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Bayesian Estimation using Lindley's Approximation and Prediction of Generalized Exponential Distribution Based on Lower Record Values

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Abstract: The two parameter generalized exponential distribution (which is denoted by $GE(\alpha,\lambda)$) was introduced by Gupta and Kundu (1999). In this article, maximum likelihood estimators (MLE's) for the two unknown parameters of the generalized exponential (GE) distribution are obtained based on lower record values. Also, we obtained the Bayes Estimators of the unknown parameters of the generalized exponential distribution using Lindley's approximation (L-approximation) under symmetric and asymmetric loss functions. Further, we have derived the Bayesian prediction for the future record values. Numerical computations are presented to illustrate the results of estimation and prediction using R software.

Keywords: Bayesian inference, independent prior, maximum likelihood estimation, Bayes estimation, Bayes prediction, record values, generalized exponential distribution, squared error loss function, linear exponential (LINEX) loss function, general entropy loss function, Bayes prediction.

1 Introduction

Record values and their basic properties were first introduced by Chandler [1]. The study of record values and associated statistics are of great significance in multiple real life situations such as meteorology, seismology, athletic events, economics and life testing. Numerous papers on record values and their distributional properties have appeared in the statistical literature, among them [2,3,4,5,6]. Moreover, Raqab and Ahsanullah [7] studied the properties of order and record statistics, and their inferences, respectively.

Let $X_1, X_2, ...$ be a sequence of independent and identically distributed *(iid)* random variables *(rv's)* with a cumulative distribution function *(cdf)* $F(x)$ and a probability density function *(pdf)* $f(x)$. Let $Y_n = min\{X_1, X_2, ..., X_n\}$, for $n \geq 1$; then, the observation X_j is a lower record value of $\{X_n, n \geq 1\}$, if it is smaller than all the preceding observations; in other words, if $Y_j < Y_{j-1}, j > 1$. By definition, X_1 is a lower, as well as an upper record value, called the base record value.

The two parameter generalized exponential has a distribution function of the form

$$F(x) = (1-e^{-\lambda x})^{\alpha}, \qquad x \geq 0, \quad \alpha,\lambda > 0, \tag{1}$$

and the density function corresponding to (1) is

$$f(x) = \alpha\lambda e^{-\lambda x}(1-e^{-\lambda x})^{\alpha-1}, \qquad x \geq 0, \quad \alpha,\lambda > 0. \tag{2}$$

Here, α is the shape parameter, and λ is the scale parameter. When $\alpha = 1$, the GE distribution coincides with the exponential distribution. It was observed that the GE distribution can be used in situations where a skewed distribution for a non-negative random variable is needed. The GE is a unimodal density function. and for a fixed scale parameter, as the shape parameter increases, it becomes more and more symmetric. In addition, the GE distribution has a good physical interpretation. It has been studied extensively by Gupta and Kundu [8,9,10,11,12].

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For Bayesian inference, a frequent choice of loss function is a squared error loss function (SELF), because it is symmetric, and easy to implement and tractable. An asymmetric loss function is also found useful in many situations. A useful asymmetric loss function, known as the linear-exponential (LINEX) loss function, was introduced by Varian [13] and has been widely used by several authors, [14,15,16,17,18,19]. Moreover, we use a more flexible loss function, General entropy loss function (GELF), because different choices of parameter values involved in the loss function can produce both symmetric and asymmetric loss functions.

However, Bayesian estimation under the loss function is not frequently discussed, perhaps, because the estimators under symmetric and asymmetric loss function involve integral expressions, which are not analytically solvable. Therefore, one has to use the numerical techniques or certain approximation methods for the solution. Lindley's approximation and Tierney and Kadane approximation technique are the methods suitable for solving such problems. There has been a significant amount of research done in statistical inference of several distributions based on lower record values, see, [20,21,22,23,24,25]. For GE distribution, Sarhan and Tadj [26] presented the Bayesian inferences of the unknown parameters of the GE distribution based on upper record values.

Also, prediction of future records becomes a problem of great interest. For example, while studying the record rainfalls or snowfalls, having observed the record values until the present time, we will be naturally interested in predicting the amount of rainfall or snowfall that is to be expected when the present record is broken for first time in the future, See [2]. We may be interested in predicting (either point or interval) the value of the next record (X_{m+1}), or more generally, the value of the $s-th$ record (X_s), for some $s > m$. Prediction of future records has been studied by a number of statisticians, See [5,27,28,29,30,31]. The prediction of future record value based on given records is a useful research component involved in many applications.

In this paper, the maximum likelihood estimators of the parameters are obtained. Here we observed that the MLE's cannot be obtained in a closed form, we, therefore, propose to use the Newton-Raphson numerical approximation method to compute the MLE's via the Taylor series, and the proposed method works quite well. The Bayes estimators of the parameters of the generalized exponential distribution based on lower record values are obtained under the linear exponential loss function, general entropy loss function and squared error loss function using Lindley's approximation technique. Bayesian prediction of the $s-th$ lower record, either point or interval, is also presented. Moreover, Numerical computations using R software are given to illustrate the results.

2 Estimation of The Parameters

In this section, we are going to derive the MLE's of the parameters from a record data. We will also obtain the Bayes estimators under symmetric and asymmetric loss functions.

2.1 Maximum Likelihood Estimator

Suppose we observe m lower record values $X_{L(1)} = x_1, X_{L(2)} = x_2, ..., X_{L(m)} = x_m$, arising from a sequence of *iid* $GE(\alpha,\lambda)$ with *pdf* $f(x)$ (2), Arnold [2] gives the likelihood function as

$$l(\alpha,\lambda|\underline{x}) = f(x_{L(m)};\alpha,\lambda)\prod_{i=1}^{m-1}\frac{f(x_{L(i)};\alpha,\lambda)}{F(x_{L(i)};\alpha,\lambda)}. \tag{3}$$

Substituting (1) and (2) in (3), we get

$$l(\alpha,\lambda|\underline{x}) = \alpha^m\lambda^m exp(\alpha log(1-e^{-\lambda x_m}))t(\underline{x}), \tag{4}$$

where
$t(\underline{x}) = \prod_{i=1}^{m}\frac{e^{-\lambda x_i}}{(1-e^{-\lambda x_i})}$ and $\underline{x} = (x_1, x_2, ..., x_m)$.
The log-likelihood function is

$$L(\alpha,\lambda|\underline{x}) = m(log\alpha + log\lambda) + \alpha log(1 - exp(-\lambda x_m))$$

$$-\lambda\sum_{i=1}^{m}x_i - \sum_{i=1}^{m}log(1-exp(-\lambda x_i)). \tag{5}$$

By differentiating the equation (5) with respect to α and λ and equating to zero, we get

$$\frac{\partial L(\alpha,\lambda|\underline{x})}{\partial\alpha}=\frac{m}{\alpha}+log(1-exp(-\lambda x_m))=0, \tag{6}$$

$$\frac{\partial L(\alpha,\lambda|\underline{x})}{\partial\lambda}=\frac{m}{\lambda}-\sum_{i=1}^{m}x_i+\alpha\frac{x_m exp(-\lambda x_m)}{(1-exp(-\lambda x_m))}$$
$$-\sum_{i=1}^{m}\frac{x_i exp(-\lambda x_i)}{(1-exp(-\lambda x_i))}=0. \tag{7}$$

