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Bayesian Prediction Limits for Inverse Weibull Distribution when Observations are Mid Type II Censored

Vastoshpati Shastri [1], *Rakesh Ranjan* [2,*] *and Deependra Singh Pal*[1]

[1]Department of Statistics, Government Arts & Science College, Ratlam, India
[2]DST Centre for Interdisciplinary Mathematical Sciences, Banaras Hindu University, Varanasi, India

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Abstract: In this paper, the problem of prediction is discussed for inverse Weibull distribution. Posterior distribution is obtained using different informative priors when observations are mid type II censored. When the shape parameter of the model is known, prediction interval is obtained using predictive probability density function method. Whereas, when both parameters are known, inferences from the posterior distribution are drawn using Bayes computation. Comparisons have been made on the basis of simulated data set for the smallest ordered future observation.

Keywords: Prior, posterior, Bayesian Prediction, Bayes computation, Metropolis-Hasting, Gibbs sampler, simulation

1. Introduction

Weibull model has a significant position in analyzing reliability and lifetime data for reliability practitioners and statisticians. Its spontaneous failure rate or hazard rate may be constant, monotone increasing or monotone decreasing. Depending upon its parameters, the density may be increasing, decreasing or unimodal. If a product consists of several small devices, and each of its devices has an identical hazard rate, with the breakdown of the weakest device, the product will fail finally. In this situation, Weibull model is a suitable probability distribution for such failure pattern (See Nelson (1982)). Numerous works has been done on Weibull distribution in classical as well as in Bayesian perspective. Some important references on the model include works of Johnson et al. (1995), Nordman and Meeker (2002), Kundu (2008) and Jia et. al. (2016) among others.

If random variable Z follows the Weibull distribution with scale parameter α and shape parameter λ, having probability density function (pdf) as

$$f(z;\alpha,\lambda) = \alpha\lambda z^{\alpha-1}e^{-\lambda z^{\alpha}};z>0,\alpha,\lambda>0$$

Transforming the above random variable Z to a new random variable X by means of transformation $X=\frac{1}{Z}$ has an inverse Weibull distribution with pdf

$$f(x;\alpha,\lambda) = \alpha\lambda e^{-\lambda x^{-\alpha}}x^{-(\alpha+1)};x>0\alpha>0,\lambda>0 \tag{1}$$

Here α is the shape parameter, and λ is the scale parameter, and its cumulative distribution function (cdf) can be expressed as

$$F(x;\alpha,\lambda) = e^{-\lambda x^{-\alpha}};x>0\alpha>0,\lambda>0 \tag{2}$$

In the reliability and life testing application, inverse Weibull distribution is an important lifetime probability model. Inverse Weibull model is acceptable in a variety of failure characteristics such as in wear out period of machines, cost of

* Corresponding author e-mail: rakesh.stats88@gmail.com

maintenance of devices and also for several devices which include regulators, capacitors, among several others. Adaptability of the distribution along with measure of central tendencies; dispersion and Pearsonian coefficients were studied by Khan et. al (2008).Whereas, Kundu and Howlader (2010) provided predictive inferences of the model when observations are right type-II censored in Bayesian perspective. Performance of Bayes estimators of inverse Weibull scale parameter have been studied by Yahgmaei et.al. (2013) using uniform, gamma and quasi priors, when loss function was symmetric as well as asymmetric.

For ordered observations statistical prediction, in life testing and reliability problem, we can see how far a lot of products or items might perform until all get crashed or failed. For making inferences about future sample characteristics, predictive density is determined by combining the posterior distribution with the pdf of future characteristics given parameters. Predictive distribution for the future characteristics, is obtained when integrating with respect to each of the parameters. This provides the information about future samples, considering knowledge supplied by the given sample.

Bayesian prediction for exponential distribution was discussed by Upadhyay and Pandey (1989). Prediction of future samples based on censored observation drawn from inverse Weibull model was dealt by Calabria and Pulcini (1994), without a prior information. Kundu and Howlader (2010) constructed two sided prediction interval for inverse Weibull distribution when the observations are censored. Singh et. al. (2013) obtained the Bayesian methods for predicting future samples from inverse Weibull distribution when observations are compounded by hybrid censoring mechanism. Recently, Xiuyun and Zaizai (2016) considered estimation and prediction of parameters when observations are general progressive censored. For the prediction problem related to inverse Weibull distribution, when the data are mid type II censored, it is untouched in the literature.

Mid type II censoring was discussed by Upadhyay et. al. (1996) while estimating exponential scale parameter, as a particular case of multiply type II censoring. Mid censoring arises when an experimenter could not observe some middle observations due to some unforeseen reasons. This means he records only two sets of observations, i.e., one set, recording of failure times in the beginning of the experiment, thereafter, he misses some observations and, finally, the another set, at the end till all items get failed.

To draw Bayesian inferences for inverse Weibull distribution, a difficulty is faced due to involvement of intractable integrals, therefore, Bayes computation is sought for drawing inferences from posterior distribution. The major impediment with the inverse Weibull distribution lies when the available data is censored. When the shape parameter of the model is known, the routine method is implemented for obtaining predictive inference. Whereas, when both parameters are unknown, likelihood function (LF) is not that easy because of the involvement of complex function, hence, we have implemented one of Markov Chain Monte Carlo techniques for drawing predictive inferences. Predictive intervals are obtained using Metropolis-Hastings technique. A brief description of Bayes computation is given in the next section which contains the Metropolis-Hastings algorithm.

