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Size- biased Poisson-Ishita Distribution and its Applications to Thunderstorm Events

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Abstract: In the present paper, a size-biased Poisson-Ishita distribution (SBPID) has been proposed, which is the size-biasing of the Poisson-Ishita distribution (PID) introduced by Shukla and Shanker (2019). The moments about origin and moments about mean have been obtained and hence expressions for coefficient of variation, skewness, kurtosis and index of dispersion have been derived and their behavior explained graphically. The estimation of its parameter has been discussed using method of moments and maximum likelihood estimation. The applications of SBPID have been explained through real datasets relating to thunderstorm events. The goodness of fit of SBPID has been found satisfactory over size-biased Poisson distribution (SBPD) and size-biased Poisson-Lindley distribution (SBPLD).
Keywords: Ishita distribution, Poisson-Ishita distribution, Moments, Properties, Estimation of parameter, Thunderstorm events, Applications.

1 Introduction

[1] firstly introduced weighted distributions to model ascertainment biases which were later formalized by [2] in a unifying theory for problems where the observations fall in non-experimental, non-replicated and non-random categories. In recent past years, many researchers studied and proposed different size-biased distributions and applied to biological science, medical science, migration, engineering as well as agricultural data such as [3], [4], [5],[6],[7],[8],[9],[10],[11],[12],[13],[14],[15],[16],[17], and [18] are some among others.

In field applications, size-biased distributions can arise either because individuals are sampled with unequal probability by design or because of unequal detection probability. Size-biased distributions come into play when organisms occur in groups, and group size influences the probability of detection. Size-biased distributions have applications in environmental science, econometrics, social science, biomedical science, human demography, ecology, geology, forestry etc.

The main aim of the present paper is to propose a model and apply to model thunderstorm events. Thunderstorms are related with combinations of environmental conditions, including unstable air with high moisture content, and some type of lifting action during different months in the year at any part of world. As per United States weather observation, thunderstorm is reported whenever thunder is heard at weather station. As [19] reported in their paper that thunderstorms are collected along with other atmospheric phenomenon on the standard weather observer's form WBAN-10 when thunder is first heard and ends 15 minutes after thunder is last heard. They applied negative binomial model for describing frequencies of thunderstorm events.

Let a random variable X has probability distribution $P_0\left(x;\theta\right)$; $x=0,1,2,...,\theta>0$. If sample units are weighted or selected with probability proportional to x^{α}, then the corresponding size-biased distribution of order α is given by its probability mass function (pmf)

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$$P_1(x;\theta)=\frac{x^{\alpha}\cdot P_0(x;\theta)}{\mu'_{\alpha}} \tag{1.1}$$

where $\mu'_{\alpha}=E\left(X^{\alpha}\right)=\sum_{x=0}^{\infty}x^{\alpha}P_0(x;\theta)$. When $\alpha=1$, the distribution is known as simple size-biased distribution and is applicable for size-biased sampling and for $\alpha=2$, the distribution is known as area-biased distribution and is applicable for area-biased sampling in forestry, earth sciences and geography.

The probability density function (pdf) of Ishita distribution defined as

$$f(x;\theta)=\frac{\theta^3}{\theta^3+2}\left(\theta+x^2\right)e^{-\theta x}\quad ;x>0,\ \theta>0 \tag{1.2}$$

has been proposed by [20]. The detailed discussion about its various properties, estimation of parameter and applications can be seen in [20]. A simulation study has also been done on Ishita distribution by [21]. The distribution (1.2) can be easily expressed as a mixture of exponential (θ) and gamma $(3,\theta)$ distributions with mixing proportion $\frac{\theta^3}{\theta^3+2}$. We have

$$f(x,\theta)=p\,g_1(x;\theta)+(1-p)\,g_2(x;\theta)$$

where $p=\frac{\theta^3}{\theta^3+2}$, $g_1(x;\theta)=\theta e^{-\theta x}$, and $g_2(x;\theta)=\frac{\theta^3x^2e^{-\theta x}}{2}$.