From equation (6), we get

$$\hat{\alpha}=-\frac{m}{log(1-exp(-\lambda x_m))}. \tag{8}$$

First, we shall find $\hat{\lambda}$ so that $\hat{\alpha}$ can be determined. Note that the estimators of α and λ are not in closed form. So that we propose to find $\hat{\lambda}$ by using Newton-Raphson method as given below.
Let $f(\lambda)$ be the same as equation (7), and taking the first differential of $f(\lambda)$, we have

$$f'(\lambda)=-\frac{m}{\lambda^2}-\frac{\alpha x_m^2 exp(-\lambda x_m)}{(1-exp(-\lambda x_m))^2}+\sum_{i=1}^{m}\frac{x_i^2 exp(-\lambda x_i)}{(1-exp(-\lambda x_i))^2}. \tag{9}$$

By substituting equation (8) into equation (7), we call $f(\lambda)$ as

$$f(\lambda)=\frac{m}{\lambda}-\frac{m x_m exp(-\lambda x_m)}{(1-exp(-\lambda x_m))log(1-exp(-\lambda x_m))}$$
$$-\sum_{i=1}^{m}x_i-\sum_{i=1}^{m}\frac{x_i exp(-\lambda x_i)}{(1-exp(-\lambda x_i))}. \tag{10}$$

Substituting equation (8) into equation (9), we obtain

$$f'(\lambda)=-[\frac{m}{\lambda^2}-\frac{m x_m^2 exp(-\lambda x_m)}{(1-exp(-\lambda x_m))^2 log(1-exp(-\lambda x_m))}$$
$$-\sum_{i=1}^{m}\frac{x_i^2 exp(-\lambda x_i)}{(1-exp(-\lambda x_i))^2}]. \tag{11}$$

Therefore, λ is obtained from the equation below by carefully choosing an initial value λ as λ_i and iterating the process till it converges:

$$\lambda_{i+1}=\lambda_i-\frac{\frac{m}{\lambda}-\sum_{i=1}^{m}x_i-\frac{m x_m e^{-\lambda x_m}}{(1-e^{-\lambda x_m})log(1-e^{-\lambda x_m})}-\sum_{i=1}^{m}\frac{x_i e^{-\lambda x_i}}{(1-e^{-\lambda x_i})}}{-\left[\frac{m}{\lambda^2}-\frac{m x_m^2 e^{-\lambda x_m}}{(1-e^{-\lambda x_m})^2 log(1-e^{-\lambda x_m})}-\sum_{i=1}^{m}\frac{x_i^2 e^{-\lambda x_i}}{(1e^{-\lambda x_i})^2}\right]}. \tag{12}$$

We find the MLE's of the parameters of $GE(\alpha,\lambda)$ distribution by using R software.

2.2 Bayes Estimator

In this subsection, we investigate the Bayes estimators for parameters α and λ. Under the assumption that both parameters α and λ are unknown, the priors for parameters α and λ may be taken as [32]

$$g_1(\alpha)\propto\frac{1}{\alpha},\qquad 0\le\alpha<\infty$$

and

$$g_2(\lambda)\propto\frac{b^c\lambda^{c-1}exp(-b\lambda)}{\Gamma c},\qquad \lambda\ge 0,\quad b,c>0,$$

respectively, to give the joint prior distribution for α and λ as

$$g(\alpha,\lambda)=\frac{b^c\lambda^{c-1}exp(-b\lambda)}{\alpha\Gamma c}, \qquad \alpha,\lambda\geq 0, \quad b,c>0, \tag{13}$$

where b and c are positive real numbers.

By Bayes Theorem, the posterior density function of (α,λ) is given by

$$\Pi(\alpha,\lambda|\underline{x})=Cl(\alpha,\lambda|\underline{x})g(\alpha,\lambda), \tag{14}$$

where C is a normalizing constant.

Using (4), (13) and applying Bayes Theorem, the joint posterior density function of α and λ is given by

$$\Pi(\alpha,\lambda|\underline{x})=\frac{1}{\Gamma m\phi(b,c-1,x_m)}\alpha^{m-1}\lambda^{m+c-1}t(\underline{x})exp(\alpha log(1-e^{-\lambda x_m})-b\lambda), \tag{15}$$

where

$$\phi(b,c-1,x_m)=\int_0^\infty \frac{\lambda^{m+c-1}t(\underline{x})exp(-b\lambda)}{(log(1-exp(-\lambda x_m))^{-1})^m}d\lambda. \tag{16}$$

Now, the Bayes estimators of the parameters with different loss functions are given below:

Linear Exponential (LINEX) Loss Function

This loss function according to Soliman [33] rises approximately exponentially on one side of zero and approximately linearly on the other side. There is an overestimation if $c_1>0$ and an underestimation if $c_1<0$, but when $c_1\cong 0$, LINEX Loss Function is approximately the squared error loss function. LINEX loss function with parameter c_1 is given by

$$L(\hat{\theta}-\theta)\propto exp(c_1(\hat{\theta}-\theta))-c_1(\hat{\theta}-\theta)-1, \tag{17}$$

where $\hat{\theta}$ is the estimate of the parameter θ.

Under the above loss function, the Bayes estimator $\hat{\theta}_{BL}$ of θ is given by

$$\hat{\theta}_{BL}=-\frac{1}{c_1}lnE_\theta(e^{-c_1\theta}|\underline{x}), \tag{18}$$

where E_θ stands for posterior expectation. The sign of shape parameter (c_1) reflects the direction of asymmetry, and its magnitude reflects the degree of asymmetry. The Bayes estimators for the parameters α and λ of the GE distribution under LINEX loss function may be defined as

$$\hat{\alpha}_{BL}=-\frac{1}{c_1}lnE_\alpha(e^{-c_1\alpha}|\underline{x}), \tag{19}$$

where

$$E_\alpha(e^{-c_1\alpha}|\underline{x})=\int_{(\alpha,\lambda)}e^{-c_1\alpha}\Pi(\alpha,\lambda|\underline{x})d(\alpha,\lambda) \tag{20}$$

and

$$\hat{\lambda}_{BL}=-\frac{1}{c_1}lnE_\lambda(e^{-c_1\lambda}|\underline{x}), \tag{21}$$

where

$$E_\lambda(e^{-c_1\lambda}|\underline{x})=\int_{(\alpha,\lambda)}e^{-c_1\lambda}\Pi(\alpha,\lambda|\underline{x})d(\alpha,\lambda), \tag{22}$$

respectively, provided that $E_\alpha(e^{-c_1\alpha}|\underline{x})$ and $E_\lambda(e^{-c_1\lambda}|\underline{x})$ exist and are finite.