2. Bayes Computation

One of the major difficulties with the Bayesian method is the involvement of high dimensional numerical integration, in drawing the posterior based inferences, and, thereby, providing a very challenging task from the conventional numerical quadrature perspective. The problem becomes even worse if complexities such as censored data or constrained parameter are taken into account. In reliability analysis, most widely used method is Gaussian quadrature. Other significant methods for treating sophisticated numerical integration are those based on Laplace and Lindley's approximation techniques, reparameterization strategies leading to iterative quadrature, and Monte Carlo techniques, etc. For related references, one may refer to Smith (1991), Upadhyay and Mukherjee (2008) and Chen et. al. (2012), among others. But the implementation of these techniques requires an insight of mathematical sophistication, and, perhaps, the knowledge of a specialist use of software. As an alternative to these techniques, sample based approaches have started growing in recent years. Among the various approaches, the developments include the refinement in the techniques of standard Monte Carlo importance sampling and growth of interest in Markov chain Monte Carlo methods, such as Gibbs sampler, Metropolis-Hastings versions of algorithm and some hybrid strategies combining the different algorithms (c.f. Gelfand et. al. (1990), Smith and Roberts (1993), etc.). Although these algorithms have been developed with the aim of exploring high dimensional posterior surfaces, a work by Upadhyay and Smith (1993) has demonstrated the algorithm for low dimensional models.

Metropolis algorithm is one of the popular algorithms in Bayesian literature to draw a sample from high dimensional posterior surfaces. This algorithm is similar to accept reject algorithm and needs a symmetric proposal density, $q\left(\theta^{'}|\theta\right)$. Suppose one has to draw a sample from $f(\theta)$. He/She, first, generates an observation θ' from $q\left(\theta^{'}|\theta^{*}\right)$ for the given value of θ^{*}, where θ^{*} is the previous realization of parameter θ, then he will calculate the acceptance probability, $\rho =$

$\min\left\{1, \frac{f(\theta')}{f(\theta^*)}\right\}$. Metropolis algorithm accepts the proposed realization, θ', with probability ρ. If the proposed realization is rejected, then it sets previous realization θ^* as the current realization. Starting with an initial value of θ, we can generate a single chain of $\theta's$ with stationary distribution $f(\theta)$. Thus, this chain can be used to draw inferences about distribution $f(\theta)$.

3. Prediction Limits

Let $x_1, x_2, \ldots, x_n$ items be put on a life test, where experimenter records first r observations, $x_1 < \ldots < x_r$. After recording first r observations, he/she is not in a position to record l middle observations due to some unforeseen events, namely $x_{r+1} < \ldots < x_{r+l}$ observations, and, thereafter, he/she records the last $(n-r-l)$ observations, $x_{r+l+1} < \ldots < x_n$ till the end of the experimentation. Failure times obtained for $(n-l)$ observations, is a special case of multiply type II censored sample, known as mid type II censored sample.
Likelihood function for mid type II censored observations is expressed as

$$L(x;\alpha,\lambda) = \frac{n!}{l!}\left[F(x_{r+l+1}) - F(x_r)\right]^l \prod_{i=1}^{r} f(x_i) \prod_{i=r+l+1}^{n} f(x_i) \tag{3}$$

On substituting values from (1) and (2) and solving, we have

$$L(x;\alpha,\lambda) = \frac{n!}{l!}(\alpha\lambda)^{n-l} \prod_{i=1}^{r} x_i^{-(\alpha+1)} \prod_{i=r+l+1}^{n} x_i^{-(\alpha+1)} \sum_{g=0}^{l} \Omega_g \exp\left[-\lambda\left\{(l-g)x_{r+l+1}^{-\alpha} + g x_r^{-\alpha} + \sum_{i=1}^{r} x_i^{-\alpha} + \sum_{i=r+l+1}^{n} x_i^{-\alpha}\right\}\right] \tag{4}$$

where $\Omega_g = (-1)^g \binom{l}{x}$. We have considered two different cases for parameters in the next subsections.

3.1 *Case I : When Shape Parameter α is Known*

When shape parameter α is known, we consider prior distribution for scale parameter λ as

$$g_1(\lambda|c_0, d_0) = \frac{d_0^{c_0}}{\Gamma(c_0)} \lambda^{c_0-1} e^{-d_0\lambda} \quad ; \lambda > 0, c_0, d_0 > 0 \tag{5}$$

Posterior of parameter λ is

$$p_1(\alpha,\lambda|x) = \frac{L(x;\alpha,\lambda)\, g_1(\lambda|c_0,d_0)}{\int L(x;\alpha,\lambda)\, g_1(\lambda|c_0,d_0)\, d\lambda}$$