The corresponding cumulative distribution function (cdf) of (1.2) is given by

$$F(x;\theta)=1-\left[1+\frac{\theta x(\theta x+2)}{\theta^3+2}\right]e^{-\theta x}\ ;x>0,\theta>0 \tag{1.3}$$

The present paper is divided into seven sections. The background of size-biased distributions and their applications have been discussed in the first section. Size-biased Poisson-Ishita distribution (SBPID) has been derived and proposed in the second section. In the third section, important statistical and mathematical properties of SBPID based on moments have been discussed. The unimodality and the increasing hazard rate for the proposed distribution have been presented in the fourth section. In the fifth section, method of moment and method of maximum likelihood for the estimation of its parameter have been discussed. The goodness of fit of SBPID has been illustrated in the sixth section. The concluding remarks of the proposed distribution have been presented in the seventh section.

2 Size- Biased Poisson-Ishita Distribution

The Poisson-Ishita distribution (PID) having pmf

$$P_0(x;\theta)=\frac{\theta^3}{\theta^3+2}\frac{x^2+3x+\left(\theta^3+2\theta^2+\theta+2\right)}{(\theta+1)^{x+3}}\quad ;x=0,1,2,...,\ \theta>0 \tag{2.1}$$

has been introduced by [22] for modeling count data in various fields of knowledge. The PID can be obtained as a Poisson mixture of Ishita distribution (1.2) when the parameter λ of the Poisson distribution follows Ishita distribution (1.2). They discussed its important statistical properties, estimation of parameter and applications. It has been observed by [21] that PID gives better fit than Poisson distribution (PD), Poisson-Lindley distribution (PLD) of [23] which is a Poisson mixture of [24] distribution and Poisson-Akash distribution (PAD) of [25], a Poisson mixture of Akash distribution proposed by [26].

The pmf of the size-biased Poisson-Ishita distribution (SBPID) with parameter θ can be obtained as

$$P_1(x;\theta)=\frac{x\cdot P_0(x;\theta)}{\mu_1'}=\frac{\theta^4}{\theta^3+6}\frac{\left\{x^3+3x^2+\left(\theta^3+2\theta^2+\theta+2\right)x\right\}}{(\theta+1)^{x+3}}\quad ;x=1,2,3,..,\theta>0 \tag{2.2}$$

where $\mu_1'=\dfrac{\theta^2+6}{\theta\left(\theta^3+2\right)}$ is the mean of the PID with pmf (2.1). The nature of SBPID for varying values of the parameter θ has been shown in figure 1.

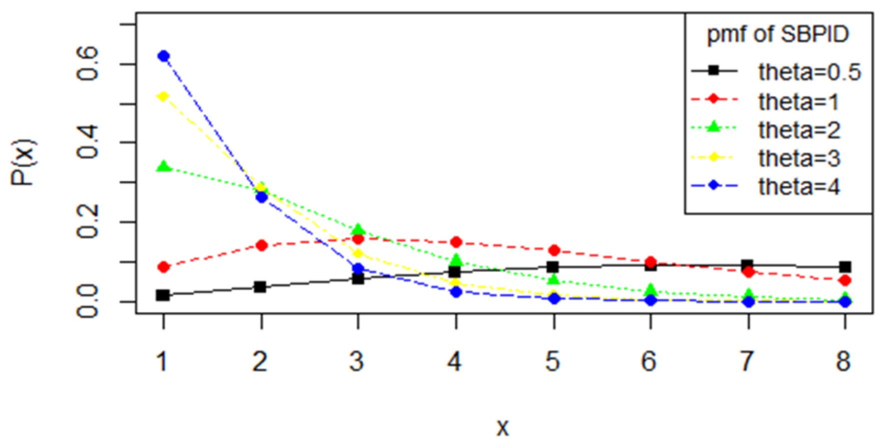

Fig. 1: Pmf plot of SBPID for different values of theta

3 Results and Discussion

The pmf (2.2) of SBPID can also be obtained from the size-biased Poisson distribution (SBPD) with pmf

$$g(x\mid\lambda)=\frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!}\quad ;x=1,2,3,....;\lambda>0 \tag{2.3}$$

when its parameter λ follows size-biased Ishita distribution (SBID) with pdf

$$h(\lambda;\theta)=\frac{\theta^4}{\theta^3+6}\lambda\left(\theta+\lambda^2\right)e^{-\theta\lambda}\quad ;\lambda>0,\ \theta>0 \tag{2.4}$$

