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# Pancharatnam Phase of Non-Hermitian Hamiltonian

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**Abstract:** The considered system is two-level atoms interacting with a single mode electromagnetic field (EMF) in dissipative cavity, taking into account the coupling function between atoms and field to be time-dependent, Stark shift effect and Kerr-like medium parameter. The system's corresponding state function is obtained when the two atoms are initially made in superposition of states and the field is in the coherent state. The results show that, the evolution of atomic inversion, Pancharatnam phase, Von Neumann entropy and phase properties are decaying in the presence of damping. We show the the long-life behavior for Pancharatnam phase. This result indicates that quantum correlations quantified by the phase can be protected effectively. Using the effect of the field decay parameter $\Gamma$, the entropy quickly reaches the non-entangled state (death of entanglement). On contrast, the atomic decay parameter $\gamma$, the entropy is very weak.

**Keywords:** two-level atoms, Pancharatnam phase, non-Hermitian, Jaynes-Cummings (JC) model, two-level atom system.

## 1 Introduction

Quantum optics is mainly based on three important processes. Namely atom-atom interaction, field- field interaction and atom-field interaction [1]. Real systems are frequently approximated by simple models, which can be solved exactly. One of these models is the Jaynes-Cummings (JC) model, where a single two-level atom interacting with a single cavity mode [2–5]. The JC model is the simplest model describing the atom-field interaction, which shows important nonclassical properties for the interaction as squeezing, collapse and revival phenomenon in the atomic population inversion and entanglement [6, 8]. The quantum mechanical system involving a single atom interacting with electromagnetic radiation can be described relativity easily [9]. In fact, this JC model is of fundamental importance in the field of quantum optics [10, 11] and It can be achieved to a good approximation in experiments with Rydberg atoms in high-quality super conducting cavities [12]. We can also see that the two-photon JC model is closely related to studying the conjugation of a single atom and a single-mode cavity field, in which the two-photon transition occurs. Here, we recall the cavitation field spectra of two photons JC model which is also mentioned, however, in the presence of a Stark shift and coupling depending on the severity [13]. There is no doubt that the construction of the first single-photon two-photon lasers by Gauthier and others rely on the states of the dressed atoms, leading to the realization of the two-photon laser [14], opened the door to circulate the JC model in different directions. The natural generalization of such a model is an increase in the number of cavity field positions or an increase in the number of atoms. Atomic inversion gives us information about the collapsing and revival phenomenon during the interaction period [15].

Newly, a lot of attention has been focused on the properties of entanglement between the field and the atom, in particular, which maps the entropy of the system. Quantum entropy is a very useful operational measure of the purity of the quantum state, which automatically comprises every moment of the density matrix. The field information is usually inferred by measuring atomic properties. In an ideal cavity, the atom undergoes either a single or double photon transition. The field entropy of the synaptic case for one atom with two levels interacting with one electromagnetic field has been studied [16]. The quantum entropy is a natural generalization of the Boltzmann classical entropy [13, 17, 18], which is proposed by Von-Neumann. Von-Neumann entropy has been applied to measure quantum entanglement, quantum

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decoherence, photocount statistics, quantum optical correlations, purity of states and quantum noise. Lately, much attention has been driven to the Von-Neumann entropy in the presence of atomic motion. For example, the effects of atomic motion and field mode structure in JC model have been explained in Refs [19]. The motion of the atom leads to the periodic evolution of the Von-Neumann entropy. These results have been extended to include the mixed state entanglement and analysis of the pattern entanglement between a three-level atom trapped ion and laser field.

Lately, a lot of interest has been given to the quantum phases, for example, the Pancharatnam phase that was previewed in 1956 by Pancharatnam [20] in his studies of interference impacts for polarized light waves. The geometric phase that was understood, in 1984 [21], is a general feature of quantum mechanics and is based on the path which is selected in space extended by all possible quantum states of the system. Also, it has been realised that such matrix component in quantum physics has been proposed in the path integration approach to quantum mechanics [22]. This approach studies the multitude of quantum pathways that link two points in the Hilbert space: $\psi(q_j(0))$ and $\psi(q_j(t))$, in which $q_j$ represents generalized quantum coordinates. Jordan got the concept of the phase change for partial cycles [23], and ideas of Pancharatnam were used [24, 25]. In this aspect, one must note that the geometric phase is not similar to the effects of other phase in cavity quantum electrodynamics. For example tangent and cotangent states [26] are linked to the phase of the atomic population through a system of two levels and the phase effects themselves have been suggested in a three-tier system [27].

In this work we investigate the interaction between an atomic system described by two-level atoms and the time-dependent cavity field taking into account Stark shift effect, Kerr-like medium parameter and damping term. We obtain the wave function of the total system and we study the effect of decaying parameters and time-varying parameter on some statistical aspects. This paper is prepared as follows. In section (2), we propose the theoretical (mathematical) model and derive the corresponding wave function. In section (3), the influence of the system parameters on the atomic inversion is investigated. In section (4), the evolution of Pancharatnam phase is explained. In section (5), phase properties is explored. In section (6), the entropy is studied. Finally, in section (7), a conclusion is presented.

