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Solution of the Telegraph Equation Using Adomian Decomposition Method with Accelerated Formula of Adomian Polynomials

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Abstract: In this paper, we will apply the Adomian decomposition method (ADM) to three different examples of the Telegraph Equation with a nonlinear term by the two polynomials called Adomian polynomials and the new accelerated Adomian polynomials proposed by El-kalla [12] that called El-kalla polynomials and compare the solution with the exact solution, we found that the new accelerated polynomials is easier and converges rapidly than Adomaian polynomials, also we found that the error between the exact solution and the solution using the new accelerated polynomials is less than the error between the exact solution and the solution using Adomian polynomials.

Keywords: Telegraph equation; Adomian polynomial; El-kalla polynomial

1 Introduction

There are many applications of the telegraph equation for example, in wave phenomena and also in wave propagation of electric signals in a cable transmission line
.
Some studies were done to solve the telegraph equation numerically or analytically as in [1], [2], [3], [4], [5], [6], [7], [14], [15], [16], [17].
ADM was discussed by the mathematician George Adomian [8], [9], [10], [11], [13]. It has been shown that ADM can solve a large class of ordinary or partial differential equations and the approximate solution converges rapidly to the accurate solutions.
The main purpose of this study is to solve the nonlinear telegraph equation by using ADM and clarify the advantages of El-kalla polynomials using the ADM for solving nonlinear telegraph equation.
The results are presented graphically to show the difference between using the two polynomials.

2 The Methods

In this section we will illustrate the main points of the Adomian decomposition method incase of nonlinear partial differential equations.

2.1 Adomian decomposition method in case of nonlinear P.D.E

Let, $u = u(x,t)$ and consider the differential equation

$$L_t u + L_x u + Ru(x,t) + f(u(x,t)) = f(x,t), \quad (1)$$

where $L_t u = \frac{dn}{dt^n}$ is the higher derivative of the t variable, $L_x u = \frac{dn}{dx^n}$is the higher derivative of the x variable, $Ru(x,t)$ is the other derivative terms, $f(u(x,t))$ will be a term of nonlinearity and $f(x,t)$ is the terms containing independent variables only. firstly we separate the higher derivative of the (x) variable or of the (t) variable in any side of the equal sign of the equation.

$$L_x u = f(x,t) + L_t u + Ru(x,t) + f(u(x,t)), \quad (2)$$

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and making the integration L_x^{-1} to the sides of the equation, where L_x^{-1} is the integration from 0 to x equal n times of the derivative, we get

$$u(x,t) = u(0,t) + L_x^{-1} f(x,t) + L_x^{-1} L_t u + L_x^{-1} Ru(x,t) + L_x^{-1} A_n. \quad (3)$$

Then the solution of Adomian will be

$$u(x,t) = \sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + u_3 + ..., \quad (4)$$

$$\begin{aligned} u_0 &= u(0,t) + L_x^{-1} f(x,t), \\ u_1 &= L_x^{-1}(L_t u_0) + L_x^{-1}(Ru_0(x,t)) + L_x^{-1}(A_0), \\ u_2 &= L_x^{-1}(L_t u_1) + L_x^{-1}(Ru_1(x,t)) + L_x^{-1}(A1), \\ u_3 &= L_x^{-1}(L_t u_2) + L_x^{-1}(Ru_2(x,t)) + L_x^{-1}(A_2), \\ &\vdots \end{aligned} \quad (5)$$

And getting the solution of the equation by the integration, where $A_0, A_1, A_2, \ldots$ called Adomian polynomials or using the polynomials called El-kalla polynomials $\bar{A}_0, \bar{A}_1, \bar{A}_2, \ldots$ as we will see later.

2.2 Adomian polynomial formula

$$A_n = \frac{1}{n!} \left(\frac{d^n}{d\lambda^n} [N(\sum_{i=0}^{n} \lambda^i u_i)] \right)_{\lambda=0}, \quad (6)$$

such that $N(u_i)$ is the term of the nonlinearity

$$\begin{aligned} A_0 &= \frac{1}{0!} \left(\frac{d^0}{d\lambda^0} [N(u_0)] \right)_{\lambda=0} = N(u_0), \\ A_1 &= \frac{1}{1!} \left(\frac{d}{d\lambda} [N(u_0 + \lambda u_1)] \right)_{\lambda=0}, \\ A_2 &= \frac{1}{2!} \left(\frac{d^2}{d\lambda^2} [N(u_0 + \lambda u_1 + \lambda^2 u_2)] \right)_{\lambda=0}, \\ &\vdots \end{aligned} \quad (7)$$

2.3 El-kalla polynomial formula

$$\bar{A}_n = f(s_n) - \sum_{i=0}^{n-1} \bar{A}_i, \quad (8)$$

where $\bar{A}_n$, are El-kalla polynomials, $\bar{A}_0, \bar{A}_1, \bar{A}_2, \ldots$.

