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On Qualitative Analysis for Time-Dependent Semi-Linear Fractional Differential Systems

Mohammed M. Matar[1], *Jehad Alzabut*[2,3,*], *Mohamed I. Abbas*[4], *Muath M. Awadallah*[5] *and Nazim I. Mahmudov*[6]

[1] Department of Mathematics, Al-Azhar University-Gaza, Gaza, Palestine
[2] Department of Mathematics and Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia
[3] Department of Industrial Engineering, OSTİM Technical University, Ankara 06374, Türkiye
[4] Department of Mathematics and Computer Science, Faculty of Science, Alexandria University, Alexandria 21511, Egypt
[5] Department of Mathematics, King Faisal University, Al Hufuf, Saudi Arabia
[6] Department of Mathematics, Eastern Mediterranean University, Gazimagusa, Mersin 10, Türkiye

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Abstract: This paper investigates some qualitative properties of solutions for fractional differential systems. Particularly, we focus on existence, uniqueness, priori bounds, and dependence on parameters for the solution. Reported results are proved via fixed point theorems and Pachpatte inequality. Unlike most of previous results, the existence theorem is proved under non-Lipschitzian condition. This work is supported with an example to validate the obtained results applicability.

Keywords: Semi-linear fractional differential system, existence, uniqueness, priori bounds, dependence on parameters, fixed point theorems.

1 Introduction

In this paper, we study some qualitative analysis of solutions for semi-linear fractional system of the form

$$\begin{cases} {}^c\mathscr{D}^{\alpha}x(t) = A(t)x(t) + \phi\left(t, x(t), {}^c\mathscr{D}^{\beta}x(t)\right), t \in (t_0, \tau) \\ M_1 x(t_0) + N_1 x(\tau) = b_1, M_2 x^{'}(t_0) + N_2 x^{'}(\tau) = b_2, \end{cases} \tag{1}$$

where ${}^c\mathscr{D}^{\alpha}$ and ${}^c\mathscr{D}^{\beta}$ are Caputo derivatives with orders $\alpha \in (1,2)$, and $\beta \in (0,1)$ such that $\alpha - \beta > 1$, $x(t) \in \mathbb{R}^n$, $t \in J = [t_0, \tau]$, $x(t_0), x^{'}(t_0), x(\tau), x^{'}(\tau), b_1, b_2 \in \mathbb{R}^n$, $A(t), M_1, M_2, N_1, N_2 \in \mathbb{R}^{n\times n}$, are $n \times n$ matrices such that $M_1 + N_1$, $M_2 + N_2$ are invertible, and $\phi : [t_0, \tau] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function satisfying $\phi(t,0,0) = 0$, $t \in J$.

Very recently, it has been recognized that fractional differential equations provide a meaningful description for many real life processes in various fields of engineering, physics and economics. One can consult the applications of fractional differential equations in [1,2,3,4,5,6], and references therein.

Qualitative properties of solutions are the most interesting features that have attracted numerous researches during the last years; see for examples [7,8,9]. Due to their significance, these features have been further investigated for fractional differential equations [10,11]. In this context, many results have been reported for the purpose of studying different aspects of solutions (see [12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28]).

As is well known, the existence of solutions is essential property prior to any further investigation, the establishment of a priori bounds is of high importance for stability of solutions and the dependence of solutions on parameters is also significant for approximation theory.

Motivated by these facts, we follow this trend and investigate these properties for a type of semi-linear fractional order systems. Specifically, we investigate the existence of solution for the system (1) using the Schafaer's fixed point theorem

* Corresponding author e-mail: jalzabut@psu.edu.sa

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and by Banach fixed point theorem, we obtain sufficient conditions for uniqueness of this system. Moreover, we obtain a priori bound for the solutions of the system using Pachpatte inequality and various investigations on dependence of parameters are performed on the system. Finally, the results are supported by practical example to validate the theoretical results.

It is worthy mentioning that the novelty of results of this article is due to establishing various analytical approaches under non-Lipschitzian condition on semi-linear time dependent fractional differential system with matrices nonlocal conditions.

