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Tutte polynomial for a small world connected copies of Farey graphs

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Abstract: The Tutte polynomial has an essential rule in several applications such as networks and many areas of science, for example, combinatorics, biology, and statistical mechanics. In this paper, we investigate a two-variable polynomial graph invariant of the Tutte polynomial of a graph. Using the two new models, an established form of the Farey graph is given, and a modification of the basic theory for these two new models is introduced. The Tutte polynomial is used to determine the number of spanning trees, as well as the number of connected spanning subgraphs. A deduction of the exact expressions for the chromatic polynomial and the reliability polynomial of graphs are presented. Moreover, we apply our models to establish the form of the Koch curve.

Keywords: Tutte polynomial, Farey graphs, Koch curve

1 Introduction

In 1952, Potts in [1], introduced the initiated model for partition function for analyzing the properties of graphs and networks. It had the inquiry into graph polynomials and their applications with the Tutte polynomial. The model was developed by Potts [2]. Y. Wang (2019) investigated a few enumeration and combinatorial issues related to the self-similar structure of the Farey graphs. Also determined the independence, matching, and dominance numbers. C. Beke (2023) concentrates on Brylawski's identities concerning the Tutte polynomial. F. Bencs (2022) studied the asymptotic behavior of the Tutte polynomial of large girth regular graphs. M. Kocho (2019) characterized the Tutte polynomial by means of Ehrhart polynomials, thus we can apply a reciprocity law and obtain analogous interpretations for positive values of the Tutte polynomial. M. Kochol (2020) discuss modifications of duality and convolution formulas known for the Tutte polynomial [3,4,5,6]. Aboutahoun and El-Safty [7] investigated a self-similar network model and derived the Tutte polynomial. In addition, they evaluated exact explicit formulas for the number of acyclic orientations and spanning trees of it as applications of this Tutte polynomial. Gong and Jin [8] introduced a family of recursively constructed self-similar graphs whose inner duals are of the self-similar property. By combining the dual property of the Tutte polynomial and the subgraph-decomposition trick, they showed that the Tutte polynomial of this family of graphs can be computed in an iterative way and in particular the exact expression of the formula of the number of their spanning trees is derived. Liao et al. [9] studied the Tutte polynomials of the diamond hierarchical lattices and a class of self-similar fractal models that can be constructed through graph operations. Kahl [10] determined more generally compression's effect on the Tutte polynomial, recovering the results of the graph transformation called the compression of a graph which is known to decrease the number of spanning trees, the all-terminal reliability, and the magnitude of the coefficients of the chromatic polynomial of a graph and obtaining similar results for a wide variety of other graph parameters derived from the Tutte polynomial. Moreover, The two-variable Tutte polynomial [11,12,13] is crucial in a number of scientific fields, including combinatorics, statistical mechanics, and biology. In a broad sense, it includes any graphical invariants that can be computed using deletion and

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contraction operations, which are standard contractions for models of many networks. The number of spanning trees, the number of connected spanning subgraphs, and the dimension of the bicycle space can all be determined by evaluating the Tutte polynomial $T(G;x_1,y)$ at specific positions (x_1,y_1).
Two-variable graph polynomial which has several areas of applications in sciences such as combinatorics [14], statistical mechanics [15], engineering [16], optimization [17], physics [16] and biology [18]. For lattice strips of varied fixed widths and arbitrarily long lengths, with various boundary conditions, an exact computation of flow polynomials F(G,q) is made. Triangle, square, and honeycomb lattice strips are all taken into consideration in [19]. the chromatic polynomials that Given in [20] are the zero-temperature antiferromagnetic Potts-model partition functions $PG(q)$ for $m\times n$ rectangular subsets of the square lattice, with $m\leq 8$ (free or periodic transverse boundary conditions) and arbitrary n (longitudinal boundary terms free), using a transfer matrix In the Fortuin-Kasteleyn representation. The Farey graph contains some interesting graphical properties of the variable graph that have been studied over the past years, which can be summarized as follows: It is a small world with average distances that increase logarithmically with its vertex number, whose cluster coefficient converges to a large constant ln 2, It is 3-colors with a minimum, unique Hamiltonian, outerplanar and perfect, see [21,22,23,24]. Matula and Kornerup 1979 introduced the Farey graph and Colbourn studied it in 1982. The Farey graph has been used as a deterministic network model by Zhang and Comellas [21,25]. The importance of these networks is due to the fact that they display some properties of networks in the real world. Here, there are some interesting graphical properties of the variable chart that have been studied over the past years. In this paper, we study a two-variable polynomial graph invariant of considerable called the Tutte polynomial of a graph, which is essential in statistical physics and combinatorics. In section 1, it gives a historical overview of the research topic. In section 2, the construction for H_n and D_n graphs and their properties (Algorithm, Deletion and Contraction) are introduced. In section 3, the Tutte polynomial of a class of self-similar fractal models $N(t)$ and its number of spanning trees are proposed. In section 4, a few spanning trees of Hn is given. In section 5, the chromatic polynomial $P(H;\gamma)$ is present. In section 6, the reliability polynomial$R(H;p)$ is given. In section 7, we study the Tutte polynomial of a class of self-similar fractal models $N(t)$ and its number of spanning trees. In section 8, we deduce an analytics and dissection for our results in this research, finally, section 9 contains the conclusions.