We have

$$P(X=x)=\int_0^\infty g(x|\lambda)\cdot h(\lambda;\theta)\,d\lambda$$

$$=\int_0^\infty \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!}\frac{\theta^4}{\theta^3+6}\lambda\left(\theta+\lambda^2\right)e^{-\theta\lambda}d\lambda \tag{2.5}$$

$$=\frac{\theta^4}{\left(\theta^3+6\right)(x-1)!}\int_0^\infty e^{-(\theta+1)\lambda}\left(\theta\lambda^x+\lambda^{x+2}\right)d\lambda$$

$$=\frac{\theta^4}{\left(\theta^3+6\right)(x-1)!}\left[\frac{\theta\,\Gamma(x+1)}{(\theta+1)^{x+1}}+\frac{\Gamma(x+3)}{(\theta+1)^{x+3}}\right]$$

$$=\frac{\theta^4}{\left(\theta^3+6\right)}\frac{x^3+3x^2+\left(\theta^3+2\theta^2+\theta+2\right)x}{(\theta+1)^{x+3}}\quad ;x=1,2,3,..,\theta>0$$

which is the pmf of SBPID.

It would be recalled that the pmf of size-biased Poisson-Lindley distribution (SBPLD) given by

$$P_2(x;\theta)=\frac{\theta^3}{\theta+2}\frac{x(x+\theta+2)}{(\theta+1)^{x+2}}\quad ;x=1,2,3,....;\theta>0 \tag{2.6}$$

has been introduced by [27], which is a size-biased version of Poisson-Lindley distribution introduced by [24] . They have discussed its important mathematical and statistical properties, maximum likelihood estimation and the method of moments for estimation of the parameter. [28] have discussed the applications of SBPLD for modeling data on thunderstorms and reported that SBPLD is a better model for thunderstorms than SBPD.

4 Moments, Skewness, Kurtosis, and Index of Dispersion

Using (2.5), the r th factorial moment about origin $\mu_{(r)}'$ of the SBPID (2.2) can be obtained as

$$\mu_{(r)}'=E\left[E\left(X^{(r)}|\lambda\right)\right],\text{ where } X^{(r)}=X(X-1)(X-2)...(X-r+1)$$

$$=\int_0^\infty\left[\sum_{x=1}^\infty x^{(r)}\frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!}\right]\cdot\frac{\theta^4}{\theta^3+6}\lambda\left(\theta+\lambda^2\right)e^{-\theta\lambda}d\lambda$$

$$=\int_0^\infty\left[\lambda^{r-1}\left\{\sum_{x=r}^\infty x\frac{e^{-\lambda}\lambda^{x-r}}{(x-r)!}\right\}\right]\cdot\frac{\theta^4}{\theta^3+6}\lambda\left(\theta+\lambda^2\right)e^{-\theta\lambda}d\lambda$$

Taking $x=x+r$, we get

$$\mu_{(r)}'=\int_0^\infty\left[\lambda^{r-1}\left\{\sum_{x=0}^\infty (x+r)\frac{e^{-\lambda}\lambda^{x}}{x!}\right\}\right]\cdot\frac{\theta^4}{\theta^3+6}\lambda\left(\theta+\lambda^2\right)e^{-\theta\lambda}d\lambda$$

$$=\frac{\theta^4}{\theta^3+6}\int_0^\infty \lambda^{r-1}(\lambda+r)\left(\theta\lambda+\lambda^3\right)e^{-\theta\lambda}d\lambda$$

$$=\frac{\theta^4}{\theta^3+6}\int_0^\infty \left(\lambda^r+r\lambda^{r-1}\right)\left(\theta\lambda+\lambda^3\right)e^{-\theta\lambda}d\lambda$$

Using gamma integral and a little algebraic simplification, the r th factorial moment about origin of SBPID (2.2) can be obtained as

$$\mu_{(r)}'=\frac{r!\left\{r\theta^4+(r+1)\theta^3+r(r+1)(r+2)\theta+(r+1)(r+2)(r+3)\right\}}{\theta^r\left(\theta^3+6\right)} \quad r=1,2,3,.... \tag{3.1}$$