From equation (15), it can be shown that

$$E_\alpha(e^{-c_1\alpha}|\underline{x})=\frac{1}{\phi(b,c-1,x_m)}\int_0^\infty \frac{\lambda^{m+c-1}t(\underline{x})exp(-b\lambda)}{[log(1-e^{-\lambda x_m})^{-1}+c_1]^m}d\lambda$$

and

$$E_\lambda(e^{-c_1\lambda}|\underline{x})=\frac{1}{\phi(b,c-1,x_m)}\int_0^\infty \frac{\lambda^{m+c-1}t(\underline{x})exp(-\lambda(b+c_1))}{[log(1-e^{-\lambda x_m})^{-1}]^m}d\lambda.$$

Using (19) and (21), we obtain the Bayes estimator of α and λ as

$$\hat{\alpha}_{BL} = -\frac{1}{c_1} ln\left[\frac{1}{\phi(b,c-1,x_m)}\int_0^\infty \frac{\lambda^{m+c-1} t(\underline{x}) exp(-b\lambda)}{[log(1-e^{-\lambda x_m})^{-1}+c_1]^m} d\lambda\right]$$

and

$$\hat{\lambda}_{BL} = -\frac{1}{c_1} ln\left[\frac{1}{\phi(b,c-1,x_m)}\int_0^\infty \frac{\lambda^{m+c-1} t(\underline{x}) exp(-\lambda(b+c_1))}{[log(1-e^{-\lambda x_m})^{-1}]^m} d\lambda\right].$$

General Entropy Loss Function

This is another useful asymmetric loss function that is used to determine whether there is an overestimation or an underestimation. It is a generalization of the entropy loss function. The general entropy loss function with parameter k is given by

$$L(\hat{\theta}-\theta) \propto \left(\frac{\hat{\theta}}{\theta}\right) - kln\left(\frac{\hat{\theta}}{\theta}\right) - 1, \tag{23}$$

where $\hat{\theta}$ is the estimate of the parameter θ.

Under the above loss function, the Bayes estimator $\hat{\theta}_{BG}$ of θ is given by

$$\hat{\theta}_{BG} = [E_\theta(\theta^{-k}|\underline{x})]^{-\frac{1}{k}}, \tag{24}$$

where E_θ stands for posterior expectation. The proper choice for k is a challenging task for an analyst, because it reflects the asymmetry of the loss function. The Bayes estimators for the parameters α and λ of the GE distribution under general entropy loss function may be defined as

$$\hat{\alpha}_{BG} = [E_\alpha(\alpha^{-k}|\underline{x})]^{-\frac{1}{k}}, \tag{25}$$

where

$$E_\alpha(\alpha^{-k}|\underline{x}) = \int_{(\alpha,\lambda)} \alpha^{-k} \Pi(\alpha,\lambda|\underline{x}) d(\alpha,\lambda) \tag{26}$$

and

$$\hat{\lambda}_{BG} = [E_\lambda(\lambda^{-k}|\underline{x})]^{-\frac{1}{k}}, \tag{27}$$

where

$$E_\lambda(\lambda^{-k}|\underline{x}) = \int_{(\alpha,\lambda)} \lambda^{-k} \Pi(\alpha,\lambda|\underline{x}) d(\alpha,\lambda), \tag{28}$$

respectively, provided that $E_\alpha(\alpha^{-k}|\underline{x})$ and $E_\lambda(\lambda^{-k}|\underline{x})$ exist and are finite.

From equation (15), it can be shown that

$$E_\alpha(\alpha^{-k}|\underline{x}) = \frac{\Gamma(m-k)}{\Gamma m\phi(b,c-1,x_m)}\int_0^\infty \frac{\lambda^{m+c-1} t(\underline{x}) exp(-b\lambda)}{[log(1-e^{-\lambda x_m})^{-1}]^{m-k}} d\lambda$$

and

$$E_\lambda(\lambda^{-k}|\underline{x}) = \frac{1}{\phi(b,c-1,x_m)}\int_0^\infty \frac{\lambda^{m+c-k-1} t(\underline{x}) exp(-b\lambda)}{[log(1-e^{-\lambda x_m})^{-1}]^{m}} d\lambda.$$

Using (24) and (27), we obtain the Bayes estimator of α and λ as

$$\hat{\alpha}_{BG} = \left[\frac{\Gamma(m-k)}{\Gamma m\phi(b,c-1,x_m)}\int_0^\infty \frac{\lambda^{m+c-1} t(\underline{x}) exp(-b\lambda)}{[log(1-e^{-\lambda x_m})^{-1}]^{m-k}} d\lambda\right]^{-\frac{1}{k}},$$

and

$$\hat{\lambda}_{BG} = \left[\frac{1}{\phi(b,c-1,x_m)}\int_0^\infty \frac{\lambda^{m+c-k-1} t(\underline{x}) exp(-b\lambda)}{[log(1-e^{-\lambda x_m})^{-1}]^{m}} d\lambda\right]^{-\frac{1}{k}}.$$

Squared Error loss function

This loss function is symmetric in nature as it gives equal weightage to both over and under estimation. In real life, we encounter many situations where over estimation may be more serious than under estimation or vice versa. The squared error loss function is given by

$$L(\hat{\theta}-\theta) \propto (\hat{\theta}-\theta)^2, \tag{29}$$

where $\hat{\theta}$ is the estimate of the parameter θ.

Under the above loss function, the Bayes estimator $\hat{\theta}_{BS}$ of θ is given by

$$\hat{\theta}_{BS} = [E_\theta(\theta|\underline{x})], \tag{30}$$

where E_θ stands for posterior expectation. The Bayes estimators for the parameters α and λ of the GE distribution under squared error loss function is the posterior mean, we have

$$\hat{\alpha}_{BS} = [E_\alpha(\alpha|\underline{x})] = \int_{(\alpha,\lambda)} \alpha \Pi(\alpha,\lambda|\underline{x}) d(\alpha,\lambda), \tag{31}$$

and

$$\hat{\lambda}_{BS} = [E_\lambda(\lambda|\underline{x})] = \int_{(\alpha,\lambda)} \lambda \Pi(\alpha,\lambda|\underline{x}) d(\alpha,\lambda). \tag{32}$$

From equation (15), it can be shown that

$$[E(\alpha|\underline{x})] = \hat{\alpha}_{BS} = m\frac{\phi(b,c-2,x_m)}{\phi(b,c-1,x_m)}$$

and

$$[E(\lambda|\underline{x})] = \hat{\lambda}_{BS} = \frac{\phi(b,c,x_m)}{\phi(b,c-1,x_m)}.$$

Because the Bayes estimators under symmetric and asymmetric loss functions involve the complicated integral function $\phi(b,c-1,x_m)$, we consider using the Lindley's approximation to calculate the approximate Bayes estimators.