$f(s_n)$: is making a substitution of the summation of the solutions in the term of the nonlinearity n times,

for instance if the nonlinear function is $f(u) = sin(u)$ then

$$\begin{aligned} f(u_0) &= sin(u_0), \\ f(u_0 + u_1) &= sin(u_0 + u_1), \\ f(u_0 + u_1 + u_2) &= sin(u_0 + u_1 + u_2), \\ &\vdots \end{aligned}$$

so,

$$\begin{aligned} \bar{A}_1 &= f(u_0) - \bar{A}_0, \\ \bar{A}_2 &= f(u_0 + u_1) - (\bar{A}_0 + \bar{A}_1), \\ \bar{A}_3 &= f(u_0 + u_1 + u_2) - (\bar{A}_0 + \bar{A}_1 + \bar{A}_2), \\ &\vdots \end{aligned} \quad (9)$$

For example, consider the nonlinear term y^3, we clarify both polynomials, the Adomian and El-kalla as in Table (1) and it is clear that the terms of the polynomials proposed by El-kalla has higher accuracy than the polynomials of Adomian

3 Numerical Examples

3.1 Example 1

Consider the nonlinear telegraph equation

$$\begin{aligned} u_{xx} &= u_{tt} + 2u_t + u^2 - e^{(2x-4t)} + e^{(x-2t)}, \\ u(x,0) &= e^x, \quad u_t(x,0) = -2e^x. \end{aligned} \quad (10)$$

Appling the ADM by El-kalla and Adomian

For the Adomian polynomials, the solution will be as following: First we will assume the solution as following

$$u(x,t) = \sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + u_3 + ..., \quad (11)$$

$$u_{tt} = u_{xx} - 2u_t - u^2 + e^{(2x-4t)} - e^{(x-2t)}. \quad (12)$$

Then, we integrate the two sides from 0 to t,

$$\begin{aligned} u_t &= u_t(x,0) + \int_0^t u_{xx} dt - \int_0^t 2u_t dt \\ &- \int_0^t u^2 dt + \int_0^t \left(e^{(2x-4t)} - e^{(x-2t)} \right) dt, \end{aligned} \quad (13)$$

Then, we integrate the two sides from 0 to t again, to get

$$\begin{aligned} u &= u(x,0) + \int_0^t u_t(x,0) dt + \int_0^t \int_0^t u_{xx} dt dt - \int_0^t \int_0^t 2u_t dt dt \\ &- \int_0^t \int_0^t u^2 dt dt + \int_0^t \int_0^t \left(e^{(2x-4t)} - e^{(x-2t)} \right) dt dt, \end{aligned} \quad (14)$$

$$\begin{aligned} u_0 &= e^x + \int_0^t -2e^x dt + \int_0^t \int_0^t \left(e^{(2x-4t)} - e^{(x-2t)} \right) dt dt, \\ u_1 &= \int_0^t \int_0^t u_{0xx} dt dt - \int_0^t \int_0^t 2u_{0t} dt dt - \int_0^t \int_0^t A_0 dt dt, \\ u_2 &= \int_0^t \int_0^t u_{1xx} dt dt - \int_0^t \int_0^t 2u_{1t} dt dt - \int_0^t \int_0^t A_1 dt dt, \\ &\vdots \end{aligned} \quad (15)$$

In the given problem the nonlinear term is u^2, calculating $A_0, A_1, A_2, \ldots$ using Equation (6), we get

$$\begin{aligned} u &= u(x,0) + \int_0^t u_t(x,0)dt + \int_0^t \int_0^t u_{xx} dtdt - \int_0^t \int_0^t 2u_t dtdt \\ &\quad - \int_0^t \int_0^t u^2 dtdt + \int_0^t \int_0^t \left(e^{(2x-4t)} - e^{(x-2t)}\right) dtdt, \\ A_0 &= (e^x + \frac{(e^{2x}(4t+e^{-4t}-1))}{16} - \frac{(e^x(2t+e^{-2t}-1))}{4} - 2te^x)^2 \\ A_1 &= \frac{1}{1!}\left(\frac{d}{d\lambda}[(u_0+\lambda u_1)^2]\right)_{\lambda=0}, \\ A_2 &= \frac{1}{2!}\left(\frac{d^2}{d\lambda^2}[(u_0+\lambda u_1+\lambda^2 u_2)^2]\right)_{\lambda=0}, \\ &\vdots \end{aligned} \tag{16}$$

