \end{array}\right\} \Rightarrow \begin{array}{l} R(z^2) = 2cos(z) \\ Q(z^2) = 1 \end{array}$$

for more details see ([5])

Lemma* 1For the derivation of a RKN method with nullification of phase lag, amplification error and phase lag's derivative, we must satisfy the conditions:*

$$\begin{aligned} R(z^2) &= 2cos(z), \\ Q(z^2) &= 1, \\ R'(z^2) &= -2sin(z) \end{aligned} \quad (8)$$

4. Construction of the new modified RKN method

In this section we demonstrate the procedure for the derivation of the new modified RKN method, which is a three-stage explicit Runge-Kutta-Nyström method of third algebraic order. From equations 2 and 4, the three-stage explicit modified RKN method can be written in the following form:

$$\begin{aligned} y_n &= y_{n-1} + hy'_{n-1} + h^2(b_1 f_1 + b_2 f_2 + b_3 f_3, \\ y'_n &= y'_{n-1}G + h(b'_1 f_1 + b'_2 f_2 + b'_3 f_3), \end{aligned} \quad (9)$$

where

$$\begin{aligned} f_1 &= f(t_{n-1}, y_{n-1}), \\ f_2 &= f(t_{n-1}+c_2 h, y_{n-1} + c_2 hy'_{n-1} + h^2 a_{21} f_1), \\ f_3 &= f(t_{n-1}+c_3 h, y_{n-1} + c_3 hy'_{n-1} \\ &\quad + h^2(a_{31} f_1 + a_{32} f_2)), \end{aligned} \quad (10)$$

At this point we consider the third-stage explicit RKN method which is presented by the Butcher tableau 1.

From the Butcher tableau 1, equation 9 is transformed as follows:

$$\begin{aligned} y_n &= y_{n-1} + hy'_{n-1} + h^2(\frac{1}{6}f_1 + \frac{2}{6}f_2), \\ y'_n &= y'_{n-1}G + h(\frac{1}{6}f_1 + b'_2 f_2 + b'_3 f_3), \end{aligned} \quad (11)$$

where

$$f_1 = f(t_{n-1}, y_{n-1}),$$
$$f_2 = f(t_{n-1} + \frac{1}{2}h, y_{n-1} + \frac{1}{2}hy'_{n-1} + \frac{1}{8}h^2 f_1),$$
$$f_3 = f(t_{n-1} + h, y_{n-1} + hy'_{n-1} + \frac{1}{2}h^2 f_2), \tag{12}$$

In order to obtain the expressions of the coefficients b'_2, b'_3 and G, we apply numerical method 11 to the test equation 4, and thus we compute the polynomials $A, \dot{A}, B, \dot{B}$ in terms of the modified Runge-Kutta-Nyström parameters. From these polynomials we obtain the expressions of $R(z^2)$ and $Q(z^2)$. Then, according to Lemma 1 we solve the system of four equations ($R(z^2) = 2cos(z)$, $Q(z^2) = 1$, $R'(z^2) = -2sin(z)$) and thus we obtain the expressions of the coefficients which are fully depended from the product of the step-length h and the frequency w.

$$b'_2 = -1/3\,(384\,z^3\sin(z) - 54\,z^6 - 960\,z^2 + 304\,z^4 + 1152\,z^2\cos(z) + 3\,z^8 - 84\,z^5\sin(z) + 6\,z^7\sin(z) + 24\,z^6\cos(z) - 336\,z^4\cos(z) - 576\,z\sin(z) + 1152 - 1152\,\cos(z))/(z^2\left(88\,z^2 - 96 - 18\,z^4 + z^6\right))$$

$$b'_3 = -1/6\,(1152\,z\sin(z) + 56\,z^4 - 1152 + 96\,z^2 + 1152\,\cos(z) - 16\,z^6 - 336\,z^3\sin(z) + 24\,z^5\sin(z) + z^8 + 48\,z^4\cos(z) - 576\,z^2\cos(z))\,/(z^2\left(88\,z^2 - 96 - 18\,z^4 + z^6\right)$$

$$G = -1/12\,(-1152 + 480\,z^2 - 120\,z^4 - 4\,z^6 + 2304\,\cos(z) + 1152\,z\sin(z) - 480\,z^3\sin(z) + 48\,z^5\sin(z) + 144\,z^4\cos(z) - 1536\,z^2\cos(z) + z^8)/(88\,z^2 - 96 - 18\,z^4 + z^6)$$

For small values of x the following Taylor series expansions are used

$$b'_2 = \frac{2}{3} - \frac{1}{240}\,z^4 - \frac{29}{20160}\,z^6 - \frac{2753}{1814400}\,z^8 - \frac{57221}{53222400}\,z^{10} - \frac{41764193}{58118860800}\,z^{12}$$

$$b'_3 = \frac{1}{6} + \frac{1}{96}\,z^4 + \frac{11}{1920}\,z^6 + \frac{731}{201600}\,z^8 + \frac{68237}{29030400}\,z^{10} + \frac{41163389}{26824089600}\,z^{12}$$

$$G = 1 + \frac{1}{180}\,z^6 + \frac{11}{4480}\,z^8 + \frac{10411}{7257600}\,z^{10} + \frac{108551}{119750400}\,z^{12} + \frac{68305253}{116237721600}\,z^{14}$$

5. Numerical illustrations

In this section we will apply our method to three well known orbital problems. We are going to compare our results with other methods designed for solving second order ordinary differential equations. The methods used in the comparison have been denoted by:

- MRKN3: The new third-order MRKN method with three stages, derived in Section 4.
- RKN3: The classical third-order RKN method with three stages from which we were using the coefficients .
- PL6RKN3: The third-order RKN method with phase lag of order six and zero amplification error of *van der Houwen and Sommeijer [1].*
- EFRKN3: The fourth-order exponential fitted RKN method with three stages, of *J.M. Franco [2]* .