Substituting from (4) and (5) and solving, yields the posterior

$$p_1(\lambda|x) = \frac{\sum_{g=0}^{l} \Omega_g \exp\left[-\lambda(S_x + d_0)\right] \lambda^{n-l+c_0-1}}{\Gamma(n-l+c_0) \sum_{g=0}^{l} \Omega_g (S_x + d_0)^{-(n-l+c_0)}} \tag{6}$$

here, $S_x = \sum_{i=1}^{r} x_i^{\alpha} + \sum_{i=r+l+1}^{n} x_i^{\alpha} + (l-g)x_{r+l+1}^{-\alpha} + g x_r^{-\alpha}$.
Let $y_1, y_2, \ldots, y_m$ be m sized future observations which are independently drawn from inverse Weibull distribution given in (1), then the density of k$^{\text{th}}$ ordered future observation, where $1 \le k \le m$, will be obtained by

$$f\left(y_{(k)}|\alpha,\lambda\right) = \frac{m!}{(k-1)!(m-k)!}\left[F\left(y_{(k)}\right)\right]^{k-1} f\left(y_{(k)}\right)\left[1 - F\left(y_{(k)}\right)\right]^{m-k}$$

Substituting values from (1) and (2),

$$f\left(y_{(k)}|\alpha,\lambda\right) = \beta^{-1}(k, m-k+1)\,\alpha\lambda y_{(k)}^{-(\alpha+1)} \sum_{i=0}^{m-k} \Omega_i \exp\left[-\lambda(i+k)y_{(k)}^{-\alpha}\right] \tag{7}$$

where, $\Omega_i = (-1)^i \binom{m-k}{i}$. Predictive pdf is expressed as

$$h_1\left(y_{(k)}|x\right) = \int f\left(y_{(k)}|\alpha,\lambda\right) p_1\left(\lambda|\alpha,x\right) d\lambda$$

Using (6) and (7), we get

$$h_1\left(y_{(k)}|x\right) = \int \beta^{-1}\left(k,m-k+1\right)\alpha\lambda y_{(k)}^{-(\alpha+1)} \sum_{i=0}^{m-k} \Omega_i \exp\left[-\lambda\left(i+k\right)y_{(k)}^{-\alpha}\right] \frac{\sum_{g=0}^{l}\Omega_g \exp\left[-\lambda\left(S_x+d_0\right)\right]\lambda^{n-l+c_0-1}}{\Gamma\left(n-l+c_0\right)\sum_{g=0}^{l}\Omega_g\left(S_x+d_0\right)^{-(n-l+c_0)}} d\lambda$$

which, on simplification, gives

$$h_1\left(y_{(k)}|x\right) = \frac{\left(n-l+c_0\right)\beta^{-1}\left(k,m-k+1\right)\alpha y_{(k)}^{-(\alpha+1)}}{\sum_{g=0}^{l}\Omega_g\left(S_x+d_0\right)^{-(n-l+c_0)}} \sum_{i=0}^{m-k}\sum_{g=0}^{l}\Omega_i\Omega_g\left\{\left(i+k\right)y_{(k)}^{-\alpha}+S_x+d_0\right\}^{-(n-l+c_0+1)} \tag{8}$$

Prediction limits is the solution of following equation:

$$P[t_{1k} \le y_{(k)} \le t_{2k}] = 1-\gamma$$

which is equivalent to the solution of following equation:

$$P\left[y_{(k)} \le t_{1k}\right] = \frac{\gamma}{2} \text{ and } P\left[y_{(k)} \ge t_{2k}\right] = \frac{\gamma}{2}$$

where γ is the level of significance.
Lower Bayes prediction limit t_{1k} is the solution of following expression:

$$\int_0^{t_{1k}} \frac{\left(n-l+c_0\right)\beta^{-1}\left(k,m-k+1\right)\alpha y_{(k)}^{-(\alpha+1)}}{\sum_{g=0}^{l}\Omega_g\left(S_x+d_0\right)^{-(n-l+c_0)}} \sum_{i=0}^{m-k}\sum_{g=0}^{l}\Omega_i\Omega_g\left\{\left(i+k\right)y_{(k)}^{-\alpha}+S_x+d_0\right\}^{-(n-l+c_0+1)} dy_{(k)} = \frac{\gamma}{2}$$

Solving integration in above expression yields lower prediction bound as

$$\frac{\beta^{-1}\left(k,m-k+1\right)}{\sum_{g=0}^{l}\Omega_g\left(S_x+d_0\right)^{-(n-l+c_0)}} \sum_{i=0}^{m-k}\sum_{g=0}^{l}\frac{\Omega_i\Omega_g}{(i+k)}\left[\left\{(i+k)t_{1k}^{-\alpha}+S_x+d_0\right\}^{-(n-l+c_0)} - \left\{S_x+d_0\right\}^{-(n-l+c_0)}\right] = \frac{\gamma}{2}. \tag{9}$$

Similarly, upper prediction limit t_{2k} is the solution of following expression:

$$\frac{\beta^{-1}\left(k,m-k+1\right)}{\sum_{g=0}^{l}\Omega_g\left(S_x+d_0\right)^{-(n-l+c_0)}} \sum_{i=0}^{m-k}\sum_{g=0}^{l}\frac{\Omega_i\Omega_g}{(i+k)}\left[\left\{(i+k)t_{2k}^{-\alpha}+S_x+d_0\right\}^{-(n-l+c_0)} - \left\{S_x+d_0\right\}^{-(n-l+c_0)}\right] = 1-\frac{\gamma}{2} \tag{10}$$

Above equations are solved using bisection method.