## 2 Theoretical model

The considered system is two-level atoms interacting with a single mode EMF where the coupling is time-dependent. The two-level atom is expressed as excited, and ground levels by $|g\rangle$ and $|e\rangle$ with energies $\omega$. Also, we consider that before the interaction the EMF in dissipative cavity passes through a Kerr-like medium and the atoms in the presence of damping term is effected by Stark shift contributions by applying an external electric field. Under these postulates, the effective Hamiltonian can be written in rotating wave approximation (RWA) as ($\hbar = 1$):

$$\hat{H}_{eff} = \hat{H}_{A+F} + \hat{H}_{kerr} + \hat{H}_{stark} + \hat{H}_{Damping} + \hat{H}_I, \tag{1}$$

where

$$\hat{H}_{A+F} = \omega\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\Omega\hat{\sigma}_z, \tag{2}$$

$$\hat{H}_{kerr} = \chi\hat{a}^{\dagger 2}\hat{a}^2, \tag{3}$$

$$\hat{H}_{stark} = \hat{a}^{\dagger}\hat{a}\left(\beta_1|e\rangle\langle e| + \beta_2|g\rangle\langle g|\right), \tag{4}$$

$$\hat{H}_{Damping} = -i\Gamma\hat{a}^{\dagger}\hat{a} - i\frac{\gamma}{2}\hat{\sigma}_z, \tag{5}$$

and

$$\hat{H}_I = \lambda(t)(\hat{a}^{\dagger 2}\hat{\sigma}_- + \hat{a}^2\hat{\sigma}_+). \tag{6}$$

wherever $\Omega$ is the frequency between the two-level atom, with $\gamma$ is the atomic corresponding decay rate, operator $\hat{a}^{\dagger}$ ($\hat{a}$) is the creation (annihilation) operator and $\omega$ is the field frequency with a corresponding decay rate $\Gamma$ [15, 28], they follow the commutation relation $[\hat{a}, \hat{a}^{\dagger}] = 1$. $\chi$ is the dispersive parts of the non-linearity of the Kerr-like medium. $\lambda(t) = \bar{\lambda}\cos(\varepsilon t)$ is the coupling function between the atom and field-mode with $\bar{\lambda}$ is coupling constant and $\varepsilon$ is coupling time-modulation parameter and $\beta_j$ are the effective Stark shift parameters.

### 2.1 The state vector

The state vector is determined by a linear combination of the state of the atomic field product $|\eta, \xi\rangle = |\eta\rangle \otimes |\xi\rangle$, So, the state vector is proposed to take the following formula:

$$|\psi(t)\rangle = \sum_{n=0}^{+\infty} q_n \left[A(n,t)e^{-i\bar{\alpha}_1 t}|e,n\rangle + B(n,t)e^{-i\bar{\alpha}_2 t}|g,n+2\rangle\right]. \tag{7}$$

where $q_n$ refers to the initial amplitude of the number state $|n\rangle$, the functions $A(n,t)$ and $B(n,t)$ are the probability amplitudes. $\bar{\alpha}_\ell$ ($\ell = 1, 2.$), and the detuning parameter are given as:

$$\bar{\alpha}_1 = \omega n + \frac{\Omega}{2}, \tag{8}$$

$$\bar{\alpha}_2 = \omega(n+2) - \frac{\Omega}{2}, \tag{9}$$

$$\Delta = \Omega - 2\omega. \tag{10}$$

where $|\eta\rangle$ implies the $\eta$-th level and $|\xi\rangle$ represents the elementary field states that relate to number states $|n\rangle$ through the relationship

$$|\xi\rangle = \sum_{n=0}^{+\infty} q_n |n\rangle, \qquad q_n = \langle n|\xi\rangle. \tag{11}$$

We consider that the field may initially be prepared in coherent states that correspond to a minimum of uncertainty whose center classically swings into a symmetric well and retains its shape, where

$$q_n = \exp\left(-\frac{|\alpha|^2}{2}\right)\frac{\alpha^n}{\sqrt{n!}},$$

in which $\bar{n} = |\alpha|^2$ imply the average photons number (light Intensity) of the mode. By applying the time-dependent Schrödinger equation $i\frac{\partial}{\partial t}|\psi(t)\rangle = \mathscr{H}_{eff}|\psi(t)\rangle$ we obtain the following system of non-linear differential equations.