so, the solution will be

$$u(x,t) = \sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + \ldots, \tag{17}$$

$$u(x,t) = e^x + \frac{(e^{2x}(4t+e^{-4t}-1))}{16} - \frac{(e^x(2t+e^{-2t}-1))}{4} - 2te^x + \cdots, \tag{18}$$

where the exact solution is

$$u(x,t) = e^{x-2t}. \tag{19}$$

3.1.1 For the polynomials called El-kalla polynomials the solution will be as following:

The steps will be the same as previous Equations (11), (12), (13), (14), (15) but, for El-kalla polynomials it will be as follow,

$$\begin{aligned} \bar{A}_0 &= u_0^2 = (e^x + \frac{(e^{2x}(4t+e^{-4t}-1))}{16} \\ &\quad - \frac{(e^x(2t+e^{-2t}-1))}{4} - 2te^x)^2, \\ \bar{A}_1 &= 2u_0u_1 + u_1^2, \\ \bar{A}_2 &= u_2^2 + 2u_0u_2 + 2u_1u_2, \\ &\vdots \end{aligned} \tag{20}$$

$$\begin{aligned} u_0 &= e^x + \frac{(e^{2x}(4t+e^{-4t}-1))}{16} - \frac{(e^x(2t+e^{-2t}-1))}{4} - 2te^x \\ u_1 &= \int_0^t \int_0^t u_{0xx} dtdt - \int_0^t \int_0^t 2u_{0t} dtdt - \int_0^t \int_0^t A_0 dtdt, \\ u_2 &= \int_0^t \int_0^t u_{1xx} dtdt - \int_0^t \int_0^t 2u_{1t} dtdt - \int_0^t \int_0^t A_1 dtdt, \\ &\vdots \end{aligned} \tag{21}$$

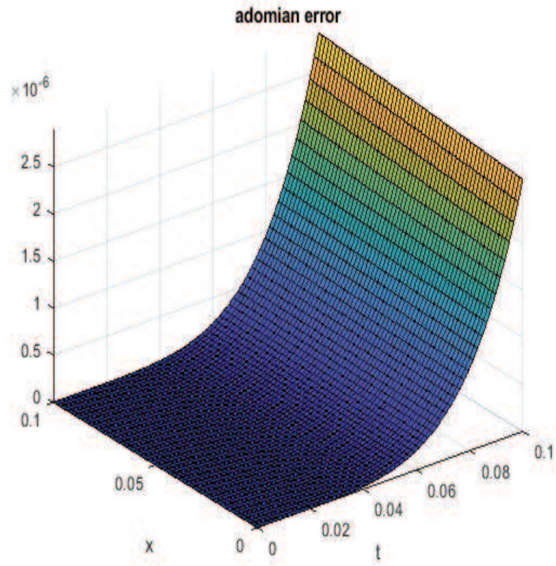

Fig. 1: The difference between the exact solution and the solution with Adomian polynomial of $u_{xx} = u_{tt} + 2u_t + u^2 - e^{(2x-4t)} + e^(x-2t)$.

The solution is

$$u(x,t) = \sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + u_3 + \ldots, \tag{22}$$

$$u(x,t) = e^x + \frac{(e^{2x}(4t+e^{-4t}-1))}{16} - \frac{(e^x(2t+e^{-2t}-1))}{4} - 2te^x + \cdots, \tag{23}$$

The data in Table (1) was calculated with three terms of the solution $u(x,t) = u_0 + u_1 + u_2$, where the difference between the exact solution and solution with El-kalla polynomial called absolute relative error (ARE) and the difference between the exact solution and with solution Adomian polynomial at $x = 0.01$ for values of t in Example 1.
Also time of the program when calculating the solution by Matlab R2014a as following:
Time when solving by Adomian polynomials = 22.8280 seconds.
Time when solving by El-kalla polynomials = 18.1860 seconds.

3.2 Example 2

Consider the nonlinear telegraph equation

$$\begin{aligned} u_{tt} - u_{xx} + 2u_t - u^2 &= e^{-2t}(\cosh x)^2 - 2e^{-t}\cosh x, \\ u(x,0) &= \cosh x, \quad u_t(x,0) = -\cosh x. \end{aligned} \tag{24}$$

Applying the ADM by El-kalla and Adomian:

