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On the Singular Kernels for Fractional Derivatives. Some Applications to Partial Differential Equations

Michele Caputo[1,*] *and Mauro Fabrizio*[2]

[1] College of Geoscience, Texas A&M University, College Station TX, USA
[2] Department of Mathematics, University of Bologna, Italy

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Abstract: In this short note, we discuss some characteristics of the fractional derivatives and the sigularity of their kernels. Then, we consider the Caputo-Fabrizio fractional derivative and its properties. Hence, by means these operators we studied elastoplastic materials.

Keywords: Fractional derivative, visco-plastic operators, singular kernels, Caputo-Fabrizio.

1 Introduction

In recent years, there was some papers on the properties that a fractional derivative should have. It is known that there is no general accepted definition. However, some conditions have been proposed that a fractional derivative must satisfy. In this regard, we believe that by the fractional derivative of $f(t)$, we mean an operator with memory that depends on a fractional ratio α, which varies between 0 and 1, for $\alpha = 0$ we have the function f, while for $\alpha = 1$ we get the first derivative f'. Moreover, a list of reasonable properties of a fractional derivative has been presented by [1]. We further believe that each fractional derivative has specific characteristics and properties which are suitable and convenient for the representation of particular problems. However, it is necessary to establish a series of conditions that the operator has to satisfy, so it can be considered as a fractional derivative.

In recent papers, more restrictive conditions have been introduced, also with the aim of checking and exploring the Caputo-Fabrizio (CF) fractional derivative [2]. However, the validity of a fractional derivative also depends on its ability to describe relevant phenomena [3]-[4]-[5] and not only on its original new mathematical properties. Among the papers that study the some properties of fractional derivatives, it is appropriate to mention the work [6] and [7], which poses some observations, such as the need for singular kernels, and the problem of the initial conditions.

In this paper, we prove that the original CF derivative can be represented by a fractional operator, whose connected differential problem has a singular kernel. Moreover, it is able to well provide classical initial conditions.The reason of this misunderstanding is due that in these papers [6] - [7], the definition of CF derivative leaves from an incorrect expression of our fractional integral. Instead, if we use our representation for the differential problem, we can obtain an expression with a singular kernel and general initial conditions. Our definition is more general. As an example in this paper, we will prove that our model allows us to describe visco-plastic materials [8]. Otherwise if we use the definition considered in [6] and [7], it is not possible to obtain these materials, but only visco-elastic materials, which are a particular case of visco-plastic ones.

2 On the kernel of the CF fractional derivative

We begin by recalling the definition of CF fractional derivative, for a smooth function $f(t) : [a, \infty) \to \mathscr{R}$, given by

$$^{CF}D_t^\alpha f(t) = \frac{1}{(1-\alpha)} \int_a^t f'(\tau) e^{-\frac{\alpha}{1-\alpha}(t-\tau)} d\tau, \tag{1}$$

* Corresponding author e-mail: mic.caput27@gmail.com

where the scalar $a \in [-\infty, 0]$, while $t \geq 0$.

It is convenient to observe, that $a \leq 0$. But only for $a = 0$ we obtain a classical operator with fading memory, as considered in [6] - [7]. Otherwise, we are interested to study more general materials. For which, we assume $a < 0$, while $t > 0$, so the integral

$$\int_a^t f'(\tau)e^{-\frac{\alpha}{1-\alpha}(t-\tau)}d\tau = \int_a^0 f'(\tau)e^{-\frac{\alpha}{1-\alpha}(t-\tau)}d\tau + \int_0^t f'(\tau)e^{-\frac{\alpha}{1-\alpha}(t-\tau)}d\tau, \tag{2}$$

will be supposed given in $(a,0)$. Instead, in the papers that dispute the definition (1), this derivative is defined by

$${}^{CF}D_t^{\alpha} f(t) = \frac{1}{(1-\alpha)}\int_0^t f'(\tau)e^{-\frac{\alpha}{1-\alpha}(t-\tau)}d\tau. \tag{3}$$

As we will see, the representation (3) (which holds only when $a = 0$) can be misleading and leads to the wrong remarks of some recent papers as [6]-[7]. Indeed, in the correct definition contained in the papers [2] - [9], the time interval is given by $[a,t)$.

Furthermore, according to the Definition 1 of the paper [2], the function $f'(\tau)$ is assigned in the interval $[a,0]$ with $f'(a) = 0$. Under these conditions, if the inferior limit is given by the scalar $a = 0$, then the initial conditions are null, because $f'(a=0) = 0$.

In the following, we prove that there is a meaningful difference between the definition (1) and (3). Indeed, the equation (3) describes the constitutive equation of a viscoelastic material, while (1) represents a visco-plastic material.