Lindley [34] proposed an approximation procedure to evaluate the ratio of two integrals such that

$$I(x) = E[u(\alpha,\lambda|\underline{x})] = \frac{\int_{(\alpha,\lambda)} u(\alpha,\lambda) e^{L(\alpha,\lambda)+G(\alpha,\lambda)} d(\alpha,\lambda)}{\int_{(\alpha,\lambda)} e^{L(\alpha,\lambda)+G(\alpha,\lambda)} d(\alpha,\lambda)}, \tag{33}$$

where
$u(\alpha,\lambda)$ = function of α and λ only;
$L(\alpha,\lambda)$= log of likelihood;
$G(\alpha,\lambda)$ = log of joint prior of α, and λ
can be evaluated as

$$\begin{aligned} I(x) = u(\hat{\alpha},\hat{\lambda}) + \frac{1}{2}[(\hat{u}_{\lambda\lambda}+2\hat{u}_\lambda\hat{p}_\lambda)\hat{\sigma}_{\lambda\lambda} + (\hat{u}_{\alpha\lambda}+2\hat{u}_\alpha\hat{p}_\lambda)\hat{\sigma}_{\alpha\lambda} \\ +(\hat{u}_{\lambda\alpha}+2\hat{u}_\lambda\hat{p}_\alpha)\hat{\sigma}_{\lambda\alpha} + (\hat{u}_{\alpha\alpha}+2\hat{u}_\alpha\hat{p}_\alpha)\hat{\sigma}_{\alpha\alpha}] \\ +\frac{1}{2}[(\hat{u}_\lambda\hat{\sigma}_{\lambda\lambda}+\hat{u}_\alpha\hat{\sigma}_{\lambda\alpha})(\hat{L}_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{\lambda\alpha\lambda}\hat{\sigma}_{\lambda\alpha}+\hat{L}_{\alpha\lambda\lambda}\hat{\sigma}_{\alpha\lambda}+\hat{L}_{\alpha\alpha\lambda}\hat{\sigma}_{\alpha\alpha}) \\ +(\hat{u}_\lambda\hat{\sigma}_{\alpha\lambda}+\hat{u}_\alpha\hat{\sigma}_{\alpha\alpha})(\hat{L}_{\alpha\lambda\lambda}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{\lambda\alpha\alpha}\hat{\sigma}_{\lambda\alpha}+\hat{L}_{\alpha\lambda\alpha}\hat{\sigma}_{\alpha\lambda}+\hat{L}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha})], \end{aligned} \tag{34}$$

where
$\hat{\alpha}= MLE$ of α;
$\hat{\lambda}= MLE$ of λ;
$\hat{u}_\lambda = \frac{\partial u(\hat{\alpha},\hat{\lambda})}{\partial\hat{\lambda}}$; $\hat{u}_\alpha = \frac{\partial u(\hat{\alpha},\hat{\lambda})}{\partial\hat{\alpha}}$; $\hat{u}_{\lambda\alpha} = \frac{\partial^2 u(\hat{\alpha},\hat{\lambda})}{\partial\hat{\lambda}\partial\hat{\alpha}}$; $\hat{u}_{\alpha\lambda} = \frac{\partial^2 u(\hat{\alpha},\hat{\lambda})}{\partial\hat{\alpha}\partial\hat{\lambda}}$;
$\hat{u}_{\lambda\lambda} = \frac{\partial^2 u(\hat{\alpha},\hat{\lambda})}{\partial\hat{\lambda}^2}$; $\hat{u}_{\alpha\alpha} = \frac{\partial^2 u(\hat{\alpha},\hat{\lambda})}{\partial\hat{\alpha}^2}$; $\hat{L}_{\lambda\lambda\alpha} = \frac{\partial^3 L(\hat{\alpha},\hat{\lambda})}{\partial\hat{\lambda}\partial\hat{\lambda}\partial\hat{\alpha}}$; $\hat{L}_{\lambda\lambda\lambda} = \frac{\partial^3 L(\hat{\alpha},\hat{\lambda})}{\partial\hat{\lambda}\partial\hat{\lambda}\partial\hat{\lambda}}$;
$\hat{L}_{\lambda\alpha\lambda} = \frac{\partial^3 L(\hat{\alpha},\hat{\lambda})}{\partial\hat{\lambda}\partial\hat{\alpha}\partial\hat{\lambda}}$; $\hat{L}_{\alpha\alpha\lambda} = \frac{\partial^3 L(\hat{\alpha},\hat{\lambda})}{\partial\hat{\alpha}\partial\hat{\alpha}\partial\hat{\lambda}}$; $\hat{L}_{\alpha\lambda\lambda} = \frac{\partial^3 L(\hat{\alpha},\hat{\lambda})}{\partial\hat{\alpha}\partial\hat{\lambda}\partial\hat{\lambda}}$; $\hat{L}_{\lambda\alpha\alpha} = \frac{\partial^3 L(\hat{\alpha},\hat{\lambda})}{\delta\hat{\lambda}\partial\hat{\alpha}\partial\hat{\alpha}}$;
$\hat{L}_{\alpha\alpha\alpha} = \frac{\partial^3 L(\hat{\alpha},\hat{\lambda})}{\partial\hat{\alpha}\partial\hat{\alpha}\partial\hat{\alpha}}$; $\hat{L}_{\alpha\lambda\alpha} = \frac{\partial^3 L(\hat{\alpha},\hat{\lambda})}{\partial\hat{\alpha}\partial\hat{\lambda}\partial\hat{\alpha}}$; $\hat{p}_\alpha = \frac{\partial G(\hat{\alpha},\hat{\lambda})}{\partial\hat{\alpha}}$; $\hat{p}_\lambda = \frac{\partial G(\hat{\alpha},\hat{\lambda})}{\partial\hat{\lambda}}$;
This approximation procedure has been used by several authors to obtain the approximate Bayes estimators for various distributions, such as [35,36,37,38].