2 Construction for H_n and D_n graphs and their properties

We generate two new self-similar copies created from Farey Graph, we donate these two graphs as H_n and D_n where $n\in\{1,2,3,\cdots\}$. The first one H_n can be donated in the following iterative way: For $n=0$, H_0 has two vertices and two parallel edges joining them. For $n\geq 1$, H_n is obtained from H_{n-1} by adding to every outer face edge introduced at step $n-1$ a new two edges connected with a new vertex that forms a path of length two. see figure 2. The second one D_n

2.1 Algorithm

- Step 1. At $n=0$, H_0 will have two vertices and two parallel edges joining them, as in H_0. It will be possible to put $H_0=\{e_1,e_2\},V_0=\{v_0,v_0\}$.
- Step 2. begin with H_0, and from e_1, its endpoint can be connected with vertex v_3 edges e_{11} and e_{12}.Similarly, e_2 can be joined to vertex e_4 and edges e_{21} and e_{22}. In this case, we get $H_1=\{e_1,e_2\}\bigcup\{e_{11},e_{12},e_{21},e_{22}\},V_1=\{v_1,v_2,v_3,v_4\}$.
- Step 3. From H_1, we get $H_1=\{e_1,e_2\}\bigcup\{e_{11},e_{12},e_{21},e_{22}\}\bigcup\{e_{111},e_{112},e_{121},e_{122},e_{21},e_{21},e_{22}\},V_1=\{v_1,v_2,v_3,v_4,v_5,v_6,v_7,v_8\}$. see Figer
- Step 4. Counting for step 3. Generally, for each $n\in N$, we have H_n.

Step 1. At $n=0$, H_0 will be two vertices and two parallel edges joining them as in H_0. It will be possible to put $H_0=\{e_1,e_2\},V_0=\{v_0,v_0\}$.

Definition 1.*Deletion and Contraction Edge deletion and contractions are essential for studying the Tutte polynomial. The graph by deleting an edge $e\in E$ is just $G-e$. The graph obtained by contracting an edge e in G results from selecting endpoints of e followed by removing e. see figure 1 for more explanation.*

2.2 Deletion and Contraction

Edge deletion and contractions are essential for studying the Tutte polynomial. The graph was obtained by deleting the edge of e. which is denoted by the symbol $G-e$, In different words, deletion means deleting an edge e from the set of edges $E(e)$. e.i.$G-e=G(V(G),E(G)-e)$. The graph obtained by contracting an edge e in G results from deleting the edge e and merging the two endpoints of e to one vertex. and is denoted G/e. See figure. 1.
The Tutte polynomial of a finite graph by contraction and deletion by the following terminologies:
The following recursive definition of the function $T(G;x_1,y_1)$ of a graph G and two independent variables

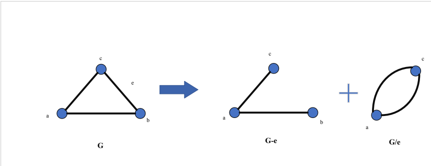

Fig. 1: Diagrams of $G-e$ and G/e.

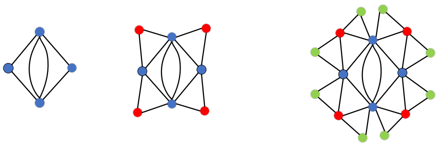

Fig. 2: The graph H at $n = 1,2,3$.