3.2 *Case II : When Shape and Scale Parameters are Both Unknown*

For the shape parameter as well as scale the parameter, separate prior have been considered as

$$g_2\left(\alpha|a_0,b_0\right) = \frac{b_0^{a_0}}{\Gamma\left(a_0\right)}\alpha^{a_0-1}e^{-b_0\alpha} \quad ; \alpha > 0, a_0, b_0 > 0 \tag{11}$$

$$g_3\left(\lambda|c_0,d_0\right) = \frac{d_0^{c_0}}{\Gamma\left(c_0\right)}\lambda^{c_0-1}e^{-d_0\lambda} \quad ; \lambda > 0, c_0, d_0 > 0 \tag{12}$$

Combining likelihood function with the priors, we get the posterior as

$$p_2\left(\alpha,\lambda|x\right) = \frac{L\left(x;\alpha,\lambda\right)g_2\left(\alpha|a_0,b_0\right)g_3\left(\lambda|c_0,d_0\right)}{\int\int L\left(x;\alpha,\lambda\right)g_2\left(\alpha|a_0,b_0\right)g_3\left(\lambda|c_0,d_0\right)d\alpha d\lambda}$$

Substituting LF (4) and priors (11) and (12), and by solving, the posterior can be expressed as

$$p_2(\alpha,\lambda|x) = \frac{\prod_{i=1}^{r} x_i^{-\alpha} \prod_{i=r+l+1}^{n} x_i^{-\alpha} \alpha^{n-l+a_0-1} e^{-b_0\alpha} \lambda^{n-l+c_0-1} \sum_{g=0}^{l} \Omega_g \exp\left[-\lambda\left(S_x+d_0\right)\right]}{\Gamma\left(n-l+c_0\right) \int_\alpha \prod_{i=1}^{r} x_i^{-\alpha} \prod_{i=r+l+1}^{n} x_i^{-\alpha} \alpha^{n-l+a_0-1} e^{-b_0\alpha} \sum_{g=0}^{l} \Omega_g \left(S_x+d_0\right)^{-(n-l+c_0)} d\alpha}$$

Posterior up to proportionality is

$$p_2\left(\alpha,\lambda|x\right) \propto \prod_{i=1}^{r} x_i^{-\alpha} \prod_{i=r+l+1}^{n} x_i^{-\alpha} \alpha^{n-l+a_0-1} e^{-b_0\alpha} \lambda^{n-l+c_0-1} \sum_{g=0}^{l} \Omega_g \exp\left[-\lambda\left\{S_x+d_0\right\}\right]$$

Density for k$^{\text{th}}$ order future observation as discussed in case I is

$$f\left(y_{(k)}|\alpha,\lambda\right) = \beta^{-1}\left(k,m-k+1\right)\alpha\lambda y_{(k)}^{-(\alpha+1)} \sum_{i=0}^{m-k} \Omega_i \exp\left[-\lambda\left(i+k\right)y_{(k)}^{-\alpha}\right]$$

Hence, the predictive pdf is

$$h_2\left(y_{(k)}|x\right) = \int\int f\left(y_{(k)}|\alpha,\lambda\right) p_2\left(\alpha,\lambda|x\right) d\alpha d\lambda$$

Above predictive pdf can not be solved using routine implementation of numerical quadrature. Hence in this situation, the Monte Carlo Markov chain methods with Gibbs sampler and Metropolis-Hastings algorithm shall be used to compute predictive estimates.

These techniques have already been comprehensively discussed in the previous section. Gibbs algorithm requires full conditional, we have the full conditional of predictive pdf as

$$h_2\left(y_{(k)}|x\right) \propto \int_\alpha\int_\lambda \prod_{i=1}^{r} x_i^{-\alpha} \prod_{i=r+l+1}^{n} x_i^{-\alpha} y_{(k)}^{-(\alpha+1)} \alpha^{n-l+a_0} \lambda^{n-l+c_0} e^{-b_0\alpha} \sum_{i=0}^{m-k}\sum_{g=0}^{l} \Omega_i\Omega_g \exp\left[-\lambda\left\{S_x+d_0+\left(i+k\right)y_{(k)}^{-\alpha}\right\}\right] d\alpha d\lambda$$

$$h_2\left(y_{(k)}|x\right) \propto \int_\alpha \prod_{i=1}^{r} x_i^{-\alpha} \prod_{i=r+l+1}^{n} x_i^{-\alpha} y_{(k)}^{-(\alpha+1)} \alpha^{n-l+a_0} e^{-b_0\alpha} \Gamma\left(n-l+c_0+1\right) \sum_{i=0}^{m-k}\sum_{g=0}^{l} \Omega_i\Omega_g \left\{\left(S_x+d_0\right)+\left(i+k\right)y_{(k)}^{-\alpha}\right\}^{-(n-l+c_0+1)} d\alpha$$

or

$$h_2\left(y_{(k)}|x\right) \propto \int_\alpha \prod_{i=1}^{r} x_i^{-\alpha} \prod_{i=r+l+1}^{n} x_i^{-\alpha} y_{(k)}^{-(\alpha+1)} \alpha^{n-l+a_0} e^{-b_0\alpha} \sum_{i=0}^{m-k}\sum_{g=0}^{l} \Omega_i\Omega_g \left\{\left(S_x+d_0\right)+\left(i+k\right)y_{(k)}^{-\alpha}\right\}^{-(n-l+c_0+1)} d\alpha \tag{13}$$