Overview of paper: In Section 2, we recall notations, basic concepts and preparatory results. In Section 3, we state and prove the existence of the solution for system (1). In Section 4, We obtain sufficient conditions for uniqueness of solution for system (1). In section 5, we obtain a priori bound for the solutions of system. In section 6, some results on dependence of parameters are studied. Finally, the results are supported by practical example.

2 Essential preliminaries

Some fundamental results that will be used later in the sequel are stated in this section.

Definition 1. *The Riemann-Liouville (left-sided) fractional integral of $\phi \in C(J)$ is defined by*

$$I^{\alpha}\psi(t) = (I^{\alpha}\psi(s))(t) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\psi(s)\,ds,\ \alpha > 0.$$

Definition 2. *The Caputo derivative (left-sided) of a function $\psi \in C^{(n)}(J)$ is defined as*

$${}^{c}\mathscr{D}^{\alpha}\psi(t) = \begin{cases} I^{n-\alpha}\psi^{(n)}(t),\ n-1<\alpha<n, \\ \psi^{(n)}(t),\ \alpha = n. \end{cases}$$

The composition of the fractional integral and derivative is given as

$${}^{c}\mathscr{D}^{\alpha}I^{\alpha}\phi(t) = \psi(t),$$
$$I^{\alpha}\,{}^{c}\mathscr{D}^{\alpha}\psi(t) = \psi(t) + \sum_{k=0}^{n-1}c_k(t-t_0)^k.$$

The next result is due to Pachpatte [29] that includes a Volterra and Fredholm integrals.

Lemma 1. *Let $u(t) \in C(J,\mathbb{R}_+)$, $a(t,s)$, $b(t,s) \in C(D,\mathbb{R}_+)$ and $a(t,s)$, $b(t,s)$ be nondecreasing in t for each $s \in J$ and suppose that*

$$u(t) \le c + \int_{t_0}^{t}a(t,s)u(s)\,ds + \int_{t_0}^{\tau}b(t,s)u(s)\,ds,\ t \in J,$$

where $c \ge 0$, and $D = \{(t,s) \in J^2 : t_0 \le s \le t \le \tau\}$. If

$$p(t) = \int_{t_0}^{\tau}b(t,s)\exp\left(\int_{t_0}^{s}a(s,\tau)\,d\tau\right)ds < 1,$$

then

$$u(t) \le \frac{c}{1-p(t)}\exp\left(\int_{t_0}^{t}a(t,s)\,ds\right),\ t \in J.$$

Lemma 2. *Let M_k+N_k, $k=1,2$, be invertible matrices, then the integral solution of the linear fractional differential system*

$$\begin{cases} {}^{c}\mathscr{D}^{\alpha}x(t) = A(t)x(t)+\phi(t),\ t \in (t_0,\tau),\ \alpha \in (1,2), \\ M_1x(t_0)+N_1x(\tau) = b_1,\ M_2x'(t_0)+N_2x'(\tau) = b_2, \end{cases} \tag{2}$$

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is given by

$$
\begin{aligned}
x(t) = & \left(M_1+N_1\right)^{-1} b_1 \\
& +\left[(t-t_0)I_n-(\tau-t_0)\left(M_1+N_1\right)^{-1} N_1\right]\left(M_2+N_2\right)^{-1} b_2 \\
& -\left(M_1+N_1\right)^{-1} N_1 I^\alpha\left(A(\tau)x(\tau)+\phi(\tau)\right)+I^\alpha\left(A(t)x(t)+\phi(t)\right) \\
& +\left[(\tau-t_0)\left(M_1+N_1\right)^{-1} N_1-(t-t_0)I_n\right]\left(M_2+N_2\right)^{-1} N_2 \\
& \times I^{\alpha-1}\left(A(\tau)x(\tau)+\phi(\tau)\right). \qquad (3)
\end{aligned}
$$