$$i\dot{A}(t) = \Gamma_1 A(t) + \tilde{g}e^{i\Delta t}B(t),$$
$$i\dot{B}(t) = \Gamma_2 B(t) + \tilde{g}e^{-i\Delta t}A(t).$$

where

$$\Gamma_1 = \chi n(n-1) + \beta n - i\Gamma n - i\frac{\gamma}{2}, \tag{12}$$

$$\Gamma_2 = \chi(n+1)(n+2) + \beta_2(n+2) - i\Gamma(n+2) + i\frac{\gamma}{2}, \tag{13}$$

$$\nu = \frac{\bar{\lambda}}{2}\sqrt{\frac{(n+2)!}{n!}}, \tag{14}$$

$$\tilde{g} = 2\nu\cos(\varepsilon t). \tag{15}$$

The cosine function in $\tilde{g}$ can be represented in an exponential form. There occur exponential functions with two various powers in the differential equations, $e^{\pm i(\Delta+\varepsilon)t}$ and $e^{\pm i(\Delta-\varepsilon)t}$. Somewhere, we can eliminate the register oscillating terms $e^{\pm i(\Delta+\varepsilon)t}$. This parataxis is identical to the RWA and has been agreeable physically for numerous models [29]. So, the differential equations take the following formula

$$i\dot{A}(t) = \Gamma_1 A(t) + \nu e^{i(\Delta-\varepsilon)t}B(t),$$
$$i\dot{B}(t) = \Gamma_2 B(t) + \nu e^{-i(\Delta-\varepsilon)t}A(t).$$

let $\bar{A} = e^{-i(\Delta-\varepsilon)t}A(t)$, the equations became

$$i\dot{\bar{A}}(t) = (\Gamma_1 + \Delta - \varepsilon)\bar{A}(t) + \nu B(t),$$
$$i\dot{B}(t) = \Gamma_2 B(t) + \nu\bar{A}(t).$$

Assuming that the initial state of the atom is in superposition atomic state:

$$|\psi(0)\rangle_{atom} = \cos(\theta)|e\rangle + e^{-i\phi}\sin(\theta)|g\rangle,$$

Thus it is easy to get the probability amplitudes.

### *2.2 The atomic reduced density matrix*

To ventilate some statistical respects in the following sections, we calculate the reduced density matrix of the atom, which is given by

$$\begin{aligned}\hat{\rho}_A(t) &= tr_F\Big(|\psi(t)\rangle\langle\psi(t)|\Big)\\ &= tr_F\Big[\sum_{n_1=0}^{+\infty}\sum_{m_1=0}^{+\infty}\Big(A_1(n,t)A_1^*(m,t)|n,e\rangle\langle e,m|\\ &\quad + A_1(n,t)A_2^*(m,t)|n,e\rangle\langle g,m+2|\\ &\quad + A_2(n,t)A_1^*(m,t)|n+2,g\rangle\langle e,m|\\ &\quad + A_2(n,t)A_2^*(m,t)|n+2,g\rangle\langle g,m+2|\Big)\Big]\end{aligned} \tag{16}$$

where $A_1(n,t) = A(n,t)e^{-i\bar{\alpha}_1 t}$ and $A_2(n,t) = B(n,t)e^{-i\bar{\alpha}_2 t}$, $\hat{\rho}_A(t)$ takes the following form as

$$\hat{\rho}_A(t) = \begin{pmatrix}\rho_{11} & \rho_{12}\\ \rho_{21} & \rho_{22}\end{pmatrix},$$

where the elements of the matrix are given by

$$\rho_{jj} = \sum_{n=0}^{+\infty} A_j(n,t)A_j^*(n,t),$$

$$\rho_{12} = \sum_{n=0}^{+\infty} A_1(n+2,t)A_2^*(n,t) = \rho_{21}^*,$$

## 3 Atomic Population Inversion

The difference between the uppermost states and the lowest states of the probability amplitudes is denoted by the atomic population inversion. It gives information about the behavior of the interaction between atom and field out of the collapses and revivals phenomenon. For this model, the atomic inversion $W(t)$ is given by

$$W(t) = \rho_{11}(t) - \rho_{22}(t), \tag{17}$$

where $\rho_{\ell\ell}$ are the diagonal elements of reduced density matrix.

In fig (1), we plot the evolution of atomic inversion $W(t)$ vs the scaled time $\bar{\lambda}t$ in different cases for the system parameters with constant values of $\chi = 0.01\bar{\lambda}$, $r = 0$ and $\Delta = 0$. Generally, we realize that the phenomenon of revival and collapse is evident in whole figures and the inversion is bounded between $(-1)$ and $(+1)$ in fig.(1a), we identify the atomic population inversion at $\chi = 0.01\bar{\lambda}$, $\Gamma = 0$, $\gamma = 0$ and $\varepsilon = 0$. In fig.(1b), we set $\varepsilon = 3\pi\bar{\lambda}$, we note that oscillation becomes about (0.2) i.e. the base-line is shifted up which means more energy is stored in the atomic system that occurs because of the presence of time-varying parameters and also the oscillation amplitude decreases during the revival period. In fig.(1c), we set $\varepsilon = 8\pi\bar{\lambda}$. We