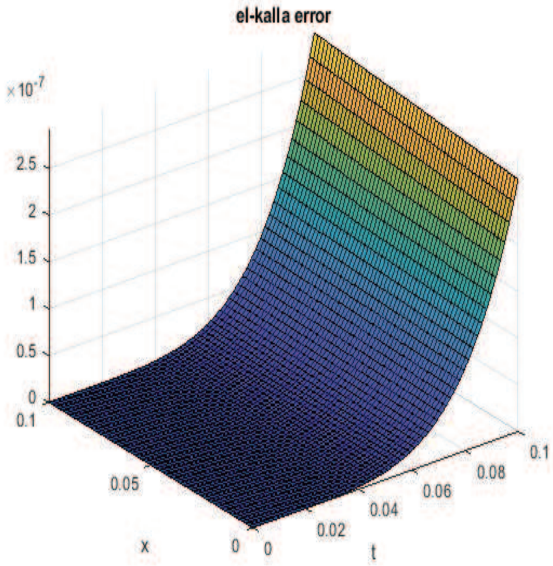

Fig. 2: The difference between the exact solution and the solution with El-kalla polynomial of $u_{xx} = u_{tt} + 2u_t + u^2 - e^{(2x-4t)} + e^(x - 2t)$.

3.2.1 For the Adomian polynomials the solution will be as following

We will assume the solution as following,

$$u(x,t) = \sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + ..., \tag{25}$$

$$u_{tt} = u_{xx} - 2u_t + u^2 - e^{-2t}(\cosh x)^2 - 2e^{-t}cosh(x), \tag{26}$$

integrate the two sides from 0 to t,

$$u_t = u_t(x,0) + \int_0^t u_{xx}dt - \int_0^t 2u_t dt + \int_0^t u^2 dt + \int_0^t e^{-2t}(\cosh x)^2 - 2e^{-t}cosh(x)dt, \tag{27}$$

integrate the two sides from 0 to t,

$$u = u(x,0) + \int_0^t u_t(x,0)dt + \int_0^t\int_0^t u_{xx}dtdt - \int_0^t\int_0^t 2u_t dtdt - \int_0^t\int_0^t u^2 dtdt + \int_0^t\int_0^t \left[e^{-2t}(\cosh x)^2 - 2e^{-t}\cosh x\right]dtdt, \tag{28}$$

$$\begin{aligned} u_0 &= \cosh x + \int_0^t -\cosh x dt \\ &+ \int_0^t\int_0^t e^{-2t}(\cosh x)^2 - 2e^{-t}\cosh x dtdt, \\ u_1 &= \int_0^t\int_0^t u_{0xx}dtdt - \int_0^t\int_0^t 2u_{0t}dtdt - \int_0^t\int_0^t A_0 dtdt, \\ u_2 &= \int_0^t\int_0^t u_{1xx}dtdt - \int_0^t\int_0^t 2u_{1t}dtdt - \int_0^t\int_0^t A_1 dtdt, \\ &\vdots \end{aligned} \tag{29}$$

In the given problem the nonlinear term is u^2, calculating $A_0, A_1, A_2, \ldots$ using Equation (6)

$$\begin{aligned} A_0 &= (\cosh x + \frac{t}{8}(\frac{e^{-x}}{2} + e^x)^2 - \\ &t\cosh x + \frac{1}{16}(\frac{e^{-x}}{2} + e^x)^2(e^{-2t} - 1) - \\ &e^{-x-t}(e^{2x} + 1)(te^t - e^t + 1))^2, \\ A_1 &= \frac{1}{1!}\left(\frac{d}{d\lambda}[(u_0 + \lambda u_1)^2]\right)_{\lambda=0}, \\ A_2 &= \frac{1}{2!}\left(\frac{d^2}{d\lambda^2}[(u_0 + \lambda u_1 + \lambda^2 u_2)^2]\right)_{\lambda=0}, \\ &\vdots \end{aligned} \tag{30}$$

so, the solution is

$$u(x,t) = \sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + u_3 + ..., \tag{31}$$

$$\begin{aligned} u(x,t) &= (\cosh x + \frac{t}{8}(\frac{e^{-x}}{2} + e^x)^2 \\ &- t\cosh x + \frac{1}{16}(\frac{e^{-x}}{2} + e^x)^2(e^{-2t} - 1) \\ &- e^{-x-t}(e^{2x} + 1)(te^t - e^t + 1))^2 + \cdots, \end{aligned} \tag{32}$$

where the exact solution

$$u(x,t) = e^{-t}\cosh x. \tag{33}$$

3.2.2 For the polynomials called El-kalla polynomials the solution will be as following

The steps will be the same as previous equations (25), (26), (27), (28), (29) but, for El-kalla polynomials it will be as follow,