Proof. Applying the fractional integral operator I^α to both sides of equation (2), we have

$$x(t)=a_0+a_1(t-t_0)+I^\alpha\left(A(t)x(t)+\phi(t)\right), \qquad (4)$$

for some vectors $a_0, a_1 \in \mathbb{R}^n$. Differentiating $x(t)$, we have

$$x'(t)=a_1+I^{\alpha-1}\left(A(t)x(t)+\phi(t)\right).$$

The boundary conditions imply

$$\begin{cases} \left(M_1+N_1\right)a_0+(\tau-t_0)N_1 a_1=b_1-N_1 I^\alpha\left(A(\tau)x(\tau)+\phi(\tau)\right), \\ a_1=\left(M_2+N_2\right)^{-1} b_2-\left(M_2+N_2\right)^{-1} N_2 I^{\alpha-1}\left(A(\tau)x(\tau)+\phi(\tau)\right). \end{cases}$$

Hence

$$
\begin{aligned}
a_0 = & \left(M_1+N_1\right)^{-1} b_1-\left(M_1+N_1\right)^{-1} N_1 I^\alpha\left(A(\tau)x(\tau)+\phi(\tau)\right) \\
& -(\tau-t_0)\left(M_1+N_1\right)^{-1} N_1\left(M_2+N_2\right)^{-1} b_2 \\
& +(\tau-t_0)\left(M_1+N_1\right)^{-1} N_1\left(M_2+N_2\right)^{-1} N_2 I^{\alpha-1}\left(A(\tau)x(\tau)+\phi(\tau)\right).
\end{aligned}
$$

Substituting a_0, and a_1 in (4) we obtain (3). This finishes the proof.

If $\phi=b_1=b_2=0$, then system (2) has a zero solution. The semigroup property of the fractional integral $I^\alpha I^\beta=I^{\alpha+\beta}$, can be used to obtain the following property:

$$
\begin{aligned}
\left(I^\alpha\phi(t)-I^{\alpha-\varepsilon}\phi_\varepsilon(t)\right) & = \int_{t_0}^{t}\left(\frac{(t-s)^{\alpha-1}\phi(s)}{\Gamma(\alpha)}-\frac{(t-s)^{\alpha-\varepsilon-1}\phi_\varepsilon(s)}{\Gamma(\alpha-\varepsilon)}\right)\phi(s)ds \\
& = \int_{t_0}^{t}(t-s)^{\alpha-\varepsilon-1}\left(\frac{(t-s)^{\varepsilon}\phi(s)}{\Gamma(\alpha)}-\frac{\phi_\varepsilon(s)}{\Gamma(\alpha-\varepsilon)}\right)\phi(s)ds \\
& = I^{\alpha-\varepsilon}\left(\left(\frac{\Gamma(\alpha-\varepsilon)(t-s)^{\varepsilon}\phi(s)}{\Gamma(\alpha)}-\phi_\varepsilon(s)\right)\right)(t). \qquad (5)
\end{aligned}
$$

3 Existence of solution

Consider the Banach space

$$\mathscr{C}=\left\{x: x(t)\in C(J,\mathbb{R}^n), {}^c\mathscr{D}^\beta x(t)\in C(J,\mathbb{R}^n), t\in J\right\},$$

equipping the norm

$$\|x\|_{\mathscr{C}}=\max\left\{\|x\|, \left\|{}^c\mathscr{D}^\beta x\right\|\right\}=\max\left\{\max_{t\in J}\|x(t)\|, \max_{t\in J}\left\|{}^c\mathscr{D}^\beta x(t)\right\|\right\},$$

where $\|x(t)\|$ and $\left\|{}^c\mathscr{D}^\beta x(t)\right\|$ are norms on $\mathbb{R}^n$. Define the operator equation

$$\mathscr{F}x=x,\ x\in\mathscr{C},$$

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where $\mathscr{F}:\mathscr{C}\to\mathscr{C}$ is defined by

