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A Gronwall Inequality of Fractional Variable Order

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Abstract: This paper presents the first generalized fractional variable order Gronwall inequality.

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1 Introduction

The following generalized Gronwall inequality for fractional differential equations (of constant order) was established in [1]:

Theorem 1. Suppose that $a(t)$ is a nonnegative function locally integrable on $[0,T)$ (for some $T \leq \infty$), $g(t)$ is a nonnegative, nondecreasing, and bounded continuous function defined on $[0,T)$ and $\beta_0 > 0$. If $u(t)$ is nonnegative and locally integrable on $[0,T)$ satisfying

$$u(t) \leq a(t) + g(t) \int_0^t (t-s)^{\beta_0 - 1} u(s) ds, \qquad 0 \leq t < T, \tag{1}$$

then

$$u(t) \leq a(t) + \int_0^t \left\{ \sum_{n=1}^{\infty} \frac{[\Gamma(\beta_0) g(t)]^n}{\Gamma(n\beta_0)} (t-s)^{n\beta_0 - 1} a(s) \right\} ds, \qquad 0 \leq t < T, \tag{2}$$

where $\Gamma(\cdot)$ is the Gamma function.

The idea of the proof is to introduce the Volterra-type (linear) operator

$$B\phi(t) := g(t) \int_0^t (t-s)^{\beta_0 - 1} \phi(s) ds, \qquad 0 \leq t < T, \tag{3}$$

so that (1) can be written as

$$u(t) \leq a(t) + Bu(t), \qquad 0 \leq t < T, \tag{4}$$

and hence, by repeated iteration of (4),

$$u(t) \leq \sum_{k=0}^{n-1} B^k a(t) + B^n u(t), \qquad 0 \leq t < T. \tag{5}$$

The remaining part of the proof of Theorem 1 [1] is the inductive justification of the inequality

$$B^n \phi(t) \leq \frac{[\Gamma(\beta_0) g(t)]^n}{\Gamma(n\beta_0)} \int_0^t (t-s)^{n\beta_0 - 1} \phi(s) ds, \qquad 0 \leq t < T, \quad n = 1, 2, \ldots, \tag{6}$$

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for any function $\phi(t) \geq 0$ which is locally integrable on $[0,T)$. An immediate implication of (6) is that

$$B^n \phi(t) \to 0 \quad \text{as} \ \ n \to +\infty \qquad \text{for any} \ \ t \in [0,T), \tag{7}$$

and the validity of (2) follows, by using (6) and (7) in (5).

Remark 1. In the statement of Theorem 1, the assumption that $g(t)$ is bounded is not necessary, since for any fixed $t \in [0,T)$ the monotonicity of g implies that $g(s) \leq g(t) < \infty$ for $0 \leq s \leq t$. Also, the assumption that $g(t)$ is continuous is not used in the proof of the theorem, hence it, too, is not necessary. Therefore, the theorem remains true for any nonnegative increasing function $g(t)$ defined on $[0,T)$. Actually, there is an immediate extension of Theorem 1 to the case where $g(t)$ is any function which is locally bounded in $[0,T)$. In this case we can just set

$$G(t) := \sup_{s \in [0,t]} \{g(s), 0\}, \qquad 0 \leq t < T, \tag{8}$$

so that $G(t)$ is nonnegative, increasing, and satisfies $G(t) \geq g(t)$ for $t \in [0,T)$, and then apply Theorem 1 with $G(t)$ in place of $g(t)$.

2 Main results

In this short note we propose an extension of Theorem 1 to the case where the constant order β_0 is replaced by a strictly positive variable order $\beta(t)$. Our motivation came from the recent monograph [2], which contains an extensive discussion on fractional integrals and derivatives of variable order and their applications.

Theorem 2. Suppose that $a(t)$ is a nonnegative function locally integrable on $[0,T)$ for some $T < \infty$, $g(t)$ is a nonnegative and nondecreasing function defined on $[0,T)$, and $\beta(t)$ is a (Lebesgue) measurable function satisfying

$$0 < \beta_0 \leq \beta(t) \leq A < \infty, \qquad 0 \leq t < T. \tag{9}$$

If $u(t)$ is nonnegative and locally integrable on $[0,T)$ satisfying the inequality

$$u(t) \leq a(t) + g(t) \int_0^t (t-s)^{\beta(s)-1} u(s) ds, \qquad 0 \leq t < T, \tag{10}$$

then

$$\begin{aligned} u(t) &\leq a(t) + \sum_{k=1}^{\infty} L^k a(t) \\ &\leq a(t) + \int_0^t \left\{ \sum_{n=1}^{\infty} \frac{\left[\Gamma(\beta_0) K g(t)\right]^n}{\Gamma(n\beta_0)} (t-s)^{n\beta_0 - 1} a(s) \right\} ds, \qquad 0 \leq t < T, \end{aligned} \tag{11}$$