$$\begin{aligned} \bar{A}_0 &= u_0^2 = (\cosh x + \frac{t}{8}(\frac{e^{-x}}{2} + e^x)^2 - t\cosh x + \\ &\frac{1}{16}(\frac{e^{-x}}{2} + e^x)^2(e^{-2t} - 1) - e^{-x-t}(e^{2x} + 1)(te^t - e^t + 1))^2, \\ \bar{A}_1 &= 2u_0u_1 + u_1^2, \\ \bar{A}_2 &= u_2^2 + 2u_0u_2 + 2u_1u_2, \\ &\vdots \end{aligned} \tag{34}$$

$$u_0 = \cosh x + \int_0^t -\cosh x dt + \int_0^t \int_0^t \left[e^{-2t} (\cosh x)^2 - 2e^{-t} \cosh x\right] dt dt,$$
$$u_1 = \int_0^t \int_0^t u_{0xx} dt dt - \int_0^t \int_0^t 2u_{0t} dt dt - \int_0^t \int_0^t A_0 dt dt,$$
$$u_2 = \int_0^t \int_0^t u_{1xx} dt dt - \int_0^t \int_0^t 2u_{1t} dt dt - \int_0^t \int_0^t A_1 dt dt,$$
$$\vdots \quad (35)$$

The solution is

$$u(x,t) = \sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + ..., \quad (36)$$

$$u(x,t) = (\cosh x + \frac{t}{8}(\frac{e^{-x}}{2} + e^x)^2 - t\cosh x + \frac{1}{16}(\frac{e^{-x}}{2} + e^x)^2 (e^{-2t} - 1) - e^{-x-t}(e^{2x} + 1)(te^t - e^t + 1))^2 + \cdots, \quad (37)$$

The data in the Table (3) calculated with three terms of the solution $u(x,t) = u_0 + u_1 + u_2$.

In the Table (3), we clear the difference between the solution using El-kalla polynomials and the exact solution that called absolute relative error (ARE)and the difference between the solution using Adomian polynomials and the Exact solution at $x = 0.01$ for values of t in Example 2. Also time of the program when calculating the solution in Matlab R2014a as following Time when solving by Adomian polynomials= 63.0717 seconds.
Time when solving by El-kalla polynomials= 57.1515 seconds.

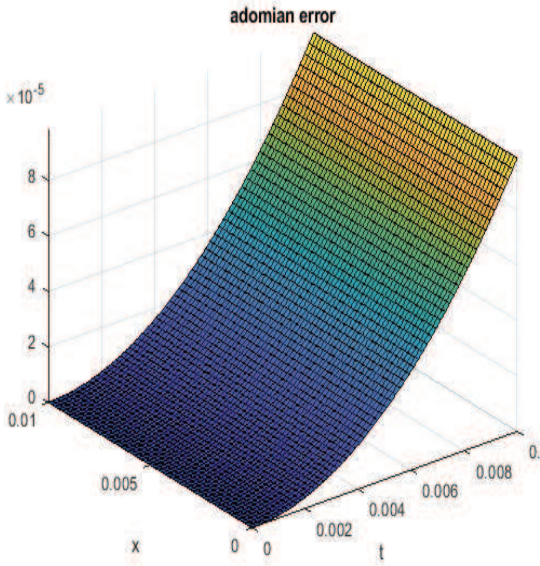

Fig. 3: The difference between the exact solution and the solution with El-kalla polynomial of The difference between the exact solution and solution with Adomian polynomial of $u_{tt} - u_{xx} + 2u_t - u^2 = e^{-2t}(coshx)^2 - 2e^{-t}\cosh x$.

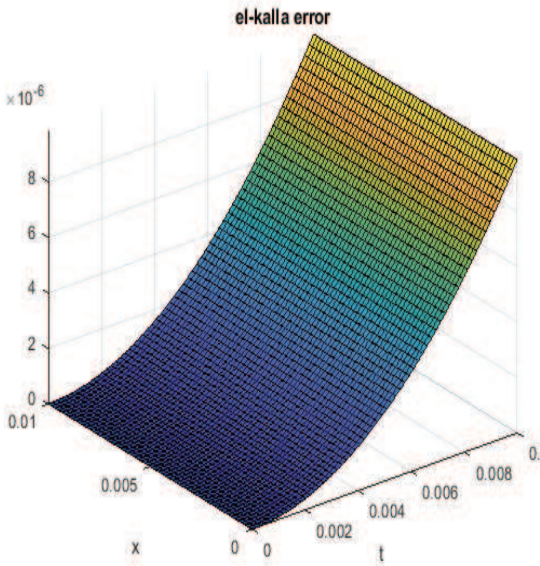

Fig. 4: The difference between the exact solution and the solution with El-kalla polynomial of The difference between the exact solution and solution with Adomian polynomial of$u_{tt} - u_{xx} + 2u_t - u^2 = e^{-2t}(\cosh x)^2 - 2e^{-t}\cosh x$.