2.2.1 Bayes Estimator of α Under LINEX Loss Function

We see the Bayes estimator of α under the LINEX loss function in equation (19). Now, substituting the value of $\Pi(\alpha,\lambda|\underline{x})$ from (15) in equation (20), we have

$$E_\alpha(e^{-c_1\alpha}|\underline{x}) = \frac{\int_{(\alpha,\lambda)} u(\alpha,\lambda)e^{L(\alpha,\lambda)+G(\alpha,\lambda)}d(\alpha,\lambda)}{\int_{(\alpha,\lambda)} e^{L(\alpha,\lambda)+G(\alpha,\lambda)}d(\alpha,\lambda)}, \tag{35}$$

where
$u(\alpha,\lambda) = e^{-c_1\alpha}$;
$L(\alpha,\lambda) = m(log\alpha + log\lambda) + \alpha log(1-exp(-\lambda x_m)) - \lambda\sum_{i=1}^m x_i - \sum_{i=1}^m log(1-exp(-\lambda x_i))$;
$G(\alpha,\lambda) = clogb + (c-1)log\lambda - b\lambda - log\alpha - logc$.
It can easily be verified that
$\hat{u}_\alpha = -c_1e^{-c_1\alpha}, \quad \hat{u}_{\alpha\alpha} = c_1^2e^{-c_1\alpha}, \quad \hat{u}_\lambda = \hat{u}_{\lambda\alpha} = \hat{u}_{\lambda\lambda} = 0,$
$\hat{p}_\lambda = \frac{(c-1)}{\lambda} - b, \quad \hat{p}_\alpha = -\frac{1}{\alpha}$ and
$\hat{L}_\lambda = \frac{m}{\lambda} - \sum_{i=1}^m x_i + \alpha\frac{x_m exp(-\lambda x_m)}{(1-exp(-\lambda x_m))} - \sum_{i=1}^m \frac{x_i exp(-\lambda x_i)}{(1-exp(-\lambda x_i))},$
$\hat{L}_\alpha = \frac{m}{\alpha} + log(1-exp(-\lambda x_m)), \quad \hat{L}_{\lambda\lambda} = -\frac{m}{\lambda^2} - \frac{\alpha x_m^2 exp(-\lambda x_m)}{(1-exp(-\lambda x_m))^2} + \sum_{i=1}^m \frac{x_i^2 e(-\lambda x_i)}{(1-e(-\lambda x_i))^2},$
$\hat{L}_{\lambda\alpha} = \frac{x_m exp(-\lambda x_m)}{(1-exp(-\lambda x_m))}, \quad \hat{L}_{\alpha\alpha} = -\frac{m}{\alpha^2}, \quad \hat{L}_{\lambda\alpha\alpha} = \hat{L}_{\alpha\lambda\alpha} = \hat{L}_{\alpha\alpha\lambda} = 0, \quad \hat{L}_{\alpha\alpha\alpha} = \frac{2m}{\alpha^3}, \quad \hat{L}_{\lambda\lambda\alpha} = -\frac{x_m^2 exp(-\lambda x_m)}{(1-exp(-\lambda x_m))^2}$
and $\quad \hat{L}_{\lambda\lambda\lambda} = \frac{2m}{\lambda^3} + \frac{\alpha x_m^3(1+e^{-\lambda x_m})exp(-\lambda x_m)}{(1-exp(-\lambda x_m))^3} - \sum_{i=1}^m \frac{x_i^3(1+e^{-\lambda x_i})exp(-\lambda x_i)}{(1-exp(-\lambda x_i))^3}.$
Again, because α and λ are independent, $\hat{\sigma}_{\lambda\alpha} = 0; \quad \hat{\sigma}_{\lambda\lambda} = -\frac{1}{\hat{L}_{\lambda\lambda}} \quad and \quad \hat{\sigma}_{\alpha\alpha} = -\frac{1}{\hat{L}_{\alpha\alpha}}.$
Evaluating u-terms, L-terms and p-terms mentioned above at point $(\hat{\alpha},\hat{\lambda})$ and using (34), we get

$$E_\alpha(e^{-c_1\alpha}|\underline{x}) = e^{-c_1\alpha} + \frac{1}{2}\frac{\alpha^2}{m}c_1e^{-c_1\alpha}$$

$$\left[\left(c_1 + \frac{2}{\alpha}\right) - \left(\frac{2}{\alpha} - \frac{\frac{x_m^2 exp(-\lambda x_m)}{(1-exp(-\lambda x_m))^2}}{\frac{m}{\lambda^2} + \frac{\alpha x_m^2 exp(-\lambda x_m)}{(1-exp(-\lambda x_m))^2} - \sum_{i=1}^m \frac{x_i^2 e(-\lambda x_i)}{(1-e(-\lambda x_i))^2}}\right)\right]$$

Thus, the Bayes estimator of α under the LINEX loss function is

$$\hat{\alpha}_{BL} = -\frac{1}{c_1}ln[e^{-c_1\alpha} + \frac{1}{2}\frac{\alpha^2}{m}c_1e^{-c_1\alpha}$$

$$((c_1 + \frac{2}{\alpha}) - (\frac{2}{\alpha} - \frac{\frac{x_m^2 exp(-\lambda x_m)}{(1-exp(-\lambda x_m))^2}}{\frac{m}{\lambda^2} + \frac{\alpha x_m^2 exp(-\lambda x_m)}{(1-exp(-\lambda x_m))^2} - \sum_{i=1}^m \frac{x_i^2 e(-\lambda x_i)}{(1-e(-\lambda x_i))^2}}))].$$

2.2.2 Bayes Estimator of λ Under LINEX Loss Function

We see the Bayes estimator of λ under the LINEX loss function in equation (21). Now, substituting the value of $\Pi(\alpha,\lambda|\underline{x})$ from (15) in equation (22), we have

$$E_\lambda(e^{-c_1\lambda}|\underline{x}) = \frac{\int_{(\alpha,\lambda)} u(\alpha,\lambda)e^{L(\alpha,\lambda)+G(\alpha,\lambda)}d(\alpha,\lambda)}{\int_{(\alpha,\lambda)} e^{L(\alpha,\lambda)+G(\alpha,\lambda)}d(\alpha,\lambda)}, \tag{36}$$

where
$u(\alpha,\lambda) = e^{-c_1\lambda}$;
$L(\alpha,\lambda)$ and $G(\alpha,\lambda)$ are the same as those given in (35).
It can easily be verified that
$\hat{u}_\lambda = -c_1e^{-c_1\lambda}, \quad \hat{u}_{\lambda\lambda} = c_1^2e^{-c_1\lambda}, \quad \hat{u}_\alpha = \hat{u}_{\lambda\alpha} = \hat{u}_{\alpha\alpha} = 0.$

Following the procedure as discussed in (2.2.1), we get after simplification,

$$E_\lambda(e^{-c_1\lambda}|\underline{x}) = e^{-c_1\lambda} + \frac{1}{2}[(\frac{c_1^2 e^{-c_1\lambda} - 2c_1(\frac{(c-1)}{\lambda} - b)exp(-c_1\lambda)}{\frac{m}{\lambda^2} + \frac{\alpha x_m^2 exp(-\lambda x_m)}{(1-exp(-\lambda x_m))^2} - \sum_{i=1}^m \frac{x_i^2 e(-\lambda x_i)}{(1-e(-\lambda x_i))^2}})$$

$$-(\frac{c_1 e^{-c_1\lambda}(\frac{2m}{\lambda^3} + \frac{\alpha x_m^3(1+e^{-\lambda x_m})exp(-\lambda x_m)}{(1-exp(-\lambda x_m))^3} - \sum_{i=1}^m \frac{x_i^3(1+e^{-\lambda x_i})exp(-\lambda x_i)}{(1-exp(-\lambda x_i))^3})}{(\frac{m}{\lambda^2} + \frac{\alpha x_m^2 exp(-\lambda x_m)}{(1-exp(-\lambda x_m))^2} - \sum_{i=1}^m \frac{x_i^2 e(-\lambda x_i)}{(1-e(-\lambda x_i))^2})^2})].$$