Taking $r=1,2,3,4$ in (3.1), the first four factorial moments about origin can be obtained and using the relationship between moments about origin and factorial moments about origin, the first four moments about origin of the SBPID (2.2) are thus obtained as

$$\mu_1'=\frac{\theta^4+2\theta^3+6\theta+24}{\theta\left(\theta^3+6\right)}$$

$$\mu_2'=\frac{\theta^5+6\theta^4+6\theta^3+6\theta^2+72\theta+120}{\theta^2\left(\theta^3+6\right)}$$

$$\mu_3'=\frac{\theta^6+14\theta^5+36\theta^4+30\theta^3+168\theta^2+720\theta+720}{\theta^3\left(\theta^3+6\right)}$$

$$\mu_4'=\frac{\theta^7+30\theta^6+150\theta^5+246\theta^4+480\theta^3+3000\theta^2+7200\theta+5040}{\theta^4\left(\theta^3+6\right)}$$

Now, using the relationship between moments about mean and the moments about origin, the moments about mean of the SBPID (2.2) are thus obtained as

$$\mu_2=\frac{2\left(\theta^7+\theta^6+18\theta^4+30\theta^3+72\theta+72\right)}{\theta^2\left(\theta^3+6\right)^2}$$

$$\mu_3=\frac{2\left(\theta^{11}+3\theta^{10}+2\theta^9+24\theta^8+108\theta^7+108\theta^6+180\theta^5+756\theta^4+432\theta^3+432\theta^2+296\theta+864\right)}{\theta^3\left(\theta^3+6\right)^3}$$

$$\mu_4=\frac{2\begin{pmatrix}\theta^{15}+13\theta^{14}+24\theta^{13}+42\theta^{12}+510\theta^{11}+1296\theta^{10}+1188\theta^9+6084\theta^8+14688\theta^7\\+10152\theta^6+29160\theta^5+62208\theta^4+38880\theta^3+49248\theta^2+93312\theta+46656\end{pmatrix}}{\theta^4\left(\theta^3+6\right)^4}$$

The coefficient of variation $(C.V)$, coefficient of Skewness $\left(\sqrt{\beta_1}\right)$, coefficient of Kurtosis (β_2) and index of dispersion (γ) of the SBPID (2.2) are thus obtained as

$$C.V=\frac{\sigma}{\mu_1'}=\frac{\sqrt{2\left(\theta^7+\theta^6+18\theta^4+30\theta^3+72\theta+72\right)}}{\left(\theta^4+2\theta^3+6\theta+24\right)}$$

$$\sqrt{\beta_1}=\frac{\mu_3}{\mu_2^{3/2}}=\frac{\left(\begin{array}{l}\theta^{11}+3\theta^{10}+2\theta^{9}+24\theta^{8}+108\theta^{7}+108\theta^{6}+180\theta^{5}+756\theta^{4}+432\theta^{3}\\+432\theta^{2}+296\theta+864\end{array}\right)}{\sqrt{2}\left(\theta^{7}+\theta^{6}+18\theta^{4}+30\theta^{3}+72\theta+72\right)^{3/2}}$$

$$\beta_2=\frac{\mu_4}{\mu_2^{2}}=\frac{\left(\begin{array}{l}\theta^{15}+13\theta^{14}+24\theta^{13}+42\theta^{12}+510\theta^{11}+1296\theta^{10}+1188\theta^{9}+6084\theta^{8}+14688\theta^{7}\\+10152\theta^{6}+29160\theta^{5}+62208\theta^{4}+38880\theta^{3}+49248\theta^{2}+93312\theta+46656\end{array}\right)}{2\left(\theta^{7}+\theta^{6}+18\theta^{4}+30\theta^{3}+72\theta+72\right)^{2}}$$

$$\gamma=\frac{\sigma^2}{\mu_1'}=\frac{2\left(\theta^{7}+\theta^{6}+18\theta^{4}+30\theta^{3}+72\theta+72\right)}{\theta\left(\theta^{3}+6\right)\left(\theta^{4}+2\theta^{3}+6\theta+24\right)}$$

It can be easily verified that SBPID is over-dispersed $\left(\mu<\sigma^2\right)$, equi-dispersed $\left(\mu=\sigma^2\right)$ and under-dispersed for $\theta<(=)>\theta^*=2.036937$. It should be noted that SBPLD is over-dispersed $\left(\mu<\sigma^2\right)$, equi-dispersed $\left(\mu=\sigma^2\right)$ and under-dispersed for $\theta<(=)>\theta^*=1.671162$.

The nature of coefficient of variation (C.V), coefficient of skewness (C.S), coefficient of kurtosis (C.K) and index of dispersion (ID) of SBPID has been shown graphically in figure 2.