Prediction limit are the solution of

$$\int_0^{t_{3k}} h_2\left(y_{(k)}|x\right) dy_{(k)} = \frac{\gamma}{2} \tag{14}$$

and

$$\int_{t_{4k}}^{\infty} h_2\left(y_{(k)}|x\right) dy_{(k)} = \frac{\gamma}{2} \tag{15}$$

Using (13), we get

$$\int_0^{t_{3k}} \int_\alpha \prod_{i=1}^{r} x_i^{-\alpha} \prod_{i=r+l+1}^{n} x_i^{-\alpha} y_{(k)}^{-(\alpha+1)} \alpha^{n-l+a_0} e^{-b_0\alpha} \sum_{i=0}^{m-k}\sum_{g=0}^{l} \Omega_i\Omega_g \left\{\left(S_x+d_0\right)+\left(i+k\right)y_{(k)}^{-\alpha}\right\}^{-(n-l+c_0+1)} dy_{(k)} d\alpha = \frac{\gamma}{2}$$

Taking $\left(S_x+d_0\right)+\left(i+k\right)y_{(k)}^{-\alpha} = z \implies -\alpha\left(i+k\right)y_{(k)}^{-\alpha-1} dy_{(k)} = dz$ and integrating out, we have lower prediction limit

$$\int_\alpha \prod_{i=1}^{r} x_i^{-\alpha} \prod_{i=r+l+1}^{n} x_i^{-\alpha} \alpha^{n-l+a_0-1} \frac{e^{-b_0\alpha}}{\left(n-l+c_0\right)} \sum_{i=0}^{m-k}\sum_{g=0}^{l} \Omega_i\Omega_g \frac{1}{\left(i+k\right)} \left[\left\{\left(S_x+d_0\right)+\left(i+k\right)t_{3k}^{-\alpha}\right\} - \left(S_x+d_0\right)^{\alpha}\right] d\alpha = \frac{\gamma}{2} \tag{16}$$

Using (15), the upper prediction limit is

$$\int_\alpha \prod_{i=1}^{r} x_i^{-\alpha} \prod_{i=r+l+1}^{n} x_i^{-\alpha} \alpha^{n-l+a_0-1} \frac{e^{-b_0\alpha}}{\left(n-l+c_0\right)} \sum_{i=0}^{m-k}\sum_{g=0}^{l} \Omega_i\Omega_g \frac{1}{\left(i+k\right)} \left[\left\{\left(S_x+d_0\right)+\left(i+k\right)t_{4k}^{-\alpha}\right\} - \left(S_x+d_0\right)^{\alpha}\right] d\alpha = 1-\frac{\gamma}{2} \tag{17}$$

4. Prediction Limits for the First Order Future Observation

For numerical study, we have dealt with the first order future observation only. On substituting k=1 in the results discussed in the previous section, we get the prediction limits for the smallest order future observation

4.1 Case I : When Parameter α is Known

On substituting k=1 in (9), lower prediction limit t_{11} is the solution of following equation:

$$\frac{m}{\sum_{g=0}^{l}\Omega_g\,(S_x+d_0)^{-(n-l+c_0)}}\sum_{i=0}^{m-1}\sum_{g=0}^{l}\frac{\Omega_i\Omega_g}{(i+1)}\left[\left\{(i+1)t_{11}^{-\alpha}+S_x+d_0\right\}^{-(n-l+c_0)}-\{S_x+d_0\}^{-(n-l+c_0)}\right]=\frac{\gamma}{2} \tag{18}$$

Similarly, the upper prediction limit t_{21} is obtained by solving the following equation using (10)

$$\frac{m}{\sum_{g=0}^{l}\Omega_g\,(S_x+d_0)^{-(n-l+c_0)}}\sum_{i=0}^{m-1}\sum_{g=0}^{l}\frac{\Omega_i\Omega_g}{(i+1)}\left[\left\{(i+1)t_{21}^{-\alpha}+S_x+d_0\right\}^{-(n-l+c_0)}-\{S_x+d_0\}^{-(n-l+c_0)}\right]=1-\frac{\gamma}{2} \tag{19}$$

Upper and lower prediction limits, t_{21} and t_{11}, are obtained numerically by solving (18) and (19), respectively, using bisection method, utilizing starting guess values.