x_1, y_1 is given by the recursive definition:
$T(G;x_1,y_1) =$

$$\begin{cases} 1 & : \textit{ if } G = \phi \\ x_1 T(G-e;x_1,y_1) & : \textit{ if e is a bridge} \\ y_1 T(G-e;x_1,y_1) & : \textit{ if e is a loop} \\ T(G-e;x_1,y_1) + T(G/e;x_1,y_1) & : \textit{ otherwise} \end{cases} \quad (1)$$

where e is an edge of G. This concept is not applicable to large networks; therefore, there are several different generating function formulations, each of which has its advantages. Then the Tutte polynomial of G has the following expansion: Tutte polynomial. Let G be a graph consisting of two sets of edges $E(G)$ and a set of vertices $V(G)$. In the following, we will introduce some important symbols and concepts to clarify the Tutte polynomial. The number of connected components of the graph G donate as $k(G)$. Let $A = (V(A), E(A))$ is a spanning subgraph. The rank of a graph G is given as $r(G) = |V| - K(G)$ and the nullity of a graph G s is given as $n(G) = |E| - |V| + K(G)$. We introduce the definition for the Tutte polynomial of a graph in terms of spanning subgraphs:

$$T(G;x_1,y_1) = \sum_{A \subseteq E} (x_1 - 1)^{r(G)-r(A)} (y_1 - 1)^{|A|-r(A)} \quad (2)$$

Graph Theory Terms
$|V|$ is the number of Vertices of graph G.
$|E|$ is the number of Edges og graph G.
$k(G)$ is the number of components of graph G.
The rank of a graph G is $r(G) = |V| - K(G)$.
The nullity of graph G is $n(G) = |E| - |V| + K(G)$.
The rank-generating function of the Tutte polynomial is defined as

$$T(G;x_1,y_1) = \sum_{A \subseteq E} (x_1 - 1)^{r(G)-r(A)} (y_1 - 1)^{|A|-r(A)} \quad (3)$$

One of the characteristics of the graph that we will use here is when two graphs are joined with one vertex, in which the Tutte polynomial satisfies the following property: $T(A*B;x_1,y_1) = T(A;x_1,y_1)T(B;x_1,y_1)$. such that $A*B$ is a graph formed by connecting two graphs A and B by a single vertex v.

In this paper, assigning values to the points of the variables (x_1,y_1)in the Tutte polynomial provides special evaluations. This will enable us to conclude many combinatorial and algebraic properties of the studied graphs. Remember some graphic terms: A spanning forest of a graph G is a subgraph of G containing the set of all vertices $|E(G)|$ which is not connected which is a forest. A Spanning subgraph is a subgraph of G containing the set of all vertices $|E(G)|$ which is connected. The special evaluations of interest are the number of spanning trees of G, given as $T(G;1,1) = NST(G)$; the number of spanning forests of G is given as $T(G;2,1) = NSF(G)$ and the number of connected spanning subgraphs of G is given as $T(G;1,2) = NCSSG(G)$.

The reliability polynomial also is one graphical polynomial as partial evaluations, the reliability polynomial is defined as the probability of a path of active edges in this random pattern between each pair of G vertices (in other words there is a path between them consisting of a sequence of connected edges of G), Let G be a connected graph with rank $r(G)$ and nullity $n(G)$ and assume that each edge is independently chosen to be active with probability p where $0 \le p \le 1$. Then the reliability polynomial by the Tutte polynomial is given as

$$R(G;p) = p^{r(G)}(1-p)^{n(G)} T(G;1,(1-p)) \quad (4)$$

One of the essential tools used to Count the number of ways to color the constrained G vertices that do not have adjacent pairs of vertices of the same color is called the chromatic polynomial $P(G;\gamma)$, the zero-temperature limit of the anti-ferromagnetic Potts model is one of the most famous complex roots uses of the chromatic polynomial. [26,27,19,20]. then the chromatic polynomial by the Tutte polynomial is given as

$$P(G;\gamma) = (-1)^{r(G)} \gamma^{k(G)} T(G;1-\gamma,0) \quad (5)$$

Corollary 1.*For each $n \ge 1$, the Tutte polynomial $T_n(G_n;x_1,y_1)$ of a small-world Farey graph G is given as follows:*

$$\begin{aligned} T_n(G;x_1,y_1) &= y_1 T_{n-1}^2(G_{n-1};x_1,y_1) \\ &+ 2T_{n-1}(G_{n-1};x_1,y_1)N_{n-1}(G_{n-1};x_1,y_1) \\ &+ (x_1-1)N_{n-1}^2(G_{n-1};x_1,y_1) \\ &+ (x_1-1)^2 N_{n-1}^2(G_{n-1};x_1,y_1) \\ &+ 2(x_1-1)T_{n-1}(G_{n-1};x_1,y_1)N_{n-1}(G_{n-1};x_1,y_1). \end{aligned}$$

with initial conditions $T_{n=0}(G_0;x_1,y_1) = 1$, $N_{n=0}(G;x_1,y_1) = 1$

Corollary 2. *The exact number of spanning trees $N_{ST}(G_n)$ of the graph G is given as follows:*

$$N_{ST}(G_n) = T_n(G_n;1,1) = (2^{n+1}-1)\prod_{i=2}^{n}(2^i-1)^{2^{n-i}} \quad (6)$$