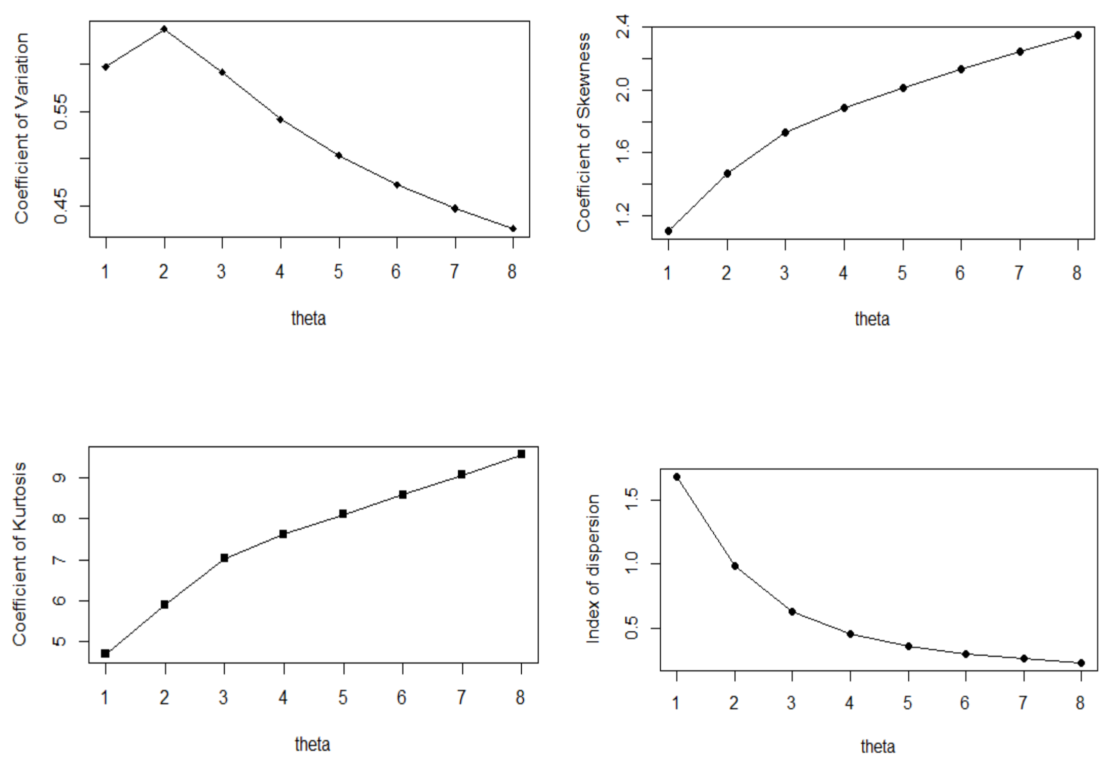

Fig. 2: CV,CS ,CK and ID of SBIPD for varying values of theta.

4.1 Unimodality and Increasing Failure Rate

Since

$$\frac{P_1(x+1;\theta)}{P_1(x;\theta)} = \left(\frac{1}{\theta+1}\right)\left(1+\frac{1}{x}\right)\left(1+\frac{2(x+2)}{x^2+3x+(\theta^3+2\theta^2+\theta+2)}\right)$$

is a deceasing function of x, $P_1(x;\theta)$ is log-concave. Therefore, SBPID is unimodal, has an increasing failure rate (IFR), and hence increasing failure rate average (IFRA). It is new better than used in expectation (NBUE) and has decreasing mean residual life (DMRL). The definitions, concepts and interrelationship between these aging concepts have been discussed in [29].

5 Estimation of Parameter

5.1 Method of Moment Estimate (MOME)

Equating the population mean to the corresponding sample mean, the method of moment estimate (MOME) $\tilde{\theta}$ of θ of SBPID (2.2) is the solution of the following fourth degree polynomial equation in θ

$$(\bar{x}-1)\theta^4 - 2\theta^3 + 6(\bar{x}-1)\theta - 24 = 0,$$

Where $\bar{x}$ is the sample mean

5.2 Maximum Likelihood Estimate (MLE)

Let $x_1, x_2, ..., x_n$ be a random sample of size n from the SBPID (2.2) and let f_x be the observed frequency in the sample corresponding to $X = x$ $(x = 1, 2, 3, ..., k)$ such that $\sum_{x=1}^{k} f_x = n$, where k is the largest observed value having non-zero frequency. The likelihood function L of the SBPID (2.2) is given by

$$L = \left(\frac{\theta^4}{\theta^3+6}\right)^n \frac{1}{(\theta+1)^{\sum_{x=1}^{k} f_x(x+3)}} \prod_{x=1}^{k}\left[x^3+3x^2+(\theta^3+2\theta^2+\theta+2)x\right]^{f_x}.$$