3.3 Example 3

Consider the following nonlinear telegraph equation

$$u_{tt} - u_{xx} = u^3 - 2u_t - u,$$
$$u(x,0) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{x}{8} + 5\right),$$
$$u_t(x,0) = \frac{3}{16} + \frac{3}{16}\left(\tanh\left(\frac{x}{8} + 5\right)\right)^2. \quad (38)$$

Applying the ADM by El-kalla and Adomian

3.3.1 For the Adomian polynomials the solution will be as following:

We will assume the solution as following

$$u(x,t) = \sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + u_3 + ..., \quad (39)$$

$$u_{tt} = u_{xx} + u^3 - 2u_t - u, \quad (40)$$

integrate the two sides from 0 to t,

$$u_t = u_t(x,0) + \int_0^t u_{xx}dt + \int_0^t u^3 dt - \int_0^t 2u_t dt - \int_0^t u dt, \quad (41)$$

integrate the two sides from 0 to t

$$\begin{aligned} u &= u(x,0) + \int_0^t u_t(x,0)dt + \int_0^t \int_0^t u_{xx}dtdt \\ &+ \int_0^t \int_0^t u^3 dtdt - \int_0^t \int_0^t 2u_t dtdt \\ &- \int_0^t \int_0^t u dtdt, \end{aligned} \quad (42)$$

$$\begin{aligned} u_0 &= \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{x}{8}+5\right) + \int_0^t \left[\frac{3}{16} + \frac{3}{16}\tanh\left(\frac{x}{8}+5\right)^2\right] dt, \\ u_1 &= \int_0^t \int_0^t u_{0xx}dtdt + \int_0^t \int_0^t A_0 dtdt \\ &- \int_0^t \int_0^t 2u_{0t}dtdt - \int_0^t \int_0^t 2u_0 dtdt, \\ u_2 &= \int_0^t \int_0^t u_{1xx}dtdt + \int_0^t \int_0^t A_1 dtdt \\ &- \int_0^t \int_0^t 2u_{1t}dtdt - \int_0^t \int_0^t 2u_1 dtdt, \\ &\vdots \end{aligned} \quad (43)$$

In the given problem the nonlinear term is u^3, calculating $A_0, A_1, A_2, \ldots$ using Equation (6)

$$\begin{aligned} A_0 &= u_0^3 = \left[\begin{matrix} \frac{1}{2}\tanh\left(\frac{x}{8}+5\right) - \\ \frac{3t}{16}\left(\left(\tanh\left(\frac{x}{8}+5\right)^2 - 1\right) + \frac{1}{2}\right) \end{matrix} \right]^3, \\ A_1 &= \frac{1}{1!}\left(\frac{d}{d\lambda}[(u_0+\lambda u_1)^3]\right)_{\lambda=0}, \\ A_2 &= \frac{1}{2!}\left(\frac{d^2}{d\lambda^2}[(u_0+\lambda u_1+\lambda^2 u_2)^3]\right)_{\lambda=0}, \\ &\vdots \end{aligned} \quad (44)$$

So, the solution is

$$u(x,t) = \sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + u_3 + \ldots, \quad (45)$$

$$\begin{aligned} u(x,t) &= \frac{1}{2}\tanh\left(\frac{x}{8}+5\right) \\ &- \frac{3t}{16}\left(\tanh\left(\frac{x}{8}+5\right)^2 - 1\right) + \frac{1}{2} + \cdots, \end{aligned} \quad (46)$$

where the exact solution

$$u(x,t) = \frac{1}{2} + \frac{1}{2}\tanh 2061\left(\frac{x}{8} + \frac{3t}{8} + 5\right). \quad (47)$$

3.3.2 For the polynomials called El-kalla polynomials the Solution will be as following

The steps will be the same as previous Equations (39), (40), (41), (42), (43) but, for El-kalla polynomials it will be as follow:

$$\begin{aligned} \bar{A}_0 &= u_0^3 = \left[\frac{1}{2}\tanh\left(\frac{x}{8}+5\right) - \frac{3t}{16}\left(\left(\tanh\left(\frac{x}{8}+5\right)^2 - 1\right) + \frac{1}{2}\right)\right]^3, \\ \bar{A}_1 &= (u_0+u_1)^3 - u_0^3, \\ \bar{A}_2 &= (u_0+u_1+u_2)^3 - \bar{A}_0 - \bar{A}_1, \\ &\vdots \end{aligned} \quad (48)$$