$$\begin{aligned}\mathscr{F}x(t) &= (M_1+N_1)^{-1}b_1+\left[(t-t_0)I_n-(\tau-t_0)(M_1+N_1)^{-1}N_1\right](M_2+N_2)^{-1}b_2\\ &-(M_1+N_1)^{-1}N_1I^{\alpha}\left(A(\tau)x(\tau)+\phi\left(\tau,x(\tau),{}^{c}\mathscr{D}^{\beta}x(\tau)\right)\right)\\ &+I^{\alpha}\left(A(t)x(t)+\phi\left(t,x(t),{}^{c}\mathscr{D}^{\beta}x(t)\right)\right)\\ &+\left[(\tau-t_0)(M_1+N_1)^{-1}N_1-(t-t_0)I_n\right](M_2+N_2)^{-1}N_2\\ &\times I^{\alpha-1}\left(A(\tau)x(\tau)+\phi\left(\tau,x(\tau),{}^{c}\mathscr{D}^{\beta}x(\tau)\right)\right). \end{aligned}\tag{6}$$

Differentiating equation (6) (β–times, $0<\beta<1$), we deduce that

$$\begin{aligned}{}^{c}\mathscr{D}^{\beta}\mathscr{F}x(t) &= I^{1-\beta}\frac{d}{dt}\mathscr{F}x(t)=I^{1-\beta}(M_2+N_2)^{-1}b_2\\ &+I^{\alpha-\beta}\left(A(t)x(t)+\phi\left(t,x(t),{}^{c}\mathscr{D}^{\beta}x(t)\right)\right)\\ &-(M_2+N_2)^{-1}N_2I^{\alpha-\beta}\left(A(\tau)x(\tau)+\phi\left(\tau,x(\tau),{}^{c}\mathscr{D}^{\beta}x(\tau)\right)\right). \end{aligned}\tag{7}$$

We observe that problem (1) has solution if the operator (6) has a fixed point. The existence result is obtained by using the Schafaer's fixed point theorem which is based on the following assumption

(H1)$\phi:J\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ is a continuous function and there exist positive real numbers $\gamma_k, k=1,2,3$, such that

$$\begin{aligned}&\left\|A(t)[x(t)-y(t)]+\left[\phi\left(t,x(t),{}^{c}\mathscr{D}^{\beta}x(t)\right)-\phi\left(t,y(t),{}^{c}\mathscr{D}^{\beta}y(t)\right)\right]\right\|\\ &\le\gamma_1+\gamma_2\|x(t)-y(t)\|+\gamma_3\left\|{}^{c}\mathscr{D}^{\beta}x(t)-{}^{c}\mathscr{D}^{\beta}y(t)\right\|,\end{aligned}\tag{8}$$

for all $t\in J$, $x(t),y(t)\in\mathbb{R}^n$. In particular, since $\phi(t,0,0)=0$, we have

$$\left\|A(t)x(t)+\phi\left(t,x(t),{}^{c}\mathscr{D}^{\beta}x(t)\right)\right\|\le\gamma_1+\gamma_2\|x(t)\|+\gamma_3\left\|{}^{c}\mathscr{D}^{\beta}x(t)\right\|,\tag{9}$$

for all $t\in J$, $x(t)\in\mathbb{R}^n$.

(H2)Assume that

$$\Gamma(\alpha+1)>\gamma_2((\theta_1+1)(\tau-t_0)+\alpha\theta_0\theta_2)(\tau-t_0)^{\alpha-1},$$
$$\Gamma(\alpha-\beta+1)>\gamma_3(1+\theta_2)(\tau-t_0)^{\alpha-\beta}.$$

*Remark.*One may observe that condition (8) is of non-Lipschitzian form. However, if $\gamma_1=0$ then it becomes the well known Lipschitz condition.

The next notations will be used to generate compact expressions.

$\vartheta_1=\left\|(M_1+N_1)^{-1}\right\|$, $\vartheta_2=\left\|(M_2+N_2)^{-1}\right\|$, $\kappa_1=\vartheta_1\|b_1\|$, $\kappa_2=\vartheta_2\|b_2\|$

$\theta_0=\max_{t\in J}\left\|(t-t_0)I_n-(\tau-t_0)(M_1+N_1)^{-1}N_1\right\|_n$, $\theta_1=\vartheta_1\|N_1\|$, $\theta_2=\vartheta_2\|N_2\|$.