4.2 Case II : When Both Parameters α and λ are Unknown

Similarly, substituting $k=1$ in (16), we get lower prediction limit t_{31} , on solving the following equation ***

$$\int_{\alpha}\prod_{i=1}^{r}x_i^{-\alpha}\prod_{i=r+l+1}^{n}x_i^{-\alpha}\alpha^{n-l+a_0-1}\frac{e^{-b_0\alpha}}{(n-l+c_0)}\sum_{i=0}^{m-1}\sum_{g=0}^{l}\Omega_i\Omega_g\frac{1}{(i+1)}\left[\left\{(S_x+d_0)+(i+1)t_{31}^{-\alpha}\right\}-(S_x+d_0)^{\alpha}\right]d\alpha=\frac{\gamma}{2} \tag{20}$$

Similarly, by substituting $k=1$ in (17), upper prediction limit t_{41} is obtained by solving the following expression:

$$\int_{\alpha}\prod_{i=1}^{r}x_i^{-\alpha}\prod_{i=r+l+1}^{n}x_i^{-\alpha}\alpha^{n-l+a_0-1}\frac{e^{-b_0\alpha}}{(n-l+c_0)}\sum_{i=0}^{m-1}\sum_{g=0}^{l}\Omega_i\Omega_g\frac{1}{(i+1)}\left[\left\{(S_x+d_0)+(i+1)t_{41}^{-\alpha}\right\}-(S_x+d_0)^{\alpha}\right]d\alpha=\frac{\gamma}{2} \tag{21}$$

For this case, prediction limits have been obtained using Metropolis Hasting technique.

5. Discussion

Numerical findings for the prediction intervals for the first order future observation are discussed based on simulated data set. For this purpose, we have randomly generated mid type II censored data from inverse Weibull distribution. Samples are generated from uniform distribution and then converted to inverse Weibull data, using cdf inversion technique. Prediction intervals are obtained as a difference of upper prediction limit and lower prediction limit at 95% confidence level. The present section is further subdivided into two subsections, where prediction intervals for two cases has been discussed.

5.1 Prediction Intervals for the Smallest Future Observation When the Parameter is Known

We have generated inverse Weibull sample using cdf inversion technique, using $\lambda=2.0$ and varying parameter $\alpha=(0.5,1.0,2.0,4.0)$. A simulation study of 1000 samples has been done in each case, using Monte Carlo technique. In the all cases width of prediction interval is reported by taking the difference between prediction limits $t_{21}-t_{11}$. Tables 1-4 summarize the prediction interval for the smallest future observation from inverse Weibull distribution for scale parameter λ, ehere shape parameter α is known at 0.5, 1.0, 2.0 and 4.0. Effect of hyperparameter d_0 is studied at 1.0, 2.0 and 4.0, keeping other hyperparameter c_0 fixed at 1.0 in all tables. Whereas, the prediction interval for the smallest future observation from inverse Weibull distribution for scale parameter, where shape parameter is known at 0.5, 1.0, 2.0 and 4.0 are presented in Tables 5-8 to study the effect on varying hyperparameter c_0 at 1.0, 2.0 and 4.0, keeping other hyperparameter d_0 fixed at 1.0. We have considered four different values of *r*, namely 2, 4, 6 and 8 and four different values of *l*, namely 2, 4, 6 and 8. We have studied nine different combinations of *r* and *l*, which are reported in Tables 1-8.

Number of censored observations in the middle is *l*, and number of observed failure time observations is (*n*-*l*). A higher value of *r* gives higher number of observations on the left than on right. While analyzing Tables 1-4, it has been figured out that by increasing the value of hyperparameter d_0, prediction limits tend to decrease. A significant change in the prediction limits has been noticed as the number of observations on the left (*r*) increases. We have noted that, as the number of middle censored observation (*l*) increases, prediction limit decreases in number of cases for smaller values of ($\alpha \leq 1$), but the trend mentioned earlier get reversed for the larger value of ($\alpha = 4$). While studying the effect of the shape parameter, we noticed that prediction limits decrease as the shape parameter increases, except when hyperparameter d_0 is high enough. We have noticed that whatever the censoring combination is, there is a sudden fall in prediction limit at $\alpha = 4.0$.

While studying Tables 5-8, we have noticed that prediction limits increase as we increase the value of hyperparameter c_0. A significant change in the prediction limits has been noticed when the number of observations on the left increases. With an increase in the number of middle censored observations (*l*), prediction limits tend to decrease in some cases, particularly, for smaller values of shape parameter $\alpha < 1$, but the mentioned trend get reversed with a larger value of parameter ($\alpha = 4$). The effect of shape parameter is observed in Tables 5-8; we noticed that prediction limits decrease as shape parameter increases, almost everywhere.

5.2 *Prediction Intervals for the Smallest Future Observation When Both Parameters are Unknown*

In order to provide the numerical illustration for both unknown parameters, we have used simulated datasets. First, we have simulated 500 datasets corresponding to each parametric value and censoring positions *l* and *r*, as given in table 9.

For the calculation of the predictive interval of the first order statistics, we have, first, simulated 1000 observations from posterior distribution using Metropolis algorithm and then simulated 1000 predictive observations of the first order statistic. Based on these predictive observations, we have obtained the highest predictive density interval and its length.

We have repeated the above process for each of the 500 simulated datasets, and obtained 500 lengths and averaged them which is given in the single cell of the table. We have calculated the average predictive length for each parameter set, and censoring point set *l* and *r* which are reported in Table 9. It is evident from the table that the average predictive length decreases everywhere as we increase the value of either parameters shape parameter or scale parameter. As far as censored observation is concerned, a clear cut indication is seen from Table 9 that as we increase the number of censored observations (*l*), a decrease in average predictive length is seen. However, no pattern has been found for the left censored observation (*r*).