One way to measure the efficiency of the method is to compute the accuracy in the decimal digits, that is $-log_{10}$ (*maximum error through the integration intervals*) versus the computational effort measured by the log_{10}(*number of function evaluations required*). The problems are tested in the interval $[0, 1000]$.

Problem 1.*(Orbit problem by Stiefel and Bettis [12])*

$$y'' = -y(t) + \epsilon exp(it), \qquad y(t) \in C$$
$$y(0) = 1, \quad y'(0) = (1 - \frac{1}{2}\epsilon)i,$$

where $\epsilon = 0.001$

The analytical solution is

$y(t) = cos(t) + \frac{1}{2}\epsilon t sin(t) + i[sin(t) - \frac{1}{2}\epsilon t cos(t)]$

Problem 2.*(Orbit problem by Franco and Palacios [11])*

$$y'' = -y(t) + \epsilon exp(i\psi t),$$
$$y(0) = 1, \quad y'(0) = i,$$

where $\epsilon = 0.001$ and $\psi = 0.01$

The analytical solution $y(t) = y_1(t) + iy_2(t)$ is given by:

$$y_1(t) = \frac{1 - \epsilon - \psi^2}{1 - \psi^2}cos(t) + \frac{\epsilon}{1 - \psi^2}cos(\psi t),$$
$$y_2(t) = \frac{1 - \epsilon\psi - \psi^2}{1 - \psi^2}sin(t) + \frac{\epsilon}{1 - \psi^2}sin(\psi t)$$

Problem 3.*(Two-Body problem)*

$$y''_1 = -\frac{y_1}{r^3}, \quad y''_2 = -\frac{y_2}{r^3}$$

where $r = \sqrt{y_1^2 + y_2^2}$,

$y_1(0) = 1, \quad y'_1(0) = 0, \quad y_2(0) = 0, \quad y'_2(0) = 1.$

The analytical solution is

$y_1(t) = cos(t)$ and $y_2(t) = sin(t)$

In the figures we display the efficiency curves, that is the accuracy versus the computational cost measured by the number of function evaluations required by each method.

Numerical results indicate that the new method derived in section 4 is much more accurate than the other methods.

More specifically the new method (MRKN3) is more accurate from the classical (RKN3) one by one decimal digit for the two-body problem and by three decimals for the rest two problems. Also the new RKN method remains more accurate than the PL6RKN3 by two decimals in all cases. Finally the new method has achived better accuracy from the EFRKN4 method by two decimals for the two body problem and by three decimals for the rest two problems.

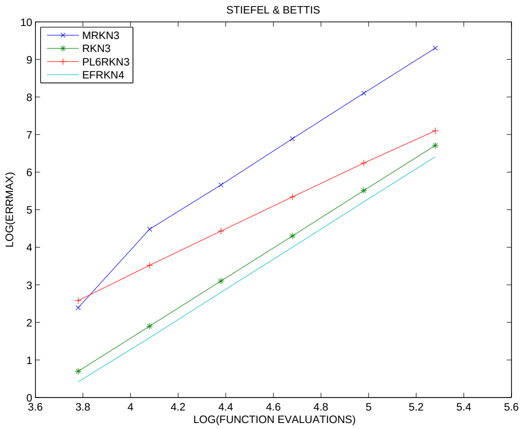

Figure 1 Efficiency for the Stiefel and Bettis Problem

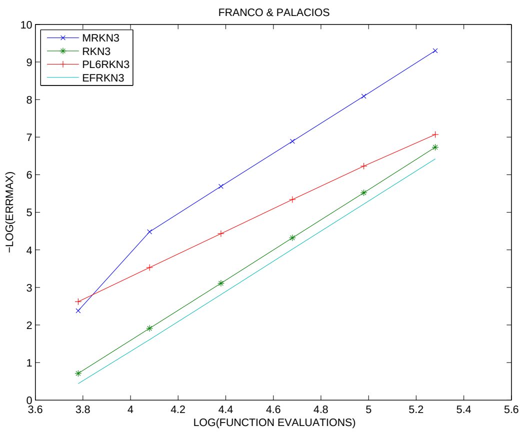

Figure 2 Efficiency for the Franco and Palacios Problem

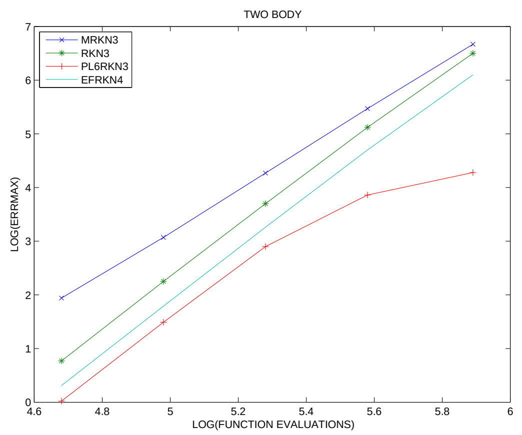

Figure 3 Efficiency for the Two Body Problem

6. Conclusions

The new modified RKN method, developed in this paper is much more efficient than all the other methods that take place, in any case. The new method remained more efficient for all the proplems and in some cases was more accurate than the other methods up to three decimals.

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