Theorem 1.*If (H1) and (H2) hold, then the system (1) has at least one solution provided that*

$$\max\left\{\frac{\gamma_2((\theta_1+1)(\tau-t_0)+\alpha\theta_0\theta_2)(\tau-t_0)^{\alpha-1}}{\Gamma(\alpha+1)},\frac{\gamma_3(1+\theta_2)(\tau-t_0)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right\}<1.$$

*Proof.*The proof will be given in several steps.

(Step 1) We show that $\mathscr{F}$ is continuous. Let x_n be a sequence such that $x_n\to x$ in $\mathscr{C}$, then for each $t\in J$, we have

$$\begin{aligned}\|\mathscr{F}x_n(t)-\mathscr{F}x(t)\| &\le\theta_1I^{\alpha}(\|A(\tau)\|\|x_n(\tau)-x(\tau)\|)\\ &+\theta_1I^{\alpha}\left\|\phi\left(\tau,x_n(\tau),{}^{c}\mathscr{D}^{\beta}x_n(\tau)\right)-\phi\left(\tau,x(\tau),{}^{c}\mathscr{D}^{\beta}x(\tau)\right)\right\|\\ &+I^{\alpha}(\|A(t)\|\|x_n(t)-x(t)\|)\\ &+I^{\alpha}\left\|\phi\left(t,x_n(t),{}^{c}\mathscr{D}^{\beta}x_n(t)\right)-\phi\left(t,x(t),{}^{c}\mathscr{D}^{\beta}x(t)\right)\right\|\\ &+\theta_0\theta_2I^{\alpha-1}(\|A(\tau)\|\|x_n(\tau)-x(\tau)\|)\\ &+\theta_0\theta_2I^{-1\alpha}\left\|\phi\left(\tau,x_n(\tau),{}^{c}\mathscr{D}^{\beta}x_n(\tau)\right)-\phi\left(\tau,x(\tau),{}^{c}\mathscr{D}^{\beta}x(\tau)\right)\right\|.\end{aligned}$$

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Similarly, we find

$$\begin{aligned}\left\|{}^{c}\mathscr{D}^{\beta}\mathscr{F}x_n(t)-{}^{c}\mathscr{D}^{\beta}\mathscr{F}x(t)\right\| \leq\ & I^{\alpha-\beta}\left(\|A(t)\|\,\|x_n(t)-x(t)\|\right)\\ &+I^{\alpha-\beta}\left\|\phi\left(t,x_n(t),{}^{c}\mathscr{D}^{\beta}x_n(t)\right)-\phi\left(t,x(t),{}^{c}\mathscr{D}^{\beta}x(t)\right)\right\|\\ &+\theta_2 I^{\alpha-\beta}\left(\|A(\tau)\|\,\|x_n(\tau)-x(\tau)\|\right)\\ &+\theta_2 I^{\alpha-\beta}\left\|\phi\left(\tau,x_n(\tau),{}^{c}\mathscr{D}^{\beta}x_n(\tau)\right)-\phi\left(\tau,x(\tau),{}^{c}\mathscr{D}^{\beta}x(\tau)\right)\right\|.\end{aligned}$$

The continuity of ϕ imply that $\left\|\phi\left(t,x_n(t),{}^{c}\mathscr{D}^{\beta}x_n(t)\right)-\phi\left(t,x(t),{}^{c}\mathscr{D}^{\beta}x(t)\right)\right\|\to 0$ as $n\to\infty$, for every $t\in J$. In virtue of dominated convergence theorem, we deduce that $\mathscr{F}$ is continuous.