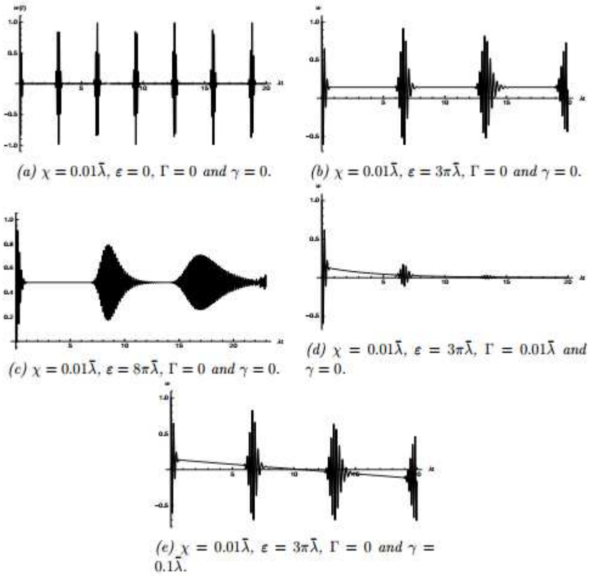


**Fig. 1:** The evolution of atomic inversion vs the scaled time

note that the mean value of the base-line is shifted upward more than the previous case and also spreading of the revival fluctuation with developing time. In fig.(1d), we set $\varepsilon = 3\pi\bar{\lambda}$, $\Gamma = 0.01\bar{\lambda}$ and $\gamma = 0.$, we note that the oscillation almost dies out as the scaled time develops. In fig.(1e), we set $\varepsilon = 3\pi\bar{\lambda}$, $\Gamma = 0$ and $\gamma = 0.01\bar{\lambda}$., we note that the behavior like the behavior in fig (1b) but the mean value of energy decreases as the scaled time develops.

## 4 Pancharatnam phase

In this section, our motivation is to investigate the Pancharatnam phase associated with the two-level atoms. For the evolution of quantum system from an initial wave function $|\psi(0)\rangle$ to a final wave function $|\psi(t)\rangle$, if the state $|\psi(t)\rangle$ cannot be obtained from $|\psi(0)\rangle$ by multiplication with a complex number, the initial and final states are distinct and the evolution is noncyclic. Suppose the initial state evolves to the final state after a certain time t. If the scalar product [30]

$$\left\langle \psi(0)|\exp[\frac{i}{\hbar}\int_0^t H(t')dt']|\psi(0)\right\rangle \qquad (18)$$

Pancharatnam prescribes the phase acquired during an arbitrary evolution of a wave function from the vector $|\psi(0)\rangle$ to $|\psi(t)\rangle$ as

$$\varphi(t) = \arg\left(\langle\psi(0)|\psi(t)\rangle\right) \qquad (19)$$

the geometric phase is obtained by subtracting the dynamical phase from the Pancharatnam phase . On the other hand, in order to shed some light on the Pancharatnam phase generalization, a simple definition based upon interference has been proposed in [31].

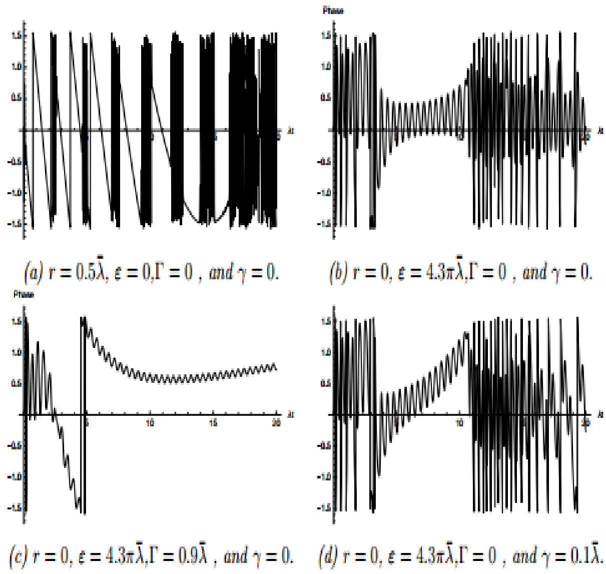


**Fig. 2:** The evolution of Pancharatnam phase vs the scaled time

In fig (2), we plot the Pancharatnam phase evolving $\varphi$ (t) vs the calculated time $\bar{\lambda}t$ in different cases for the system parameters with constant values of $\chi = 0$ and $\Delta = 0$. Mostly, we realize that the phase $\varphi$(t) is bordered between $\frac{-1}{2}\pi$ and $\frac{1}{2}\pi$, It is configured like a saw-toothed variation, and the rotation is slightly displayed. In fig (2a), we set $r = 0.5\bar{\lambda}$, $\varepsilon = 0$, $\Gamma = 0$ and $\gamma = 0$, we note that many artificial phase jumps are observed. In fig (2b), we set $r = 0$ , $\varepsilon = 4.3\pi\bar{\lambda}$, $\Gamma = 0$ and $\gamma = 0$, for $\lambda t \leq 1\pi$, the phase exposes a rabid subsequent industrial phase jumps, then we take a dominate modulation frequency (FM) effect behavior till $\lambda t \leq 3.5\pi$ that is ensued by a fluctuation that evolves to start another subsequent artificial phase jumps but quicker than the previous evolution. In fig (2c), we set $r = 0$ , $\varepsilon = 4.3\pi\bar{\lambda}$, $\Gamma = 0.9\pi\bar{\lambda}$ and $\gamma = 0$, we note that the phase take the long-life behavior after the period $\lambda t \leq 1.6\pi$. This result indicates that quantum correlations quantified by the phase can be protected effectively [32] . In fig (2d), by comparing the result with fig(2b), the modulation frequency behavior is to be slanted due to the atomic damping parameter($\gamma$).