Thus, the Bayes estimator of λ under the LINEX loss function is

$$\hat{\lambda}_{BL} = -\frac{1}{c_1} ln[e^{-c_1\lambda} + \frac{1}{2}[(\frac{c_1^2 e^{-c_1\lambda} - 2c_1(\frac{(c-1)}{\lambda} - b)exp(-c_1\lambda)}{\frac{m}{\lambda^2} + \frac{\alpha x_m^2 exp(-\lambda x_m)}{(1-exp(-\lambda x_m))^2} - \sum_{i=1}^m \frac{x_i^2 e(-\lambda x_i)}{(1-e(-\lambda x_i))^2}})$$

$$-(\frac{c_1 e^{-c_1\lambda}(\frac{2m}{\lambda^3} + \frac{\alpha x_m^3(1+e^{-\lambda x_m})exp(-\lambda x_m)}{(1-exp(-\lambda x_m))^3} - \sum_{i=1}^m \frac{x_i^3(1+e^{-\lambda x_i})exp(-\lambda x_i)}{(1-exp(-\lambda x_i))^3})}{(\frac{m}{\lambda^2} + \frac{\alpha x_m^2 exp(-\lambda x_m)}{(1-exp(-\lambda x_m))^2} - \sum_{i=1}^m \frac{x_i^2 e(-\lambda x_i)}{(1-e(-\lambda x_i))^2})^2})]].$$

2.2.3 Bayes Estimator of α Under General Entropy Loss Function

Applying the same Lindley approach here as in (34) with
$u(\hat{\alpha}, \hat{\lambda}) = \alpha^{-k}$;
$\hat{u}_\alpha = -k\alpha^{-k-1}$, $\hat{u}_{\alpha\alpha} = k(k+1)\alpha^{-k-2}$ and $\hat{u}_\lambda = \hat{u}_{\lambda\alpha} = \hat{u}_{\lambda\lambda} = 0$.
$L(\alpha, \lambda)$ and $G(\alpha, \lambda)$ are the same as those given in (35).
Following the procedure as discussed in (2.2.1), we get after simplification,

$$E_\alpha(\alpha^{-k}|\underline{x}) = \alpha^{-k} + \frac{k(k+3)\alpha^{-k}}{2m} + \frac{1}{2}\frac{k}{m}\alpha^{-k+1}$$

$$\left[\frac{\frac{x_m^2 exp(-\lambda x_m)}{(1-exp(-\lambda x_m))^2}}{\frac{m}{\lambda^2} + \frac{\alpha x_m^2 exp(-\lambda x_m)}{(1-exp(-\lambda x_m))^2} - \sum_{i=1}^m \frac{x_i^2 e(-\lambda x_i)}{(1-e(-\lambda x_i))^2}} - \frac{2}{\alpha}\right].$$

Thus, the Bayes estimator of α under the general entropy loss function is

$$\hat{\alpha}_{BG} = [\alpha^{-k} + \frac{k(k+3)\alpha^{-k}}{2m} + \frac{1}{2}\frac{k}{m}\alpha^{-k+1}$$

$$[\frac{\frac{x_m^2 exp(-\lambda x_m)}{(1-exp(-\lambda x_m))^2}}{\frac{m}{\lambda^2} + \frac{\alpha x_m^2 exp(-\lambda x_m)}{(1-exp(-\lambda x_m))^2} - \sum_{i=1}^m \frac{x_i^2 e(-\lambda x_i)}{(1-e(-\lambda x_i))^2}} - \frac{2}{\alpha}]]^{-\frac{1}{k}}.$$

2.2.4 Bayes Estimator of λ Under General Entropy Loss Function

Applying the same Lindley approach here as in (34) with
$u(\hat{\alpha}, \hat{\lambda}) = \lambda^{-k}$;
$\hat{u}_\lambda = -k\lambda^{-k-1}$, $\hat{u}_{\lambda\lambda} = k(k+1)\lambda^{-k-2}$ and $\hat{u}_\alpha = \hat{u}_{\lambda\alpha} = \hat{u}_{\alpha\alpha} = 0$.
$L(\alpha, \lambda)$ and $G(\alpha, \lambda)$ are the same as those given in (35).

Following the procedure as discussed in (2.2.1), we get after simplification,

$$E_\lambda(\lambda^{-k}|\underline{x}) = \lambda^{-k} + \frac{1}{2}[\frac{k(k+1)\lambda^{-k-2} - 2k\lambda^{-k-1}(\frac{(c-1)}{\lambda} - b)}{\frac{m}{\lambda^2} + \frac{\alpha x_m^2 exp(-\lambda x_m)}{(1-exp(-\lambda x_m))^2} - \sum_{i=1}^m \frac{x_i^2 e(-\lambda x_i)}{(1-e(-\lambda x_i))^2}}]$$

$$+\frac{1}{2}[\frac{-k\lambda^{-k-1}(\frac{2m}{\lambda^3} + \frac{\alpha x_m^3(1+e^{-\lambda x_m})exp(-\lambda x_m)}{(1-exp(-\lambda x_m))^3} - \sum_{i=1}^m \frac{x_i^3(1+e^{-\lambda x_i})exp(-\lambda x_i)}{(1-exp(-\lambda x_i))^3})}{\frac{m}{\lambda^2} + \frac{\alpha x_m^2 exp(-\lambda x_m)}{(1-exp(-\lambda x_m))^2} - \sum_{i=1}^m \frac{x_i^2 e(-\lambda x_i)}{(1-e(-\lambda x_i))^2}}].$$

Thus, the Bayes estimator of λ under the general entropy loss function is

$$\hat{\lambda}_{BG} = [\lambda^{-k} + \frac{1}{2}[\frac{k(k+1)\lambda^{-k-2} - 2k\lambda^{-k-1}(\frac{(c-1)}{\lambda} - b)}{\frac{m}{\lambda^2} + \frac{\alpha x_m^2 exp(-\lambda x_m)}{(1-exp(-\lambda x_m))^2} - \sum_{i=1}^m \frac{x_i^2 e(-\lambda x_i)}{(1-e(-\lambda x_i))^2}}]$$

$$+\frac{1}{2}[\frac{-k\lambda^{-k-1}(\frac{2m}{\lambda^3} + \frac{\alpha x_m^3(1+e^{-\lambda x_m})exp(-\lambda x_m)}{(1-exp(-\lambda x_m))^3} - \sum_{i=1}^m \frac{x_i^3(1+e^{-\lambda x_i})exp(-\lambda x_i)}{(1-exp(-\lambda x_i))^3})}{\left\{\frac{m}{\lambda^2} + \frac{\alpha x_m^2 exp(-\lambda x_m)}{(1-exp(-\lambda x_m))^2} - \sum_{i=1}^m \frac{x_i^2 e(-\lambda x_i)}{(1-e(-\lambda x_i))^2}\right\}^2}]]^{-\frac{1}{k}}.$$