(Step 2) We show that $\mathscr{F}$ maps bounded set into bounded set. Indeed, it is enough to show that for $r>0$, there exists a positive constant L, such that if $x\in B_r=\{x\in\mathscr{C}:\|x\|_{\mathscr{C}}\leq r\}$, we have $\|\mathscr{F}x\|_{\mathscr{C}}\leq L$. Using hypothesis (H1), for each $t\in J$, we have

$$\begin{aligned}\|\mathscr{F}x(t)\| \leq\ & \kappa_1+\theta_0\kappa_2+\theta_1 I^{\alpha}\left\|Ax(\tau)+\phi\left(\tau,x(\tau),{}^{c}\mathscr{D}^{\beta}x(\tau)\right)\right\|\\ &+I^{\alpha}\left\|Ax(t)+\phi\left(t,x(t),{}^{c}\mathscr{D}^{\beta}x(t)\right)\right\|+\theta_0\theta_2 I^{\alpha-1}\left\|Ax(\tau)+\phi\left(\tau,x(\tau),{}^{c}\mathscr{D}^{\beta}x(\tau)\right)\right\|\\ \leq\ & \kappa_1+\theta_0\kappa_2+\frac{\theta_1\gamma_1(\tau-t_0)^{\alpha}}{\Gamma(\alpha+1)}+\frac{\gamma_1(t-t_0)^{\alpha}}{\Gamma(\alpha+1)}+\frac{\theta_0\theta_2\gamma_1(\tau-t_0)^{\alpha-1}}{\Gamma(\alpha)}\\ &+\frac{\theta_1\gamma_2 r(\tau-t_0)^{\alpha}}{\Gamma(\alpha+1)}+\frac{\theta_1\gamma_3 r(\tau-t_0)^{\alpha}}{\Gamma(\alpha+1)}+\frac{\gamma_2 r(t-t_0)^{\alpha}}{\Gamma(\alpha+1)}+\frac{\gamma_3 r(t-t_0)^{\alpha}}{\Gamma(\alpha+1)}\\ &+\frac{\theta_0\theta_2\gamma_2 r(\tau-t_0)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\theta_0\theta_2\gamma_3 r(\tau-t_0)^{\alpha-1}}{\Gamma(\alpha)},\end{aligned}$$

and

$$\begin{aligned}\left\|{}^{c}\mathscr{D}^{\beta}\mathscr{F}x(t)\right\| \leq\ & \frac{\kappa_2(t-t_0)^{1-\beta}}{\Gamma(2-\beta)}+\frac{\gamma_1(t-t_0)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\\ &+\frac{\theta_2\gamma_1(\tau-t_0)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+\frac{\gamma_2 r(t-t_0)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+\frac{\gamma_3 r(t-t_0)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\\ &+\frac{\theta_2\gamma_2 r(\tau-t_0)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+\frac{\theta_2\gamma_3 r(\tau-t_0)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}.\end{aligned}$$

Therefore

$$\begin{aligned}\|\mathscr{F}x\|_{\mathscr{C}} \leq\ & \kappa_1+\theta_0\kappa_2+\frac{\kappa_2(\tau-t_0)^{1-\beta}}{\Gamma(2-\beta)}\\ &+\frac{(\tau-t_0)^{\alpha-1}\left((\theta_1+1)(\tau-t_0)+\alpha\theta_0\theta_2\right)(\gamma_1+r(\gamma_2+\gamma_3))}{\Gamma(\alpha)}\\ &+\frac{(\theta_2+1)(\tau-t_0)^{\alpha-\beta}(\gamma_1+r(\gamma_2+\gamma_3))}{\Gamma(\alpha-\beta+1)}.\end{aligned}$$

This shows that $\|\mathscr{F}x\|_{\mathscr{C}}\leq L$, where

$$\begin{aligned}L:=\ & \kappa_1+\theta_0\kappa_2+\frac{\kappa_2(\tau-t_0)^{1-\beta}}{\Gamma(2-\beta)}\\ &+\frac{(\tau-t_0)^{\alpha-1}\left((\theta_1+1)(\tau-t_0)+\alpha\theta_0\theta_2\right)(\gamma_1+r_2(\gamma_2+\gamma_3))}{\Gamma(\alpha)}\\ &+\frac{(\theta_2+1)(\tau-t_0)^{\alpha-\beta}(\gamma_1+r(\gamma_2+\gamma_3))}{\Gamma(\alpha-\beta+1)}.\end{aligned}$$