## 5 Phase properties

In this part, we investigate and analyze the phase properties in the existence of Stark effect, damping, Kerr-like medium and time-varying parameter. Pegg and Barnett have identified the hermitian operator of phase. By the reduced density operator of the cavity field: $\rho_f(t)$, the continuity phase properties $P(\theta,t)$ is defined as [33, 34]. By neglecting the phase angle $\theta_0$ and representing the reduced field density matrix in our calculation, then $P(\theta,t)$ is written as:

$$P(\theta,t) = \frac{1}{2\pi}\left(\left|\sum_{n=0}^{\infty} R(\theta,t)\right|^2 + \left|\sum_{n=0}^{\infty} Z(\theta,t)\right|^2\right) \qquad (20)$$

where

$$R(\theta,t) = q_n A_1(n,t)e^{-in\theta}, \quad Z(\theta,t) = q_{n+2}A_2(n+2,t)e^{-in\theta}. \qquad (21)$$

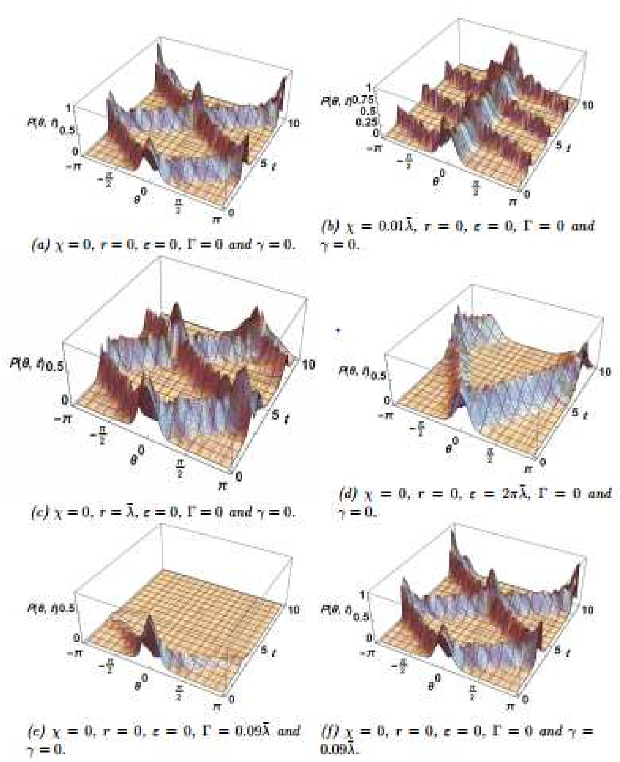


**Fig. 3:** The evolution of Pegg-Barentt phase$P(\theta,t)$ vs the scaled time $\bar{\lambda}t$ and angle $\theta$

In fig (3), we plot the evolution of Pegg-Barentt phase $P(\theta,t)$ vs the scaled time $\bar{\lambda}t$ and angle $\theta$ in different cases for the system parameters with constant values of $\chi = 0$ and $\Delta = 0$. In fig (a), we note that a single summit at $\theta = 0$ corresponding to the initial coherent state, as advance of time there are two summits moving away from each other progressively until they reach $\theta = \pm\pi$. When the scaled time $\bar{\lambda}t$ grows further the two summits start congregating again at $\theta = 0$. In fig (b), we note that a single-summit at $\theta = 0$ is repeated after the period of scaled time $\bar{\lambda}t$ equal to $\pi$ due to the impact of Kerr-like medium parameter $\chi = 0.01\bar{\lambda}$. In fig (c), we set $r = 1\bar{\lambda}$, we note that the summits became smaller than fig (a) due to Stark effect parameter. In fig (d), we set $\varepsilon = 2\pi\bar{\lambda}$, we note that the amplitude of P-B phase decreased with the rise of the scaled time and also the taken time for converging two summits is more than the first case fig (a) . In fig (e), we set $\Gamma = 0.09\bar{\lambda}$, we note that the whole peaks of P-B phase die out except the peak at zero. In fig (f), we set $\gamma = 0.09\bar{\lambda}$, we realize that the P-B phase like as the first case in fig(a).