2.2.5 Bayes Estimator of α Under Squared Error Loss Function

Applying the same Lindley approach here as in (34) with
$u(\hat{\alpha},\hat{\lambda}) = \alpha$;
$\hat{u}_\alpha = 1$ and $\hat{u}_{\alpha\alpha} = \hat{u}_\lambda = \hat{u}_{\lambda\alpha} = \hat{u}_{\lambda\lambda} = 0$.
$L(\alpha,\lambda)$ and $G(\alpha,\lambda)$ are the same as those given in (35).
Following the procedure as discussed in (2.2.1), we get after simplification,

$$E_\alpha(\alpha|\underline{x}) = \hat{\alpha}_{BS} = \alpha[1 - \frac{1}{m} + \frac{\alpha}{2m}$$

$$[-\frac{\frac{x_m^2 exp(-\lambda x_m)}{(1-exp(-\lambda x_m))^2}}{\frac{m}{\lambda^2} + \frac{\alpha x_m^2 exp(-\lambda x_m)}{(1-exp(-\lambda x_m))^2} - \sum_{i=1}^m \frac{x_i^2 e(-\lambda x_i)}{(1-e(-\lambda x_i))^2}} + \frac{2}{\alpha}]].$$

2.2.6 Bayes Estimator of λ Under Squared Error Loss Function

Applying the same Lindley approach here as in (34) with
$u(\hat{\alpha},\hat{\lambda}) = \lambda$;
$\hat{u}_\lambda = 1$ and $\hat{u}_{\alpha\alpha} = \hat{u}_\alpha = \hat{u}_{\lambda\alpha} = \hat{u}_{\lambda\lambda} = 0$.
$L(\alpha,\lambda)$ and $G(\alpha,\lambda)$ are the same as those given in (35).
Following the procedure as discussed in (2.2.1), we get after simplification,

$$E_\lambda(\lambda|\underline{x}) = \hat{\lambda}_{BS} = \lambda + \frac{\frac{(c-1)}{\lambda} - b}{\frac{m}{\lambda^2} + \frac{\alpha x_m^2 exp(-\lambda x_m)}{(1-exp(-\lambda x_m))^2} - \sum_{i=1}^m \frac{x_i^2 e(-\lambda x_i)}{(1-e(-\lambda x_i))^2}}$$

$$+\frac{1}{2}\left[\frac{\frac{2m}{\lambda^3} + \frac{\alpha x_m^3(1+e^{-\lambda x_m})exp(-\lambda x_m)}{(1-exp(-\lambda x_m))^3} - \sum_{i=1}^m \frac{x_i^3(1+e^{-\lambda x_i})exp(-\lambda x_i)}{(1-exp(-\lambda x_i))^3}}{\left\{\frac{m}{\lambda^2} + \frac{\alpha x_m^2 exp(-\lambda x_m)}{(1-exp(-\lambda x_m))^2} - \sum_{i=1}^m \frac{x_i^2 e(-\lambda x_i)}{(1-e(-\lambda x_i))^2}\right\}^2}\right].$$

3 Bayesian Prediction of Future Records

Assume that we have m lower records $X_{L(1)} = x_1, X_{L(2)} = x_2, ..., X_{L(m)} = x_m$, from the $GE(\alpha,\lambda)$. Based on such sample, prediction, either point or interval is needed for the s-th lower records, $1 < m < s$. Now let $Y = X_{L(s)}$ be the s-th lower record value, $1 < m < s$. The conditional pdf of Y for the given parameters α and λ and that the m-th record had been observed is [4]

$$f(y|x_m,\theta) = \frac{[H(y)-H(x_m)]^{s-m-1}}{\Gamma(s-m)}\frac{f(y)}{F(x_m)}, \quad 0 \le y < x_m < \infty, \tag{37}$$

where $f(.)$ and $F(.)$ are, respectively, the *pdf* and the *cdf*, $H(.) = -lnF(.)$, and "ln" is used for the natural logarithm. Combining the posterior density given by (15), and the conditional density given by (37), and integrating out the parameters α and λ, one may get the Bayesian predictive density function of $Y = X_{L(s)}$ for the given past m records, in the form

$$q(y|\underline{x}) = \int_0^\infty \int_0^\infty f(y|x_m,\alpha,\lambda)\Pi(\alpha,\lambda|\underline{x})d\alpha d\lambda$$

$$= \frac{\phi(x_m,y)}{B(s-m,m)\phi(b,c-1,x_m)}, \quad 0 < y < x_m < \infty, \tag{38}$$

where $B(.,.)$ is the complete beta function and

$$\phi(x_m,y) = \int_0^\infty \frac{\lambda^{m+c}e^{-\lambda(y+b)}t(\underline{x})}{(1-e^{-\lambda y})[log(1-e^{-\lambda y})^{-1}]^s}\left[ln(\frac{1-e^{-\lambda x_m}}{1-e^{-\lambda y}})\right]^{s-m-1} d\lambda.$$

The Bayesian predictive bounds of $Y = X_{L(s)}$ are obtained by evaluating $P(Y > z|\underline{x})$ for some given value of z. It follows from (38) that

$$P(Y_s > z|\underline{x}) = \int_z^\infty \phi(x_m,y)dy / \int_{x_m}^\infty \phi(x_m,y)dy. \tag{39}$$

Two-sided $(1-\gamma)100\%$ prediction interval for $Y = X_{L(s)}$ is obtained by evaluating both the lower, $L(\underline{x})$, and the upper, $U(\underline{x})$, limits which satisfy

$$P(Y_s > L(\underline{x})|\underline{x}) = 1 - \tfrac{\gamma}{2} \text{ and } P(Y_s > U(\underline{x})|\underline{x}) = \tfrac{\gamma}{2}.$$

Thus, one may obtain $L(\underline{x})$ and $U(\underline{x})$ by equating (39) to $1-\frac{\gamma}{2}$ and $\frac{\gamma}{2}$, respectively, and solving, numerically, the resulting equations.
For the special case, when $s = m+1$, it can be shown that

$$P(Y_{m+1} > z|\underline{x}) = \frac{\phi(b,c-1,z)}{\phi(b,c-1,x_m)}. \tag{40}$$

4 Numerical Computations

In order to illustrate the usefulness of the inference procedures discussed in the previous sections, we generate five sets of record values of sizes 10, 20, 30, 40 and 50 from the generalized exponential distribution with parameter α and λ. The Bayes estimators of α and λ under symmetric and asymmetric loss functions are shown in Table 1 and Table 2. The MLE's are also discussed in the same table.
Moreover, in order to illustrate the prediction procedures obtained in section 3, using (40), the lower and upper 95% predictive bounds for Y_{m+1}(next upper record in the X_n sequence) at prior parameters $b = 2$ and $c = 2$ are shown in Table 3, respectively.