The order and size of the Farey graph

$$(V_n, E_n)$$

are readily demonstrated by induction to be, respectively,

$$V_n(H) = 2^{n+1}, E_n(H) = 2^{n+2}-2 \quad (7)$$

It is not difficult to prove by induction that the order and

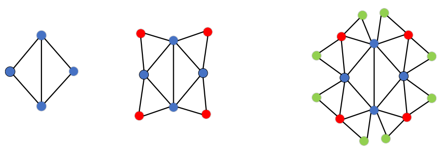

Fig. 3: The graph D at $n = 1,2,3$.

size of the Farey graph (Vn, En) are, respectively,

$$V_n(D) = 2^{n+1}, E_n(D) = 2^{n+2}-3 \quad (8)$$

By using 2.2 on graph G we get the tutte polynomial as follows:

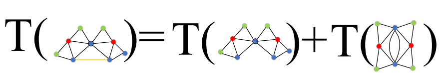

Fig. 4: $T_3(G_3;x_1,y_1) = T_2^2(G_2;x_1,y_1) + T_2(H_2;x_1,y_1)$

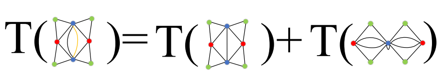

Fig. 5: $T_2(H_2;x_1,y_1) = T_2(D_2;x_1,y_1) + y_1 T_1^2(H_1;x_1,y_1)$

Theorem 1. *The Tutte polynomial of the graph H is expressed as follows:*

$$\begin{aligned}
&T_{n-1}(H_n;x_1,y_1) = y_1 T_{n-1}^2(G_{n-1};x_1,y_1)\\
&+2T_{n-1}(G_{n-1};x_1,y_1)N_{n-1}(G_{n-1};x_1,y_1)\\
&+(x_1-1)N_{n-1}^2(G_{n-1};x_1,y_1)\\
&+2(x_1-1)T_{n-1}(G_{n-1};x_1,y_1)N_{n-1}(G_{n-1};x_1,y_1)\\
&+(x_1-1)^2N_{n-1}^2(G_{n-1};x_1,y_1) - T_{n-1}^2(G_{n-1};x_1,y_1).
\end{aligned}$$

Proof. From figure 2 we git

$$T_n(G_n;x_1,y_1) = T_{n-1}^2(G_{n-1};x_1,y_1) + T_{n-1}(H_{n-1};x_1,y_1) \quad (9)$$

$$T_{n-1}(H_{n-1};x_1,y_1) = T_n(G_n;x_1,y_1) - T_{n-1}^2(G_{n-1};x_1,y_1) \quad (10)$$

By using equation 1 in equation 10 we get

$$\begin{aligned}
&T_{n-1}(H_{n-1};x_1,y_1) = y_1 T_{n-1}^2(G_{n-1};x_1,y_1)\\
&+2T_{n-1}(G_{n-1};x_1,y_1)N_{n-1}(G_{n-1};x_1,y_1)\\
&+(x_1-1)N_{n-1}^2(G_{n-1};x_1,y_1)\\
&+2(x_1-1)T_{n-1}(G_{n-1};x_1,y_1)N_{n-1}(G_{n-1};x_1,y_1)\\
&+(x_1-1)^2N_{n-1}^2(G_{n-1};x_1,y_1) - T_{n-1}^2(G_{n-1};x_1,y_1).
\end{aligned}$$

Corollary 3. *The number of spanning trees of graph H at any time step $n \geqslant 1$ is given as follows:*

$$N_{ST}(H_n) = T_n(H_n;1,1) = (2^{n+2}-1)\prod_{i=2}^{n+1}(2^i-1)^{2^{n-i+1}}$$
$$+(2^{n+1}-1)\prod_{i=2}^{n}(2^i-1)^{2^{n-i}}$$

Theorem 2. *The Tutte polynomial of the graph D is expressed as follows:*

$$T_n(D_n;x_1,y_1) = T_n(H_n;x_1,y_1) - y_1\, T_{n-2}^2(H_{n-1};x_1,y_1) \quad (11)$$

where

$$\begin{aligned}
&T_{n-1}(H_{n-1};x_1,y_1) = y_1 T_{n-1}^2(G_{n-1};x_1,y_1)\\
&+2T_{n-1}(G_{n-1};x_1,y_1)N_{n-1}(G_{n-1};x_1,y_1)\\
&+(x_1-1)N_{n-1}^2(G_{n-1};x_1,y_1)\\
&+2(x_1-1)T_{n-1}(G_{n-1};x_1,y_1)N_{n-1}(G_{n-1};x_1,y_1)\\
&+(x_1-1)^2N_{n-1}^2(G_{n-1};x_1,y_1)\\
&-T_{n-1}^2(G_{n-1};x_1,y_1).
\end{aligned}$$