We have also plotted the kernel density estimate for the predictive density for few sets of parameters which is given in Figs. 1-3. Note that these plotting were done for only a single data set, where *r*=2 and *l*=3 everywhere. The plots for $\lambda < 4$ is seen as unimodal, but the plot for $\alpha = \lambda = 4$ is bimodal. The plots for predictive densities are seen as platykurtic curves in nature, through they are close to symmetric curves.

Prediction intervals are obtained with 99% confidence level also, and it was found shorter everywhere, due to paucity of space not reported over here.

Table 1: Prediction interval for inverse Weibull scale parameter λ when the shape parameter is known at α=0.5, and hyperparameter c_0 is fixed at 1.0

r	l	$d_0 = 1$	$d_0 = 2$	$d_0 = 4$
2	4	0.05948	0.04027	0.02486
2	6	0.05594	0.03808	0.02352
2	8	0.05183	0.03893	0.02301
4	2	0.05709	0.04023	0.02410
4	4	0.05704	0.03825	0.02455
4	6	0.05483	0.03816	0.02529
6	2	0.05042	0.0462	0.02542
6	4	0.05700	0.03955	0.02454
8	2	0.06071	0.03947	0.02670

Table 2: Prediction interval for inverse Weibull scale parameter λ when the shape parameter is known at α=1.0, and hyperparameter c_0 is fixed at 1.0

r	l	$d_0 = 1$	$d_0 = 2$	$d_0 = 4$
2	4	0.04033	0.03371	0.02670
2	6	0.04128	0.03449	0.02514
2	8	0.04024	0.03375	0.02606
4	2	0.03959	0.03451	0.02715
4	4	0.04072	0.03479	0.02712
4	6	0.04134	0.03424	0.02675
6	2	0.03923	0.03309	0.02683
6	4	0.03942	0.03376	0.02668
8	2	0.04059	0.03371	0.02693

Table 3: Prediction interval for inverse Weibull scale parameter λ when the shape parameter is known at α=2.0, and hyperparameter c_0 is fixed at 1.0

r	l	$d_0 = 1$	$d_0 = 2$	$d_0 = 4$
2	4	0.02488	0.02276	0.01983
2	6	0.02428	0.02270	0.01990
2	8	0.02474	0.02243	0.02005
4	2	0.02481	0.02223	0.02014
4	4	0.02425	0.02236	0.02002
4	6	0.02443	0.02270	0.02039
6	2	0.02414	0.02252	0.02036
6	4	0.02405	0.02250	0.02021
8	2	0.02424	0.02235	0.01977

Table 4: Prediction interval for inverse Weibull scale parameter λ when the shape parameter is known at $\alpha = 4.0$, and hyperparameter c_0 is fixed at 1.0

r	l	$d_0 = 1$	$d_0 = 2$	$d_0 = 4$
2	4	0.01362	0.01305	0.01234
2	6	0.01375	0.01311	0.01231
2	8	0.01392	0.01327	0.01240
4	2	0.01348	0.01296	0.01229
4	4	0.01361	0.01304	0.01220
4	6	0.01344	0.01297	0.01224
6	2	0.01349	0.01308	0.01231
6	4	0.01356	0.01302	0.01233
8	2	0.01344	0.01305	0.01230

Table 5: Prediction interval for inverse Weibull scale parameter λ when the shape parameter is known at α=0.5, and hyperparameter d_0 is fixed at 1.0

r	l	$c_0 = 1$	$c_0 = 2$	$c_0 = 4$
2	4	0.05767	0.06562	0.07672
2	6	0.05172	0.06245	0.07985
2	8	0.05419	0.06420	0.09397
4	2	0.05414	0.05680	0.08409
4	4	0.05102	0.06626	0.07700
4	6	0.05654	0.05865	0.09464
6	2	0.05204	0.05956	0.08363
6	4	0.05748	0.06303	0.09668
8	2	0.05460	0.06356	0.08260

Table 6: Prediction interval for inverse Weibull scale parameter λ when the shape parameter is known at α=1.0, and hyperparameter d_0 is fixed at 1.0

r	l	$c_0 = 1$	$c_0 = 2$	$c_0 = 4$
2	4	0.04026	0.0435	0.04897
2	6	0.0411	0.04284	0.04886
2	8	0.0399	0.04366	0.05058
4	2	0.03928	0.04291	0.04649
4	4	0.04032	0.04224	0.04702
4	6	0.03848	0.04298	0.04736
6	2	0.04083	0.04224	0.04666
6	4	0.04044	0.04124	0.04754
8	2	0.0392	0.04316	0.04734

Table 7: Prediction interval for inverse Weibull scale parameter λ when the shape parameter is known at α=2.0, and hyperparameter d_0 is fixed at 1.0

r	l	$c_0 = 1$	$c_0 = 2$	$c_0 = 4$
2	4	0.02466	0.02527	0.02644
2	6	0.02512	0.02579	0.02657
2	8	0.02503	0.02548	0.02717
4	2	0.02481	0.02524	0.02667
4	4	0.02451	0.02507	0.02663
4	6	0.02402	0.02518	0.02628
6	2	0.02435	0.02505	0.02598
6	4	0.02431	0.02446	0.02653
8	2	0.02406	0.02496	0.02637