## 6 Von-Neumann Entropy

To investigate the dynamics of entanglement and get the degree of entanglement, the quantum mutual information (quantum entropy) is a useful criterion that leads us to the amount of entanglement [35]. In the present bipartite quantum system under our consideration (atom and field), we use the Von Neumann entropy which is defined as [36]

$$S = -tr\rho \ln\rho, \qquad (22)$$

where $\rho$ is the density matrix of the given quantum system the Boltzmann constant is assumed to be unity. Mostly, quantum entropy is difficult to calculate, because it involves large diagonal (and, in many cases, infinitely dimensional) density matrices. $S = 0$, for $\rho$ describes a pure state, and $S \neq 0$ for $\rho$ describes a mixed state. When the initial state is prepared in a pure state, so the whole atom-field closed system is isolated from its environment, and its entropy is always zero. suppose $S_A$ and $S_F$ refer to the entropy of two interacting systems [we refer to A for Atom and F for Field] and let $S$ indicate the entropy of the combined system. The von Neumann entropy of the field $S_F = -Tr_F\rho_F \ln\rho_F$ and that of the atom $S_A = -Tr_A\rho_A \ln\rho_A$. The entropy of a general two-component system are included in the Arachy and Leib theory [37] that states

$$|S_A - S_F| \leq S \leq S_A + S_F, \qquad (23)$$

A nice illustration of this inequality in the context of the Jaynes-Cummings model has been given by Knight and Phoenix [13, 18, 19]. Knight and Co-workers developed the general method for computing the various field eigenstates in simple mode. The studied system is an atom for two-level that interacts with an un-damped cavity in a coherent state at the beginning. In this case the composite entropy is initially zero and remains zero at all times because the atom-field system is isolated from its environment. Under these conditions, the last inequality in (23) is trivially satisfied and the first means that the atomic and the field entropies are equal; $S_F = S_A$. We can express the entropies of the atom and the field, when treated as isolated systems, are defined through the corresponding reduced density matrices by

$$S_{A(F)} = -tr_{A(F)}\left\{\rho_{A(F)} \ln \rho_{A(F)}\right\}. \qquad (24)$$

Recent experiences have been founded to measure the entanglement between atoms through decaying photons [38]. Since the trace is invariable under a similarity transformation, we are able to go to basis where the atomic density operator is diagonal and we can write the equation (24) as

$$DEM(t) = S_A(t) = S_F(t) = -\sum_{j=1}^{2} \Phi_j \ln \Phi_j, \qquad (25)$$

where $\Phi_j$ are the eigenvalues of the atomic density operator $\rho_A(t)$ as explained in the subsection(2.2). the reduced atomic density matrix eigenvalues for two-level atoms can be calculated as:

$$\Phi_j = \frac{1 \pm \sqrt{1 - 4(\rho_{11}\rho_{22} - \rho_{12}\rho_{21})}}{2} \qquad (26)$$

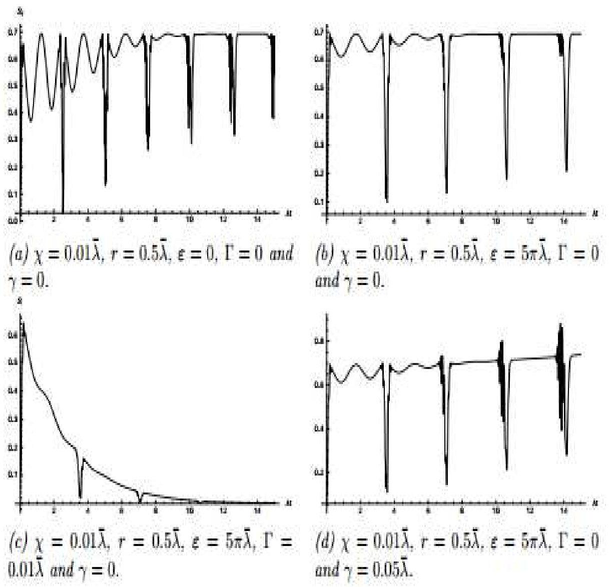


**Fig. 4:** The evolution of Von-Neumann entropy vs the scaled time

In fig (4), We plot Von-Neumann entropy evolving $S_A(t)$ vs the scaled time $\bar{\lambda}t$ in several cases for the system parameters with constant values of $r = 0.5\bar{\lambda}$, $\chi = 0.01\bar{\lambda}$ and $\Delta = 0$. In fig (a), we note that the function starts with quick fluctuations untill it reaches its extreme (maximum and minimum values) and then retracts to show oscillatory movement. In fig (b), we set $\varepsilon = 5\pi$, $\Gamma = 0$ and $\gamma = 0$, we note that the function fluctuations is decreasing, and $S_A(t)$ oscillates at 0.7 and this is the maximum of entanglement . In fig (c), we set $\varepsilon = 5\pi$, $\Gamma = 0.01\bar{\lambda}$ and $\gamma = 0$ ,we note that the function starts with maximum value and it is decreasing to confer that the function refers to disentanglement. In fig (d) ,we set $\varepsilon = 5\pi$, $\Gamma = 0$ and $\gamma = 0.1\bar{\lambda}$ ,we note that $S_A(t)$ oscillates round 0.6 this means that very-weakly entanglement occurs. Admittedly, for the interesting features studied in this paper, even with a decay rate, and field frequency with a corresponding decay rate, practical quantum applications still remains elusive [39, 45]. One major limitation is the relatively low impact of the appearance of Pancharatnam phase as an indicator of quantum entanglement, which leads to an open question of using Pancharatnam phase as a dynamical control of the quantum system.