Proof. From figure 5 we get

$$T_n(H_n;x_1,y_1) = T_n(D_n;x_1,y_1) + y_1\, T_{n-1}^2(H_{n-1};x_1,y_1) \quad (12)$$

$$T_n(D_n;x_1,y_1) = T_n(H_n;x_1,y_1) - y_1\, T_{n-2}^2(H_{n-1};x_1,y_1) \quad (13)$$

Corollary 4. *The number of spanning trees in the graph D at any step time $n \geqslant 1$ is given as follows:*

$$N_{ST}(D_n) = T_n(H_n;1,1) = (2^{n+2}-1)\prod_{i=2}^{n+1}(2^i-1)^{2^{n-i+1}}$$
$$+(2^{n+1}-1)\prod_{i=2}^{n}(2^i-1)^{2^{n-i}} - y_1(T_n(H_{n-1};1,1)$$
$$=(2^{n+1}-1)\prod_{i=2}^{n}(2^i-1)^{2^{n-i}} + (2^n-1)\prod_{i=2}^{n-1}(2^i-1)^{2^{n-i-1}})^2$$

3 Tutte polynomial of a class of self-similar fractal models $N(t)$ and its number of spanning tree

To find the Tutte polynomial of graph H_n where $T(H_n;x_1,y_1) = \xi_n(x_1,y_1)$, and the Tutte polynomial of the ferry graph is given by $T(G_n;x_1,y_1) = T_n(x_1,y_1)$, we use the following terminology: The recursive definition of the function $T(G;x_1,y_1)$ results in the following recursive definition of the graph G and the two independent variables x_1,y_1:
$T(G;x_1,y_1) =$

$$\begin{cases} 1 & : if\ G=\phi \\ x_1T(G-e;x_1,y_1) & : if\ e\ is\ a\ bridge \\ y_1T(G-e;x_1,y_1) & : if\ e\ is\ a\ loop \\ T(G-e;x_1,y_1)+T(G/e;x_1,y_1) & : otherwise \end{cases} \quad (14)$$

where e is an edge of G. by the fourth terminology in equation 14 we get

$$T_n(x_1,y_1) = T^2_{n-1}(x_1,y_1) + \xi_{n-1}(x_1,y_1) \quad (15)$$

i.e. see the following equation for example which we put $n = 3$ at Farey graph
we know the following from corollary 3 in [28]

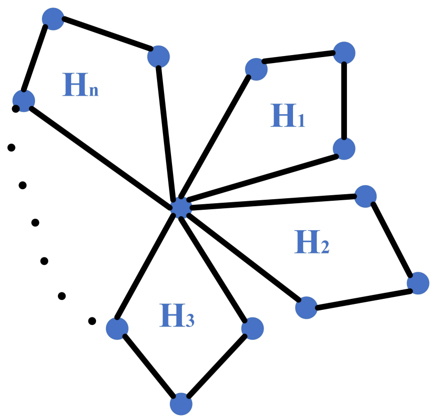

Fig. 6: $T_3(G_3;x_1,y_1) = T^2_2(G_2;x_1,y_1) + T_2(H_2;x_1,y_1)$

Theorem 3.[28] *For each $n \geq 1$, the Tutte polynomial $T_n(x_1,y_1)$ of G_n ferry graph is given by:*

$$T_n(x_1,y_1) = T_{1,n}(x_1,y_1) + (x_1-1)N_n(x_1,y_1) \quad (16)$$

where the polynomials T_n ,N_n satisfy the following recursive relations

$$T_{1,n}(x_1,y_1) = y_1 \cdot T^2_{1,n-1} + 2\cdot T_{1,n-1}\cdot N_{n-1} + (x_1-1)\cdot N^2_{n-1} \quad (17)$$

$$N_n(x_1,y_1) = 2\cdot T_{1,n-1}\cdot N_{n-1} + (x_1-1)\cdot N^2_{n-1} \quad (18)$$

Such that $T_{0,0} = 1, N_0 = 1$

Two new self-similar copies of the Farey graph are generated and denoted as graphs H_n, where n is a positive integer that falls between the intervals $\{1,2,3,\cdots\}$. Here, is an iterative method for donating the first one, H_n: with two parallel edges connecting its two vertices, H_0 has two $n = 0$. When n is greater than one, H_n is produced from H_{n-1} by appending a new pair of edges joined by a new vertex to each outer face edge added at step $n-1$. Additionally, Theorem 4 is important for deducing the number of spanning trees of H_n.