where L is the Volterra-type (linear) operator

$$L\phi(t) := g(t) \int_0^t (t-s)^{\beta(s)-1} \phi(s) ds, \qquad 0 \leq t < T, \tag{12}$$

and

$$K := \max\{1, T\}^{A-\beta_0}. \tag{13}$$

Proof. Observe that from (9) we get (since $\beta(s) \geq \beta_0$)

$$\frac{(t-s)^{\beta(s)-1}}{(t-s)^{\beta_0-1}} = (t-s)^{\beta(s)-\beta_0} \leq \max\{1, T\}^{A-\beta_0}, \qquad 0 \leq s < t < T,$$

that is

$$(t-s)^{\beta(s)-1} \leq \max\{1, T\}^{A-\beta_0} (t-s)^{\beta_0-1}, \qquad 0 \leq s < t < T. \tag{14}$$

Therefore, (10) implies

$$u(t) \leq a(t) + K g(t) \int_0^t (t-s)^{\beta_0-1} u(s) ds, \qquad 0 \leq t < T, \tag{15}$$

where K is given by (13). Since $\beta_0 > 0$ is a constant, we can apply Theorem 1 to (15), where the operator B now has the slightly different form:

$$B\phi(t) = Kg(t)\int_0^t (t-s)^{\beta_0-1}\phi(s)ds, \qquad 0 \le t < T. \tag{16}$$

As in the proof of Theorem 1,

$$\lim_{n\to+\infty} B^n u(t) = 0, \qquad 0 \le t < T. \tag{17}$$

Now, it is clear from (12), (14), and (16) that for $u(t) \ge 0$ we have

$$0 \le Lu(t) \le Bu(t), \qquad 0 \le t < T. \tag{18}$$

Hence, by using (17) in (18) we get

$$\lim_{n\to+\infty} L^n u(t) = 0, \qquad 0 \le t < T, \tag{19}$$

and since (10) can be written as

$$u(t) \le a(t) + Lu(t), \qquad 0 \le t < T, \tag{20}$$

formula (11) follows easily from (19), (18) and Theorem 1. ∎

Notice that, as in the standard Gronwall inequality, the value of (11) lies in the fact that it gives a bound for $u(t)$ in terms of $a(t)$, $g(t)$, and $\beta(t)$.

As explained in Remark 1, in the case where $g(t)$ is any locally bounded function in $[0,T)$, Theorem 2 holds by replacing $g(t)$ with $G(t)$ of (8).

Corollary 1. All as in Theorem 2 with $g(t) = b \ge 0$ constant. If

$$u(t) \le a(t) + b\int_0^t (t-s)^{\beta(s)-1}u(s)ds, \qquad 0 \le t < T, \tag{21}$$

then

$$\begin{aligned} u(t) &\le a(t) + \sum_{k=1}^{\infty} L_1^k a(t) \\ &\le a(t) + \int_0^t \left\{ \sum_{n=1}^{\infty} \frac{[\Gamma(\beta_0)Kb]^n}{\Gamma(n\beta_0)}(t-s)^{n\beta_0-1}a(s) \right\} ds, \qquad 0 \le t < T, \end{aligned} \tag{22}$$

where L_1 is the Volterra operator

$$L_1\phi(t) := b\int_0^t (t-s)^{\beta(s)-1}\phi(s)ds, \qquad 0 \le t < T. \tag{23}$$

Corollary 2. All as in Theorem 2 with $a(t)$ be a nondecreasing function on $[0,T)$. Then

$$u(t) \le a(t)E_{\beta_0}\left(Kg(t)\Gamma(\beta_0)t^{\beta_0}\right), \qquad 0 \le t < T, \tag{24}$$

where $E_{\beta_0}(\cdot)$ is the Mittag-Leffler function defined by

$$E_{\beta_0}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\beta_0+1)}.$$

Proof. The assumptions of Theorem 2 and (11) imply

$$\begin{aligned} u(t) &\le a(t)\left(1 + \int_0^t \left\{ \sum_{n=1}^{\infty} \frac{[\Gamma(\beta_0)Kg(t)]^n}{\Gamma(n\beta_0)}(t-s)^{n\beta_0-1} \right\} ds \right) \\ &= a(t)\sum_{n=0}^{\infty} \frac{[\Gamma(\beta_0)Kg(t)t^{\beta_0}]^n}{\Gamma(n\beta_0+1)} = a(t)E_{\beta_0}\left(Kg(t)\Gamma(\beta_0)t^{\beta_0}\right). \end{aligned} \tag{25}$$

∎

Gronwall inequality of fractional variable order is expected to find wide applications in the forthcoming studies of fractional differential equations of variable order.

References

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