The log likelihood function can be obtained as

$$\log L = n\log\left(\frac{\theta^4}{\theta^3+6}\right) - \sum_{x=1}^{k} f_x(x+3)\log(\theta+1) + \sum_{x=1}^{k} f_x \log\left[x^3+3x^2+(\theta^3+2\theta^2+\theta+2)x\right].$$

The first derivative of the log likelihood function is thus given by

$$\frac{d\log L}{d\theta} = \frac{4n}{\theta} - \frac{3n\theta^2}{\theta^3+6} - \frac{n(\bar{x}+3)}{\theta+1} + \sum_{x=1}^{k}\frac{(3\theta^2+4\theta+1)f_x}{x^2+3x+(\theta^3+2\theta^2+\theta+2)}$$

where $\bar{x}$ is the sample mean.

The maximum likelihood estimate (MLE), $\hat{\theta}$ of θ of SBPID (2.3) is the solution of the equation $\frac{d\log L}{d\theta} = 0$ and is given by the solution of the following non-linear equation

$$\frac{4n}{\theta} - \frac{3n\theta^2}{\theta^3+6} - \frac{n(\bar{x}+3)}{\theta+1} + \sum_{x=1}^{k}\frac{(3\theta^2+4\theta+1)f_x}{x^2+3x+(\theta^3+2\theta^2+\theta+2)} = 0$$

This non-linear equation can be solved by any numerical iteration methods such as Newton- Raphson method, Bisection method, Regula –Falsi method etc. In the present paper, Newton-Raphson method has been used to solve the above non-linear equation to find maximum MLE of the parameter using R-software.

6 Goodness of Fit

In this section the goodness of fit of SBPID, SBPLD and SBPD have been presented for four count datasets from thunderstorm activity and it was taken from [30]. The fitting of these distributions are based on maximum likelihood estimates of the parameter. The datasets in tables 1, 2, 3 and 4 are the data regarding thunderstorm events during Fall, September, May and Spring, at Cape Kennedy. From the goodness of fit, it is obvious that SBPID gives closer fit than SBPD and SBPLD.

Table 1: Distribution of number of Thunderstorms and frequencies of Thunderstorm events during Fall.

No. of thunderstorms	Frequencies of thunderstorm event	Expected Frequency		
		SBPD	SBPLD	SBPID
1	170	161.7	166.2	166.3
2	47	56.9	49.9	49.7
3	7	10.0	11.1	11.2
4	4	1.2	2.2	2.2
5	2	0.2	0.6	0.6
Total	230	230.0	230.0	230.0
ML estimate		$\hat{\theta}=0.35217$	$\hat{\theta}=6.3654$	$\hat{\theta}=5.83571$
-2logL		937.06	358.96	358.67
χ^2		2.37	0.313	0.30
d.f.		1	1	1
p-value		0.1236	0.5776	0.5838

Table 2: Distribution of Thunderstorms and their frequencies during September.

No. of thunderstorm	Frequencies of thunderstorm event	Expected Frequency		
		SBPD	SBPLD	SBPID
1	122	114.1	117.9	118.0
2	35	44.1	38.3	38.1
3	5	8.5	9.3	9.3
4	4	1.1	1.9	2.1
5	2	0.2	0.6	0.5
Total	168	168.0	168.0	168.0
ML estimate		$\hat{\theta}=0.3869$	$\hat{\theta}=5.8405$	$\hat{\theta}=5.3509$
-2logL		731.44	278.54	278.20
χ^2		2.57	0.48	0.45
d.f.		1	1	1
p-value		0.1089	0.4884	0.4999

Table 3: Distribution of number of Thunderstorms and their frequencies during May.