Theorem 4.

$$\xi_n(x_1,y_1) = (y_1-1)T^2_{1,n} + (x_1-1)N^2_n + 2T_{1,n}N_n(x_1,y_1) \quad (19)$$

s.t

$$T_{1,n}(x_1,y_1) = y_1\cdot T^2_{1,n-1} + 2\cdot T_{1,n-1}\cdot N_{n-1} + (x_1-1)\cdot N^2_{n-1} \quad (20)$$

$$N_n(x_1,y_1) = 2\cdot T_{1,n-1}\cdot N_{n-1} + (x_1-1)\cdot N^2_{n-1} \quad (21)$$

Such that $T_{0,0} = 1, N_0 = 1$

Proof. Put $n = n-1$ in equation 16 we get

$$T_{n-1}(x_1,y_1) = T_{1,n-1}(x_1,y_1) + (x_1-1)N_{n-1}(x_1,y_1) \quad (22)$$

From equations 16 and 22 in equation 15

$$\begin{aligned} \xi_{n-1}(x_1,y_1) = T_{1,n}(x_1,y_1) + (x_1-1)N_n - T^2_{1,n-1}(x_1,y_1) \\ -2(x_1-1)T_{1,n-1}(x_1,y_1)N_{n-1}(x_1,y_1) \\ -(x_1-1)^2N^2_{n-1}(x_1,y_1). \end{aligned} \quad (23)$$

From equations 14 and 18 in equation 23 we get

$$\begin{aligned} \xi_{n-1}(x_1,y_1) = (y_1-1)T^2_{1,n-1}(x_1,y_1) + (x_1-1)N^2_{n-1} \\ +2T_{1,n-1}(x_1,y_1)N_{n-1}(x_1,y_1). \end{aligned} \quad (24)$$

4 Number of spanning tree of H_n

The Tutte polynomial of the Ferry graph is $T(G_n;x_1,y_1) = T_n(x_1,y_1)$, and the Tutte polynomial of the graph Hn is $T(H_n;x_1,y_1) = T_n(x_1,y_1)$; to get these, we use the following terminology. We can find the number of spanning trees of graph H_n using the formula $x_1 = y_1 = 1$ in Theorem 4, which provides us. To obtain the number of spanning trees of graph H_n we put $x_1 = y_1 = 1$ in Theorem 4 we obtain:

$$\xi_n(1,1) = 2T_{1,n}(1,1)N_n(1,1) \quad (25)$$

s.t

$$T_{1,n}(1,1) = T^2_{1,n-1}(1,1) + 2\cdot T_{1,n-1}(1,1)\cdot N_{n-1}(1,1) \quad (26)$$

$$N_n(1,1) = 2\cdot T_{1,n-1}(1,1)\cdot N_{n-1}(1,1) \quad (27)$$

From equations 25 and 27, we get

$$\begin{aligned}\xi_n(1,1) &= 2T_{1,n}(1,1)N_n(1,1) = N_{n+1}\\ &= 2\cdot T_{1,n}\cdot 2\cdot T_{1,n-1}\cdot N_{n-2}\\ &= 2\cdot T_{1,n}\cdot 2\cdot T_{1,n-1}\cdot 2\cdot T_{1,n-2}\cdot N_{n-2}\\ &\vdots\\ &= 2^{n+1}\cdot\prod_{i=0}^{n} T_{1,i}\cdot N_0\end{aligned}$$

$$\xi_{n-1} = 2^{n+1}\cdot\prod_{i=0}^{n}(2^{i+1}-1)\prod_{j=2}^{i}(2^j-1)^{2^{i-j}} \tag{28}$$

5 The chromatic polynomial $P(H;\gamma)$

Since the chromatic polynomial $P(H;,\gamma)$ satisfies the following relation:

$$P(H;\gamma) = (-1)^{r(H)}\gamma^{k(G)}\xi_n(1-\gamma,0) \tag{29}$$

Substituting $y_1 = 0$ into equation 19 in theorem 4, we obtain the following recursive equations:

$$\xi_n(x_1,0) = -T_{1,n}^2 + (x_1-1)N_n^2 + 2T_{1,n}\cdot N_n \tag{30}$$

$$T_{1,n}(x_1,0) = 2T_{1,n}\cdot N_n + (x_1-1)N_n^2 \tag{31}$$

$$N_n(x_1,0) = 2T_{1,n}\cdot N_n + (x_1-1)N_n^2 \tag{32}$$

From Equations 31 and 32, we obtain

$$T_{1,n}(x_1,0) = N_n(x_1,0) \tag{33}$$

From equation 33 in equation32, we obtain

$$N_n(x_1,0) = (x_1+1)N_{n-1}^2 = (x_1+1)^{2^n-1} \tag{34}$$

$$\xi(x_1,0) = x_1N_n^2(x_1,0) = x_1(x_1+1)^{2^{n+1}-2} \tag{35}$$

Put the last equation 35 in equation 29, we get

$$P(H;\gamma) = -\gamma\cdot(1-\gamma)\cdot(2-\gamma)^{2^{n+1}-2} \tag{36}$$

Then, it is clear that H_n is minimally 3-colorable.