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(Step 3) We show that $\mathscr{F}$ maps bounded set into equicontinuous set of $\mathscr{C}$. Let $t_1,\ t_2 \in J$ such that $t_1 < t_2$. Let B_r be a bounded set of $\mathscr{C}$ defined as in step 2. Let $x \in B_r$, then

$$\begin{aligned}
\|\mathscr{F}x(t_2) - \mathscr{F}x(t_1)\| \le\ & \kappa_2 (t_2 - t_1) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} \left| (t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right| \\
& \times \left\| A(s)(x(s)) + \phi\left(s, x(s), {}^{c}\mathscr{D}^{\beta} x(s)\right) \right\| ds \\
& + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \left\| A(s)(x(s)) + \phi\left(s, x(s), {}^{c}\mathscr{D}^{\beta} x(s)\right) \right\| ds \\
& + \frac{\theta_2 (t_2 - t_1)}{\Gamma(\alpha - 1)} \int_{t_0}^{\tau} (\tau - s)^{\alpha-2} \left\| A(s)(x(s)) + \phi\left(s, x(s), {}^{c}\mathscr{D}^{\beta} x(s)\right) \right\| ds \\
\le\ & \kappa_2 (t_2 - t_1) + \frac{(\gamma_1 + r(\gamma_2 + \gamma_3)) \left[(t_2 - t_0)^{\alpha} - (t_1 - t_0)^{\alpha} + 2(t_2 - t_1)^{\alpha}\right]}{\Gamma(\alpha + 1)} \\
& + \frac{\theta_2 (\tau - t_0)^{\alpha-1} (\gamma_1 + r(\gamma_2 + \gamma_3)) (t_2 - t_1)}{\Gamma(\alpha)},
\end{aligned}$$

and

$$\begin{aligned}
\left\| {}^{c}\mathscr{D}^{\beta} \mathscr{F}x(t_2) - {}^{c}\mathscr{D}^{\beta} \mathscr{F}x(t_1) \right\| \le\ & \frac{1}{\Gamma(\alpha - \beta)} \int_{t_0}^{t_1} \left| (t_2 - s)^{\alpha-\beta-1} - (t_1 - s)^{\alpha-1} \right| \\
& \times \left\| A(s)(x(s)) + \phi\left(s, x(s), {}^{c}\mathscr{D}^{\beta} x(s)\right) \right\| ds \\
& + \frac{1}{\Gamma(\alpha - \beta)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-\beta-1} \left\| A(s)(x(s)) + \phi\left(s, x(s), {}^{c}\mathscr{D}^{\beta} x(s)\right) \right\| ds \\
\le\ & \frac{(\gamma_1 + r(\gamma_2 + \gamma_3)) \left[(t_2 - t_0)^{\alpha-\beta} - (t_1 - t_0)^{\alpha-\beta} + 2(t_2 - t_1)^{\alpha-\beta}\right]}{\Gamma(\alpha - \beta + 1)}.
\end{aligned}$$

As $t_1 \to t_2$, the right-hand sides of the above inequalities tend to zero. This shows the equicontinuity for the set $\mathscr{F}B_r \subset \mathscr{C}$.

As a consequence of steps 1 to 3 together with the Arzela-Ascoli theorem, we deduce that $\mathscr{F} : \mathscr{C} \to \mathscr{C}$ is completely continuous.

(Step 4) We show that the set of solutions of the system (1) is bounded. This is equivalent to showing that, for each $\lambda \in (0,1)$, the solutions of the following family of boundary value problems

$$\begin{cases}
{}^{c}\mathscr{D}^{\alpha} x(t) = \lambda A(t)x(t) + \lambda \phi\left(t, x(t), {}^{c}\mathscr{D}^{\beta} x(t)\right),\ t > t_0, \\
M_1 x(t_0) + N_1 x(\tau) = \lambda b_1, \\
M_2 x'(t_0) + N_2 x'(\tau) = \lambda b_2,
\end{cases} \tag{10}$$