No. of thunderstorm	Observed Frequency	Expected Frequency		
		SBPD	SBPLD	SBPID

1 2 3 4	87 25 5 3	83.1 30.5 5.6 0.8	85.6 26.6 6.1 1.7	85.7 26.5 6.2 1.6
Total	120	120.0	120.0	120.0
ML estimate		$\hat{\theta} = 0.3666$	$\hat{\theta} = 6.1290$	$\hat{\theta} = 5.6281$
-2logL		456.29	190.97	190.90
χ^2		1.57	0.12	0.10
d.f.		1	1	1
p-value		0.2102	0.7290	0.7518

Table 4: Distribution of number of Thunderstorms and their frequencies during spring

No. of thunderstorm	Observed Frequency	Expected Frequency		
		SBPD	SBPLD	SBPID
1 2 3 4	174 50 10 4	168.6 58.1 10.0 1.3	173.1 51.1 11.2 2.6	173.3 50.8 11.2 2.7
Total	238	238.0	238.0	238.0
ML estimate		$\hat{\theta} = 0.3445$	$\hat{\theta} = 6.4897$	$\hat{\theta} = 5.9636$
-2logL		874.75	363.44	363.40
χ^2		1.94	0.03	0.016
d.f.		1	1	1
p-value		0.1636	0.8624	0.9203

The probability plots of the fitted distributions for four datasets in tables 1, 2, 3, and 4 have been shown in figure 3.

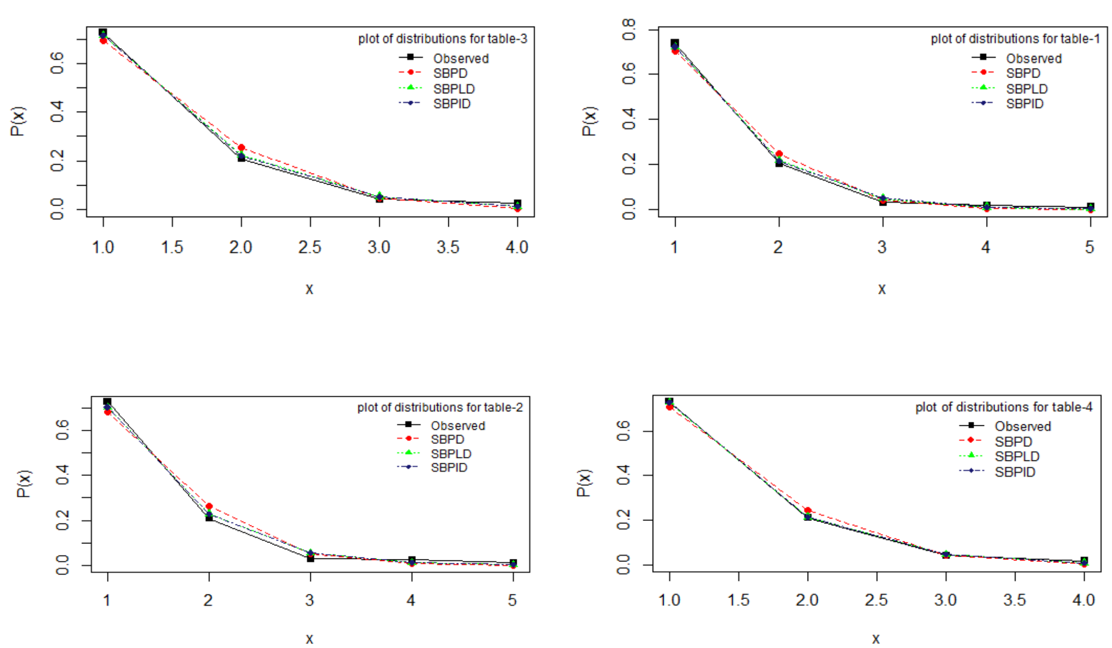

Fig.3: Probability plots of fitted distribution distributions.

7 Concluding Remarks

In the present paper, a size-biased Poisson-Ishita distribution (SBPID) has been proposed. The first four moments about origin and the central moments have been obtained and hence expressions for coefficient of variation (C.V.), skewness and kurtosis have been derived and their behavior varying values of parameter presented graphically. The estimation of its parameter has been discussed using maximum likelihood estimation and method of moments. Some thunderstorm examples of real datasets have been presented. The goodness of fit of SBPID has been found satisfactory over size-biased Poisson distribution (SBPD) and size-biased Poisson-Lindley distribution (SBPLD), and hence it can be considered as a good model for thunderstorm events.

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Conflict of Interest: The authors declare that they have no conflict of interest.

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