6 The reliability polynomial $R(H;p)$

The reliability polynomial $R(H;p) = q^{n(H)}p^{r(G)}\xi(H;1,q^{-1})$
From [28] we get

$$T_n(1,y_1) = \frac{2^{n+1}-y_1^{n+1}}{(2-y_1)^{2^{n-1}}}\cdot\prod_{i=2}^{n}(2^i-y_1^i)^{2^{n-i}} \tag{37}$$

It is clear that when $x_1 = 1$, we have $T_n(1,y_1) = T_{1,n}(1,x_1)$ from 15 but $x_1 = 1$ we get

$$\xi_n(1,y_1) = T_{n-1}(1,y_1) - T_n^2(1,y_1) \tag{38}$$

From equations 38 and 39, we get

$$\begin{aligned}\xi_n(1,y_1) = &\tfrac{2^{n+2}-y_1^{n+2}}{(2-y_1)^{2^n}}\cdot\textstyle\prod_{i=2}^{n+1}\left(2^i-y_1^i\right)^{2^{-i+n+1}}\\ &-\left(\tfrac{2^{n+1}-y_1^{n+1}}{(2-y_1)^{2^{n-1}}}\cdot\textstyle\prod_{i=2}^{n}\left(2^i-y_1^i\right)^{2^{n-i}}\right)^2\end{aligned} \tag{39}$$

$$\xi_n(1,y_1) = \frac{y_1^{n+1}-y_1^{n+2}+2^{n+1}}{(2-y_1)^{2^n}}\cdot\prod_{i=2}^{n+1}\left(2^i-y_1^i\right)^{2^{-i+n+1}} \tag{40}$$

The reliability polynomial as

$$\begin{aligned}R(H;p) =&q^{2^{n+2}-1}\cdot p^{2^{n+2}-1}\cdot\frac{y_1^{n+1}-y_1^{n+2}+2^{n+1}}{(2-y_1)^{2^n}}\\ &\cdot\prod_{i=2}^{n+1}\left(2^i-y_1^i\right)^{2^{n+1-i}}\end{aligned}$$

7 Tutte polynomial of a class of self-similar fractal models $N(t)$ and its number of spanning tree

Theorem 5. *Let G [29]be an outerplanar graph that can be divided into $H1;H2;\cdots;Hn$ sub-graph, Figure 7 shows the Tutte polynomial of each subgraph $H1;H2;\cdots;Hn$ is $T(H1);T(H2);\cdots;T(Hn)$ respectively, so the Tutte polynomial $T(G)$ of graph G is given by the following formula:*

$$T(G) = \prod_{i=1}^{n} T(Hi) \tag{41}$$

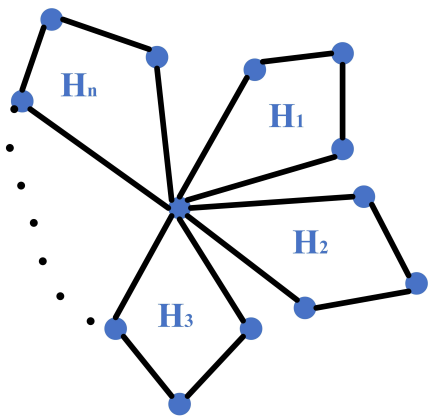

Fig. 7: $G = H1\cap H2\cap\cdots\cap Hn$

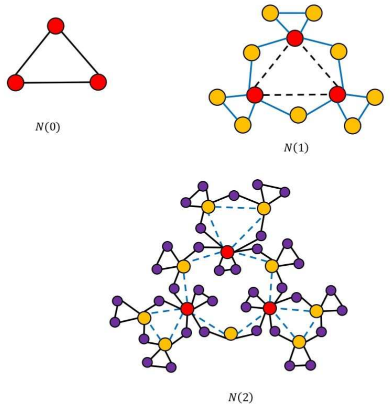

Fig. 8: This diagrams of the graphs $N(t)$ at $t = 0,1,2$

Let $T(C_n)$ be the tutte polynomial of the cycle graph C_n such that $T(C_n)\{x_1,y_1\} = x_1^{n-1} + x_1^{n-2} + \cdots + x_1 + y_1$ By using mathematical simplification, we obtain $T(C_n)\{x_1,y_1\} = \frac{x_1^n - x_1}{x_1 - 1} + y_1$,and let $T(N_t)\{x_1,y_1\}$ be the tutte polynomial of the self-similar fractal models $N(t)$. The number of edge $E(N_t) = \frac{3(\gamma^t - \eta^t)}{\gamma - \eta}$.