$$\begin{aligned} u_0 &= \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{x}{8}+5\right) + \int_0^t \left[\frac{3}{16} + \frac{3}{16}\tanh\left(\frac{x}{8}+5\right)^2\right] dt, \\ u_1 &= \int_0^t \int_0^t u_{0xx}dtdt + \int_0^t \int_0^t A_0 dtdt \\ &= -\int_0^t \int_0^t 2u_{0t}dtdt - \int_0^t \int_0^t 2u_0 dtdt, \\ u_2 &= \int_0^t \int_0^t u_{1xx}dtdt + \int_0^t \int_0^t A_1 dtdt \\ &= -\int_0^t \int_0^t 2u_{1t}dtdt - \int_0^t \int_0^t 2u_1 dtdt, \\ &\vdots \end{aligned} \quad (49)$$

The solution is

$$u(x,t) = \sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + u_3 + \ldots, \quad (50)$$

$$\begin{aligned} u(x,t) &= \frac{1}{2}\tanh\left(\frac{x}{8}+5\right) \\ &- \frac{3t}{16}\left(\tanh\left(\frac{x}{8}+5\right)^2 - 1\right) + \frac{1}{2} + \cdots, \end{aligned} \quad (51)$$

The data in the Table (4) calculated with three terms of the solution $u(x,t) = u_0 + u_1 + u_2$.

In the Table (4), we clear the difference between the exact solution and solution with El-kalla polynomial that called absolute relative error (ARE) and the difference between the exact solution and solution with Adomian polynomial at $x = 0.01$ for values of t in Example 3.

Also time of the program when calculating the solution in Matlab R2014a as following:
Time when solving by Adomian polynomials= 5.2057 seconds.
Time when solving by El-kalla polynomials= 4.7091 seconds.

Table 1: An example of the y^3 nonlinear term to clear that the polynomials of Adomian appear slower than the polynomials of El-kalla

By the traditional formula	By El-Kalla's formula
$A_0 = y_0^3$,	$\bar{A}_0 = y_0^3$,
$A_1 = 3y_0^2 y_1$,	$\bar{A}_1 = 3y_0^2 y_1 + 3y_0 y_1^2 + y_1^3$,
$A_2 = 3y_0^2 y_1 + 3y_1^2 y_0$,	$\bar{A}_2 = 3y_0^2 y_2 + 6y_0 y_1 y_2 + 3y_1^2 y_2 + 3y_0 y_2^2 + 3y_1 y_2^2 + y_2^3$,
$A_3 = y_1^3 + 3y_0^2 y_3 + 6y_0 y_1 y_2$, $A_4 = 3y_0^2 y_4 + 3y_1^2 y_2 + 3y_2^2 y_0 + 6y_0 y_1 y_3$.	$\bar{A}_3 = 3y_0^2 y_3 + 6y_0 y_1 y_3 + 3y_1^2 y_3 + 6y_0 y_2 y_3 + 6y_1 y_2 y_3 + 3y_2^2 y_3 + 3y_0 y_3^2 + 3y_1 y_3^2 + 3y_2 y_3^2 + y_3^3$.

Table 2: The difference between the exact solution and solution with El-kalla polynomial that called absolute relative error (ARE). And the difference between the exact solution and solution with Adomian polynomial at $x = 0.01$ for values of t , in Example 1

t	(ARE)of ADM at $x = 0.01$	(ARE)of El-kalla at $x = 0.01$
0.1	$6.70184245 * 10^{-5}$	$6.715700672 * 10^{-6}$
0.2	$1.054413173 * 10^{-3}$	$1.0634826002 * 10^{-4}$
0.3	$5.179816713 * 10^{-3}$	$5.2852626316 * 10^{-4}$
0.4	$1.568758671 * 10^{-2}$	$1.6290984728 * 10^{-3}$
0.5	$3.625025864 * 10^{-2}$	$3.858860689 * 10^{-3}$
0.6	$7.0250310527 * 10^{-2}$	$7.73238466857 * 10^{-3}$
0.7	0.12000282907	$1.3801903921 * 10^{-2}$
0.8	0.18596913217	$2.26388934648 * 10^{-2}$
0.9	0.26600379658	$3.4823923093 * 10^{-2}$
1	0.35467204962	$5.09435831384 * 10^{-2}$