Finally, in the study of for ${}^{CF}D_t^{\alpha} f(t)$ with $a \neq 0$, by (1) we have

$$\begin{aligned} {}^{CF}D_t^{\alpha} f(t) &= \frac{1}{(1-\alpha)}(\int_a^0 f'(\tau)e^{-\frac{\alpha}{1-\alpha}(t-\tau)}d\tau + \int_0^t f'(\tau)e^{-\frac{\alpha}{1-\alpha}(t-\tau)}d\tau) = \\ &= \frac{1}{(1-\alpha)}(g_0'(t) + \int_0^t f'(\tau)e^{-\frac{\alpha}{1-\alpha}(t-\tau)}d\tau), \end{aligned} \tag{4}$$

where

$$g_0'(t) = \int_a^0 f'(\tau)e^{-\frac{\alpha}{1-\alpha}(t-\tau)}d\tau$$

is defined according to the conditions of the past history. Furthermore, through the differential problem, we can prove that two different histories $f_1'(\tau), f_2'(\tau)$ which in $[a,0]$ are such that

$$\int_a^0 f_1'(\tau)e^{-\frac{\alpha}{1-\alpha}(t-\tau)}d\tau = \int_a^0 f_2'(\tau)e^{-\frac{\alpha}{1-\alpha}(t-\tau)}d\tau$$

provide the same function $g_0'(t)$. So, we can reduce the definition of ${}^{CF}D_t^{\alpha} f(t)$ to a single integral in $[0,t)$. Indeed, we obtain from (4)

$${}^{CF}D_t^{\alpha} f(t) = \frac{1}{(1-\alpha)}\int_0^t (g_0'(\tau)\delta(t-\tau) + f'(\tau)e^{-\frac{\alpha}{1-\alpha}(t-\tau))}d\tau, \tag{5}$$

or the equivalent representation

$${}^{CF}D_t^{\alpha} f(t) = \frac{1}{(1-\alpha)}\int_0^t \left(\frac{g_0'(\tau)}{f'(t)}\delta(t-\tau) + e^{-\frac{\alpha}{1-\alpha}(t-\tau)}\right) f'(\tau)d\tau, \tag{6}$$

so we obtain a singular kernel also for CF derivatives.

Finally, as we have shown, this formulation allows to write the differential problem for a visco-plastic material [9], [10].

For this purpose, we consider, without getting lost in the general formulations, the one-dimensional case, where the variable $u(x,t)$ denotes the displacement. Thus, the motion equation is given by the partial differential system

$$\rho_0 \frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial}{\partial x}\sigma + \rho_0 h(x,t), \tag{7}$$

where ρ_0 denotes the density, h a body force and the stress σ is defined by

$$\sigma(x,t) = A(x)\frac{\partial}{\partial x}\, {}^{CF}D_t^{\alpha} u(x,t = \frac{A(x)}{(1-\alpha)}\frac{\partial}{\partial x}\int_0^t u_t(x,t)(\tau)e^{-\frac{\alpha}{1-\alpha}(t-\tau)}d\tau, \tag{8}$$

while A is a positive coefficient. As for all systems with memory, the equation (7) can be rewritten in the form

$$\rho_0 \frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial}{\partial x} \frac{A(x)}{(1-\alpha)} \frac{\partial}{\partial x} \int_0^t u_t(x,t)(\tau) e^{-\frac{\alpha}{1-\alpha}(t-\tau)} d\tau + \frac{\partial}{\partial x} A(x) \frac{\partial}{\partial x} \frac{g_0'(x,t)}{(1-\alpha)} + \rho_0 h(x,t), \tag{9}$$

from which we can observe that now the supply of (3) is given by

$$F(x,t) = \frac{\partial}{\partial x} A(x) \frac{\partial}{\partial x} \frac{g_0'(x,t)}{(1-\alpha)} + \rho_0 h(x,t),$$

but equation (3) does not authorize to claim that the CF derivative is given by the functional (3).

Finally, we can see as the observations made for the CF fractional derivative can be extended to other operators with memory. Moreover, it is possible to find some definitions of fractional derivatives in which the integral is given in the interval $[a,t)$, but if the differential problem is defined in $[0,t)$, then we have to suppose the function f' given in $(a,0)$. Hence, we obtain the equation (4).

3 On kernel properties

We begin by recalling the definition of Caputo (C) fractional derivative [11] for a function $f(t) : [0,\infty) \to \mathscr{R}$.

The C derivative is

$$^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^{\alpha}} d\tau, \tag{10}$$

from which, if we take into account *only particular* smooth functions, we have

$$^{C}D_t^{\alpha} f(t) = \frac{-1}{(1-\alpha)\Gamma(1-\alpha)} \int_0^t f''(\tau)(t-\tau)^{1-\alpha} d\tau + \frac{f'(0)t^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}. \tag{11}$$

For this problem, when the initial conditions are null, so that $f'(0) = 0$, the last term of (11) is zero. Then, under these special hypotheses also this operator has no singular kernel.

Hence, the request of particular assumptions on the kernels is a request that can be restrictive for some mathematical problems. In fact, the features of these operators may depend on particular choices of the initial conditions.

Finally, we note that this is convenient also for distributed order fractional derivative introduced by Caputo in [11] and studied in [12]-[13] and [14].

4 Conclusion

In the paper, we discussed the role and properties of CF fractional derivative. We observe that in the study of some differential problems, this derivative describes fading memory materials with initial conditions, that summarize all past history. This allows to prove the equivalence of CF fractional derivative with a model with singular kernel. By this representation, we have studied some visco-plastic materials.

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