## 7 Conclusion

In this research article, we have investigated the time-dependent reactions coupling torque between a two-level atoms with a single-mode EMF in a dissipative cavity. We examine the atomic population inversion and we note that the oscillation baseline shifts above, and spreading of revival fluctuation and revivals are squeezed due to the increase in the value of the time-modulation coupling parameter $\varepsilon$, also by increasing the decay parameter of the field $\Gamma$, the atomic population inversion almost dies out as the scaled time developed and by increasing the atomic decay parameter $\gamma$, the mean value of energy decreases with increasing scaled time. We study the Pancharatnam phase, we observe that by increasing the value of the decay parameter of the field $\Gamma$ enhances long-life behavior period. We study the Pegg-Barentt phase, we observe that by increasing the value of the time-modulation coupling parameter $\varepsilon$, the amplitude of P-B phase decreases, also by increasing the decay parameter of the field $\Gamma$, the phase reaches zero. conversely, the atomic decay parameter $\gamma$ has almost no effect on the P-B phase. We study the Von Neumann entropy, we observe that by increasing the effect of field decay parameter $\Gamma$, the entropy quickly reaches the disentangled state (death of entanglement). Conversely, by increasing the atomic decay parameter $\gamma$, entanglement becomes weak.

## Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

## References

[1] D. A. M. Abo-Kahla and M. M. A. Ahmed, Some studies of the interaction between N-two level atoms and three level