Table 3: The difference between the exact solution and solution with El-kalla polynomial that called absolute relative error (ARE). And the difference between the exact solution and solution with Adomian polynomial at $x = 0.01$ for values of t, in Example 2

t	(ARE)of ADM at $x = 0.01$	(ARE)of El-kalla at $x = 0.01$
0.1	$8.73122605 * 10^{-3}$	$8.7313578254 * 10^{-4}$
0.2	$3.030788465 * 10^{-2}$	$3.0316217252 * 10^{-3}$
0.3	$5.868434229 * 10^{-2}$	$5.877812371 * 10^{-3}$
0.4	$8.8702292699 * 10^{-2}$	$8.92229621507 * 10^{-3}$
0.5	0.11572650561	$1.1768939792 * 10^{-2}$
0.6	0.13526819187	$1.410620088106 * 10^{-2}$
0.7	0.14259270381	$1.57039933919 * 10^{-2}$
0.8	0.13230896442	$1.641576130971 * 10^{-2}$
0.9	$9.79377771519 * 10^{-2}$	$1.61858050708 * 10^{-2}$
1	$3.1455235866 * 10^{-2}$	0.150623247937

Table 4: The difference between the exact solution and solution with El-kalla polynomial that called absolute relative error (ARE). And the difference between the exact solution and solution with Adomian polynomial at $x = 0.01$ for values of t, in Example 3

t	(ARE)of ADM at $x = 0.01$	(ARE)of El-kalla at $x = 0.01$
0.1	$6.1887411743 * 10^{-7}$	$6.18874116796 * 10^{-8}$
0.2	$2.2490574667 * 10^{-7}$	$2.249057426 * 10^{-7}$
0.3	$4.584065585 * 10^{-6}$	$4.58406514 * 10^{-7}$
0.4	$7.3605672875 * 10^{-6}$	$7.3605648385 * 10^{-7}$
0.5	$1.03570304106 * 10^{-5}$	$1.035702126 * 10^{-6}$
0.6	$1.33923610671 * 10^{-5}$	$1.3392334357 * 10^{-6}$
0.7	$1.63245368513 * 10^{-5}$	$1.632447097 * 10^{-6}$
0.8	$1.904923443947 * 10^{-5}$	$1.904909088 * 10^{-6}$
0.9	$2.149845197 * 10^{-5}$	$2.1498167436 * 10^{-6}$
1	$2.3639126642 * 10^{-5}$	$2.3638603288 * 10^{-6}$

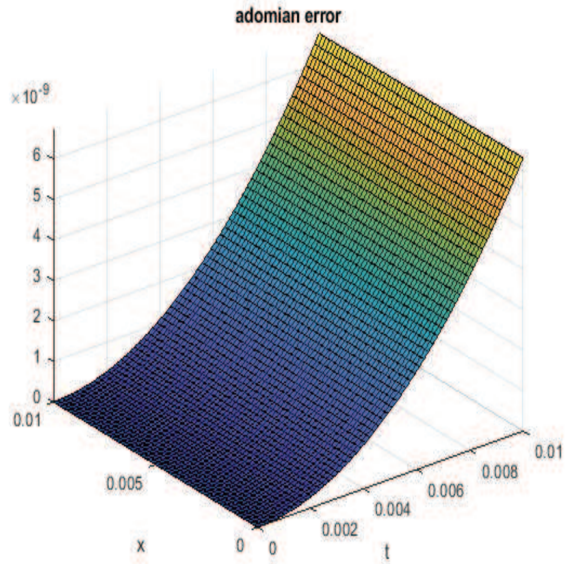

Fig. 5: The difference between the exact solution and solution with Adomian polynomial of $u_{tt} - u_{xx} = u^3 - 2u_t - u$.

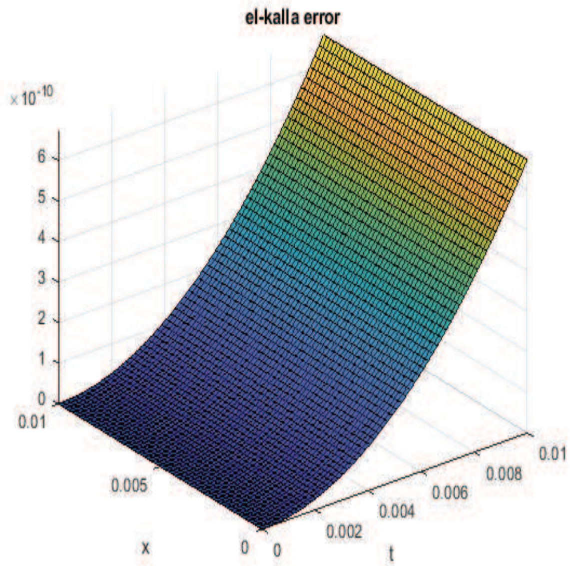

Fig. 6: The difference between the exact solution and solution with El-kalla polynomial of $u_{tt} - u_{xx} = u^3 - 2u_t - u$.

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Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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