Table 1. MLE's and Bayes estimates at prior parameters $b = 2$ and $c = 2$ based on generated record values, when the population parameters are $\alpha = 3$

n	m	$\hat{\alpha}_{ML}$	$\hat{\alpha}_{BS}$	$c_1 = k$	$\hat{\alpha}_{BL}$	$\hat{\alpha}_{BG}$
10	7	3.242715	2.999803	-0.5	3.297902	2.893620
				-0.4	3.244582	2.872885
				-0.3	3.187298	2.852418
				-0.2	3.126754	2.832257
				-0.1	3.063884	2.812442
				0.1	2.935724	2.773979
				0.2	2.872862	2.755393
				0.3	2.812330	2.737270
				0.4	2.755061	2.719634
				0.5	2.701759	2.702503
20	7	2.997946	2.999997	-0.5	3.296272	2.893811
				-0.4	3.242868	2.873074
				-0.3	3.185511	2.852605
				-0.2	3.124911	2.832442
				-0.1	3.062006	2.812625
				0.1	2.933847	2.774157
				0.2	2.871020	2.755568
				0.3	2.810544	2.737442
				0.4	2.753348	2.719803
				0.5	2.700132	2.702669
30	7	3.210332	2.997906	-0.5	3.298069	2.891758
				-0.4	3.244758	2.871037
				-0.3	3.187482	2.850587
				-0.2	3.126943	2.830446
				-0.1	3.064077	2.810652
				0.1	2.935917	2.772240
				0.2	2.873051	2.753680
				0.3	2.812513	2.735587
				0.4	2.755237	2.717982
				0.5	2.701926	2.700882
40	8	2.873693	2.999992	-0.5	3.263146	2.906975
				-0.4	3.215437	2.888755
				-0.3	3.164610	2.870740
				-0.2	3.111245	2.852963
				-0.1	3.056085	2.835451
				0.1	2.943900	2.801332
				0.2	2.888739	2.784774
				0.3	2.835375	2.768579
				0.4	2.784549	2.752766
				0.5	2.736840	2.737351

50	7	2.992030	2.999963	-0.5	3.298039	2.893778
				-0.4	3.244727	2.873041
				-0.3	3.187450	2.852572
				-0.2	3.126910	2.832410
				-0.1	3.064043	2.812593
				0.1	2.935883	2.774126
				0.2	2.873018	2.755537
				0.3	2.812481	2.737412
				0.4	2.755206	2.719774
				0.5	2.701897	2.702640

ML: Maximum Likelihood, BS: Squared Error Loss Function,
BL: LINEX Loss Function, BG: General Entropy Loss Function

Table 2. MLE's and Bayes estimates at prior parameters $b = 2$ and $c = 2$ based on generated record values, when the population parameters are $\lambda = 1$

n	m	$\hat{\lambda}_{ML}$	$\hat{\lambda}_{BS}$	$c_1 = k$	$\hat{\lambda}_{BL}$	$\hat{\lambda}_{BG}$
10	7	0.998754	0.998114	-0.5	1.079375	0.917052
				-0.4	1.063584	0.901744
				-0.3	1.047489	0.886901
				-0.2	1.031160	0.872582
				-0.1	1.014675	0.858832
				0.1	0.981559	0.833187
				0.2	0.965093	0.821339
				0.3	0.948795	0.810159
				0.4	0.932741	0.799648
				0.5	0.917003	0.789802
20	7	0.978359	0.998099	-0.5	1.079357	0.917041
				-0.4	1.063567	0.901733
				-0.3	1.047471	0.886892
				-0.2	1.031143	0.872573
				-0.1	1.014659	0.858825
				0.1	0.981545	0.833181
				0.2	0.965079	0.821334
				0.3	0.948782	0.810154
				0.4	0.932729	0.799644
				0.5	0.916991	0.789799
30	7	0.999989	0.998029	-0.5	1.079271	0.916992
				-0.4	1.063484	0.901689
				-0.3	1.047391	0.886852
				-0.2	1.031066	0.872538
				-0.1	1.014585	0.858794
				0.1	0.981479	0.833159
				0.2	0.965018	0.821315
				0.3	0.948724	0.810139
				0.4	0.932676	0.799632
				0.5	0.916943	0.789789

40	8	0.999947	0.999547	-0.5	1.080678	0.917826
				-0.4	1.064853	0.902452
				-0.3	1.048717	0.887544
				-0.2	1.032344	0.873159
				-0.1	1.015810	0.859346
				0.1	0.982585	0.833579
				0.2	0.966059	0.821675
				0.3	0.949699	0.810440
				0.4	0.933581	0.799878
				0.5	0.917778	0.789984
50	7	0.999788	0.999143	-0.5	1.080614	0.917788
				-0.4	1.064790	0.902417
				-0.3	1.048657	0.887512
				-0.2	1.032285	0.873130
				-0.1	1.015754	0.859320
				0.1	0.982534	0.833559
				0.2	0.966011	0.821658
				0.3	0.949654	0.810426
				0.4	0.933539	0.799866
				0.5	0.917739	0.789975

ML: Maximum Likelihood, BS: Squared Error Loss Function,
BL: LINEX Loss Function, BG: General Entropy Loss Function

Table 3. Lower and Upper 95% prediction bounds for Y_{m+1}

Number of Records m	Interval prediction for the next record Y_{m+1}		*Length*
	$L(\underline{x})$	$U(\underline{x})$	
3	0.02114997	0.00000117	.02114880
4	0.00926856	0.00000619	.00926237
5	0.04599991	0.00134991	.04465000
6	0.04459921	0.04456998	.00002923
7	.030254000	0.00118600	0.0290680

5 Conclusion

In this paper, we have studied the Bayesian estimator under symmetric and asymmetric loss functions, and maximum likelihood estimation as well as prediction for the two parameter generalized exponential distribution based on generated record values. Bayes estimators were obtained using Lindley approximation, while MLE's were obtained using Newton-Raphson method.

It is noted that as sample size increases, the Bayes estimators of α and λ under symmetric and asymmetric loss function increase and decrease irregularly. Also, the MLE's decrease and increase irregularly. Bayes estimators of α and λ under squared error loss function are more or less than MLE's. If the magnitude of c_1 and k, i.e., the LINEX and General Entropy loss parameter increases, then the estimator of α and λ decreases.

For negative values of c_1 and k:

(1) The Bayes estimator of λ under LINEX loss function is greater than MLE's. However, the Bayes estimator of α under LINEX loss function is more or less than MLE's.
(2) The Bayes estimator of λ under General Entropy loss function is less than MLE's. However, the Bayes estimator of α under General Entropy loss function is more or less than MLE's.
For positive values of c_1 and k:
(1) The Bayes estimator of α and λ under LINEX loss function is more or less than MLE's.
(2) The Bayes estimator of α and λ under General Entropy loss function is less than MLE's.

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Conflict of Interest

The authors declare that they have no conflict of interest.

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