Table 8: Prediction interval for inverse Weibull scale parameter λ when shape parameter is known at α=2.0 and hyperparameter d_0 is fixed at 1.0

r	l	$c_0 = 1$	$c_0 = 2$	$c_0 = 4$
2	4	0.01346	0.01373	0.01387
2	6	0.01371	0.01382	0.01412
2	8	0.01386	0.01382	0.01413
4	2	0.01337	0.01349	0.01374
4	4	0.01368	0.01355	0.01383
4	6	0.01361	0.01356	0.01403
6	2	0.01357	0.01363	0.01381
6	4	0.01371	0.01363	0.01356
8	2	0.01355	0.01362	0.01364

Table 9: Prediction interval for inverse Weibull scale and shape parameters

(r,l)	Variation in Parameters (α, λ)					
	(2,2)	(2,4)	(3,4)	(4,3)	(4,2)	(4,4)
(2,2)	1.741339	1.682455	1.033206	0.8246686	0.9875498	0.7778541
(2,3)	1.661818	1.634784	0.9863835	0.7470398	0.839274	0.7326823
(2,4)	1.575451	1.508359	0.9715497	0.7219768	0.7793068	0.7114554
(3,2)	1.666175	1.638689	1.049793	0.8375615	0.9273114	0.7674046
(3,3)	1.629811	1.617737	1.004488	0.7895658	0.859431	0.7389515
(3,4)	1.597743	1.5776	0.9901628	0.7398998	0.822865	0.7246762
(4,2)	1.709496	1.629259	1.032112	0.8167973	0.9661159	0.7695671
(4,3)	1.693198	1.673873	0.9985587	0.7729078	0.8483346	0.7616518
(4,4)	1.586376	1.557612	0.9735342	0.6986156	0.7785551	0.7209336

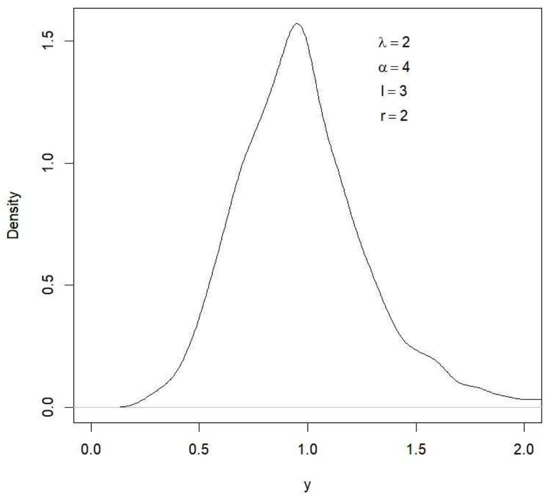

Fig. 1: Kernel density estimate for predictive density at $\alpha = 4$ and $\lambda = 2$

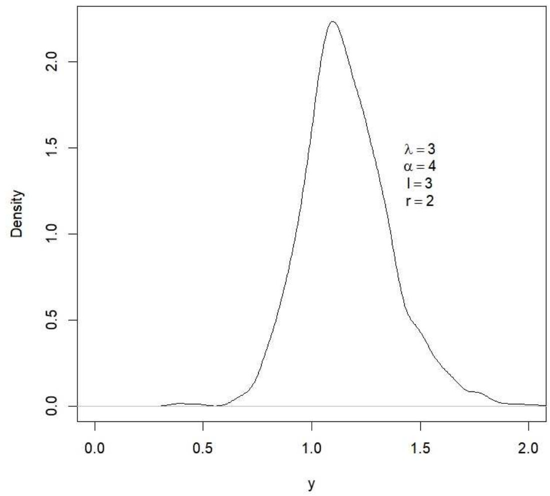

Fig. 2: Kernel density estimate for predictive density at $\alpha = 4$ and $\lambda = 3$

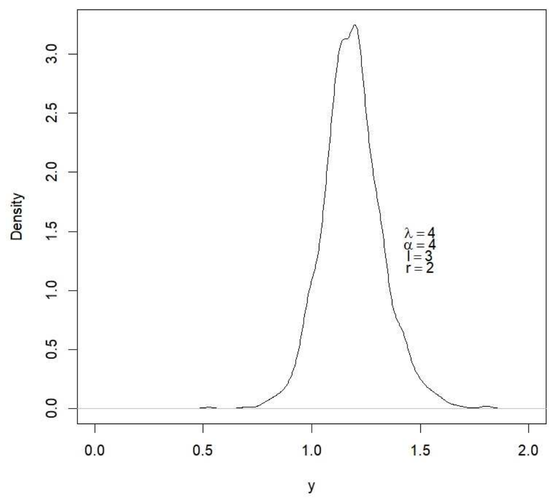

Fig. 3: Kernel density estimate for predictive density at $\alpha = 4$ and $\lambda = 4$

6. Conclusion

Prediction interval for inverse Weibull distribution is recommended with higher prediction confidence. When the scale parameter is known, then its higher values are suggested in order to obtain the shortest prediction interval. In case both parameters are unknown, Metropolis Hasting technique is suggested to obtain the predictive inferences. As the number of mid censored observations increases, a significant shorter average predictive length is found. As far as prior hyperparameters are concerned, the use of smaller values of a_0 and b_0, and larger value of c_0 and d_0 are suggested. To obtain predictive inferences, mid type II censoring scheme is suggested for use.

Conflict of Interest

The authors declare that they have no conflict of interest.

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