The number of vertex $V(N_t) = \frac{3\left((\gamma-1)\gamma^{t-1} - (\eta-1)^{t-1}\right)}{\gamma - \eta}$. where

$$\gamma = \frac{1}{2}\left(5 + \sqrt{13}\right) \tag{42}$$

$$\eta = \frac{1}{2}\left(5 - \sqrt{13}\right) \tag{43}$$

$$T(N_1) = T(C_3) = x_1^2 + x_1 + y_1,$$
$$T(N_2) = T^3(N_1)T(C_6) = T^3(C_3)T(C_6)$$
$$T(N_3) = T^3(N_2)T^3(N_1)T(C_{12}) = T^{12}(C_3)T^3(C_6)T(C_{12})$$
$$T(N_4) = T^3(N_3)T^3(N_2)T^6(N_1)T(C_{12})$$
$$T(N_5) = T^{219}(C_3)T^{51}(C_6)T^{12}(C_{12})T^3(C_{24})T(C_{48})$$

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$$T(N_n) = T^{n_{t-1}}(C_3)T^{n_{t-2}}(C_{3\cdot 2}) \cdots T^{n_{3\cdot 2^{t-2}}}(C_12)T(C_{3\cdot 2^{t-1}}).$$

$$T(N_t) = \prod_{i=0}^{t-1} T^{n_{t-i-1}}(C_{3\cdot 2^i})$$

with assumption that

$$n_i = \frac{(\gamma - 1)\gamma^{-i+t-1} - (\eta - 1)\eta^{-i+t-1}}{\gamma - \eta} \tag{44}$$

In the general case, we start with a cycle of length k and add a path of length $m+1$ the Tutte polynomial of $N_{k,m}(t)$ is given as follows:

$$T(N_{k,m}(t)) = \prod_{i=0}^{t-1} T^{n_{t-i-1}}(C_{k\cdot(m+1)^i}) \tag{45}$$

where

$$n_i = \frac{(\gamma - 1)\gamma^{-i+t-1} - (\eta - 1)\eta^{-i+t-1}}{\gamma - \eta} \tag{46}$$

where

$$\gamma = \frac{1}{2}\left(k + m + 1 + \sqrt{(k+m+1)^2 - 4k}\right) \tag{47}$$

$$\eta = \frac{1}{2}\left(k + m + 1 - \sqrt{(k+m+1)^2 - 4k}\right) \tag{48}$$

8 Analysis and dissection

The main aim is to evaluate the Tutte polynomial for two types of graphs H_n and D_n for $n \in 1,2,3,\cdots$. We concentrate particularly on the Tutte polynomial of a graph resulting from the duplication of Farey graphs. We provide an example of how the Tutte polynomial can be used to count spanning trees and connected spanning subgraphs in a graph. The chromatic polynomial and dependability polynomial of the graphs are also derivable using exact formulations. Our findings are widely applicable in several disciplines such as combinatorics, biology, and statistical mechanics. The underlying theory for the two new models built on the Farey graph was modified. In addition, we reduce the number of network models by contracting and deleting operators that contain graphical invariants. The number of spanning trees of graphs, connected subgraphs, and the dimension of the bicycle space are given by the Tutte polynomial $T(G;x_1,y)$ at a point (x_1,y). The chromatic polynomial, reliability polynomial, and flow polynomial are single-variable polynomials deduced from the special cases of the Tutte polynomial. In addition, two self-similar-structure generators induced by Farey graphs are generated, and Tutte polynomials for a family of recursive graphs are investigated. Finally, we used the Koch curve model to apply our research.

9 Conclusions

In this work, we present a technique based on the self-similar structure of the Farey graph family to produce recursive formulas for the Tutte polynomial. A class of planar graphs known as Farey graphs are produced from the Farey sequence, which is a series of fractions with denominators less than 190 or equal to a predetermined number and lying between 0 and 1. We obtained precise formulas for the dependability and chromatic polynomials of Farey graphs, which have uses in a variety of domains including network reliability and coding theory. We updated the fundamental theory for the two new models we introduced, which were based on the Farey graph. Ultimately, the model developed from the Koch curve was used to apply our research. Owing to its intriguing characteristics, this fractal curve has gained popularity as a study topic 195 across numerous disciplines. To determine the number of spanning trees, chromatic polynomials, reliability polynomials, and linked and spanning subgraphs, we further examined specific examples of these findings.

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