atom, Journal of the Egyptian Mathematical Society, **24**, 487-491 (2016).
[2] E. T. Jaynes and F. W. Cummings, Comparison of quantum and semiclassical radiation theories with application to the beam maser, Proceedings of the IEEE, **51**, 89-109 (1963).
[3] A.-H.M. Ahmed, M. N. Zakaria, N. Metwally, Teleportation in the presence of technical defects in transmission stations, Appl. Math & Info. Sci, **6**, 781 (2012).
[4] A.-H. M. Ahmed, L.Y. Cheong, N. Zakaria, N. Metwally, Dynamics of Information Coded in a Single Cooper Pair Box, International Journal of Theoretical Physics, **52**, 1979-1988 (2012).
[5] M. Zidan, A.-H. Abdel-Aty, A. El-Sadek, E. A. Zanaty, and M. Abdel-Aty, Low-cost autonomous perceptron neural network inspired by quantum computation, AIP Conference Proceedings, **1905**, 20005 (2017).
[6] K. Dong-Peng, L. Qing-Hong, A. M. Ashfaq, Entropy squeezing of an atom with a k-photon in the Jaynes-Cummings model, Chinese Physics B, **19**, 014206 (2010).
[7] M. O. Scully, M. S. Zubairy, et al, Quantum Optics, Cambridge University Press, London UK,192-210, (1997).
[8] G. Rempe, H. Walther, and N. Klein, Observation of quantum collapse and revival in a one-atom maser, Physical Review Letters, **58**, 353 (1987).
[9] A.-S. F. Obada, M. M. A. Ahmed, FK Faramawy, and EM Khalil, Entropy and entanglement of the nonlinear Jaynes-Cummings model, Chinese Journal of Physics, **42**, 79-91 (2004).
[10] H.-I. Yoo and J. H. Eberly, Dynamical theory of an atom with two or three levels interacting with quantized cavity fields, Physics Reports, **118**, 239-337 (1985).
[11] B. W. Shore and P. L. Knight, The Jaynes-Cummings model, Journal of Modern Optics **40**, 1195-1238 (1993).
[12] H. Walther, B. T. H. Varcoe, B.-G. Englert, and T. Becker, Cavity quantum electrodynamics, Reports on Progress in Physics, **69**, 1325 (2006).
[13] S. J. D. Phoenix and P. L. Knight, Fluctuations and entropy in models of quantum optical resonance, Annals of Physics, **186**, 381-407 (1988).
[14] D. J. Gauthier, Q. Wu, S. E. Morin, and T. W. Mossberg, Realization of a continuous-wave, two-photon optical laser, Physical Review Letters, **68**, 464 (1992).
[15] R. Salah, A. M. Farouk, A. Farouk, M. Abdel-Aty, H. Eleuch, and A.-S. F. Obada, Entanglement control of two-level atoms in dissipative cavities, Applied Sciences, **10**, 1510 (2020).
[16] M.-F. Fang and G.-H. Zhou, Influence of atomic coherence on the evolution of field entropy in multiphoton processes, Physics Letters A, **184**, 397-402 (1994).
[17] S. J. D. Phoenix and P. L. Knight, Establishment of an entangled atom-field state in the Jaynes-Cummings model, Physical Review A, **44**, 6023 (1991).
[18] S. J. D. Phoenix and P. L. Knight, Comment on collapse and revival of the state vector in the Jaynes- Cummings model: An example of state preparation by a quantum apparatus, Physical Review Letters, **66**, 2833 (1991).
[19] V. Bužek, H. Moya-Cessa, P. L. Knight, and S. J. D. Phoenix, Schrödinger-cat states in the resonant Jaynes-Cummings model: Collapse and revival of oscillations of the photon-number distribution, Physical Review A, **45**, 8190 (1992).
[20] S. Pancharatnam, The adiabatic phase and pancharatnams phase for polarized light, In Proc. Indian Acad. Sci., **44**, 247-262 (1956).
[21] M. V. Berry, In chaotic behavior of deterministic systems, Proc. R. Soc. London, Ser. A, **400**, 229 (1985).
[22] M. S. Swanson, Path integrals and quantum processes,Academic Press, New York, 214-220 (2014).
[23] T. F. Jordan, Berry phases for partial cycles, Physical Review A, **38**, 1590 (1988).
[24] J. Samuel, R. Bhandari and J. Samuel, General setting for Berry's phase, Physical Review Letters, **60**,2339-2342 (1988).
[25] D. Suter, K. T. Mueller, and A. Pines, Study of the Aharonov-Anandan quantum phase by NMR interferometry, Physical Review Letters, **60**, 1218 (1988).
[26] J. J. Slosser, Harmonic oscillator driven by a quantum current, Physical Review Letters, **63**, 934 (1989).
[27] M. Orszag, J. C. Retamal, and C. Saavedra, Preparation of a pure atomic state, Physical Review A, **45**, 2118 (1992).
[28] A.-S F. Obada, M. M. A. Ahmed, A. M. Farouk, and A. Salah, A moving three-level $\Lambda$-type atom in a dissipative cavity, The European Physical Journal D, **71**, 338 (2017).
[29] W. H. Louisell, A. Yariv, and A. E. Siegman, Quantum fluctuations and noise in parametric processes, Physical Review, **124(6)**, 1646 (1961).
[30] E. Sjöqvist, A. K. Pati, A. Ekert, J. S. Anandan, M. Ericsson, D. K. L. Oi, and V. Vedral, Geometric phases for mixed states in interferometry, Physical Review Letters, **85(14)**, 2845 (2000).
[31] Y. Ben-Aryeh, Berry phase and interference effects in three-level atoms, Journal of Modern Optics, **49**, 207-220 (2002).
[32] Y.-N. Guo, C. Yang, Q.-L. Tian, G.-Y. Wang, and K. Zeng, Local quantum uncertainty and interferometric power for a two-qubit system under decoherence channels with memory, Quantum Information Processing, **18**, 375 (2019).
[33] S. M. Barnett and D. T. Pegg, On the Hermitian optical phase operator, Journal of Modern Optics, **36**, 7-19 (1989).
[34] A. Luis and L. Lorenzo Sánchez-Soto, Alternative derivation of the Pegg-Barnett phase operator, Physical Review A, **47**, 1492 (1993).
[35] M. Nelsin and I. Chuang, Quantum Information and Computation, Oxford Press, London UK, 53-60, (1998).
[36] J. Von Neumann, Measurement of fiber widths by the wedge method, Melliand Textile Monthly, New York, 107-119 (1932).
[37] H. Araki and E. H. Lieb, Entropy inequalities, Communications in Mathematical Physics, **18**, 160-170 (1970).
[38] B. M. Schwarzschild, Experiment demonstrates quantum entanglement between atoms a meter apart, Physics Today, **60(11)**, 16 (2007).
[39] M. Sebawe Abdalla, S. S. Hassan, M. Abdel-Aty, Entropic uncertainty in the Jaynes-Cummings model in presence of a second harmonic generation, Optics Communications, **244**(1-6), 431-443 (2005).
[40] M. Abdel-Aty, M. S. Abdalla, A.-S. F. Obada, Entropy and phase properties of isotropic coupled oscillators interacting with a single atom: one- and two-photon processes, Journal of Optics B: Quantum and Semiclassical Optics, **4**(3), 133-141 (2002).

[41] M. Abdel-Aty, S. Abdel-Khalek, A.-S. F. Obada, Pancharatnam phase of two-mode optical fields with Kerr nonlinearity, Optical Review, **7**, 499-504 (2000).
[42] T. M. El-shahat, S. Abdel-khalek, M. Abdel-Aty, A.-S. F. Obada, Entropy squeezing of a degenerate two-photon process with a nonlinear medium, Journal of Modern Optics, **50**, 2013-2030 (2003).
[43] S. Arshed, A. Biswas, M. Abdel-Aty, S. P. Moshokoa, M. Belic, Optical soliton perturbation for Gerdjikov–Ivanov equation via two analytical techniques, Chinese Journal of Physics **56**, 2879-2886 (2018).
[44] M. S. Abdalla, M. Abdel-Aty, A.-S. F. Obada, Degree of entanglement for anisotropic coupled oscillators interacting with a single atom, Journal of Optics B: Quantum and Semiclassical Optics **4**, 396-401 (2002).
[45] J.-S. Zhang, A.-X. Chen, M. Abdel-Aty, Two atoms in dissipative cavities in dispersive limit: Entanglement sudden death and long-lived entanglement, Journal of Physics B: Atomic, Molecular and Optical Physics **43**, 025501 (2010).