are bounded, with the bound independent of λ. Let x_λ be a solution to (10) for fixed $\lambda \in (0,1)$. Then by Lemma 2, x_λ satisfies that

$$\begin{aligned}
x_\lambda(t) =\ & \lambda (M_1 + N_1)^{-1} b_1 + \lambda \left[(t - t_0) I_n - (\tau - t_0)(M_1 + N_1)^{-1} N_1\right] (M_2 + N_2)^{-1} b_2 \\
& - \lambda (M_1 + N_1)^{-1} N_1 I^{\alpha} \left(A(\tau)x(\tau) + \phi\left(\tau, x(\tau), {}^{c}\mathscr{D}^{\beta} x(\tau)\right)\right) \\
& + \lambda I^{\alpha} \left(A(t)x(t) + \phi\left(t, x(t), {}^{c}\mathscr{D}^{\beta} x(t)\right)\right) \\
& + \lambda \left[(\tau - t_0)(M_1 + N_1)^{-1} N_1 - (t - t_0) I_n\right] (M_2 + N_2)^{-1} N_2 \\
& \times I^{\alpha-1} \left(A(\tau)x(\tau) + \phi\left(\tau, x(\tau), {}^{c}\mathscr{D}^{\beta} x(\tau)\right)\right).
\end{aligned}$$

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Therefore, as in Step 2, we deduce that

$$\begin{aligned}\|x_\lambda\| \leq\ & \kappa_1+\theta_0\kappa_2+\frac{\theta_1\gamma_1(\tau-t_0)^{\alpha}}{\Gamma(\alpha+1)}+\frac{\gamma_1(\tau-t_0)^{\alpha}}{\Gamma(\alpha+1)}+\frac{\theta_0\theta_2\gamma_1(\tau-t_0)^{\alpha-1}}{\Gamma(\alpha)}\\&+\frac{\theta_1\gamma_2\|x_\lambda\|(\tau-t_0)^{\alpha}}{\Gamma(\alpha+1)}+\frac{\theta_1\gamma_3\left\|{}^c\mathscr{D}^{\beta}x_\lambda\right\|(\tau-t_0)^{\alpha}}{\Gamma(\alpha+1)}\\&+\frac{\gamma_2\|x_\lambda\|(\tau-t_0)^{\alpha}}{\Gamma(\alpha+1)}+\frac{\gamma_3\left\|{}^c\mathscr{D}^{\beta}x_\lambda\right\|(\tau-t_0)^{\alpha}}{\Gamma(\alpha+1)}\\&+\frac{\theta_0\theta_2\|x_\lambda\|(\tau-t_0)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\theta_0\theta_2\gamma_3\left\|{}^c\mathscr{D}^{\beta}x_\lambda\right\|(\tau-t_0)^{\alpha-1}}{\Gamma(\alpha)},\end{aligned}$$

and

$$\begin{aligned}\left\|{}^c\mathscr{D}^{\beta}x_\lambda\right\| \leq\ & \frac{\kappa_2(\tau-t_0)^{1-\beta}}{\Gamma(2-\beta)}+\frac{\gamma_1(\tau-t_0)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\\&+\frac{\theta_2\gamma_1(\tau-t_0)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+\frac{\gamma_2\|x_\lambda\|(\tau-t_0)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+\frac{\gamma_3\left\|{}^c\mathscr{D}^{\beta}x_\lambda\right\|(\tau-t_0)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\\&+\frac{\theta_2\gamma_2\|x_\lambda\|(\tau-t_0)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+\frac{\theta_2\gamma_3\left\|{}^c\mathscr{D}^{\beta}x_\lambda\right\|(\tau-t_0)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}.\end{aligned}$$

Hence, we get

$$\begin{aligned}\|x_\lambda\| \leq\ & \frac{(\kappa_1+\theta_0\kappa_2)\Gamma(\alpha+1)}{\Gamma(\alpha+1)-\gamma_2((\theta_1+1)(\tau-t_0)+\alpha\theta_0\theta_2)(\tau-t_0)^{\alpha-1}}\\&+\frac{\gamma_1((\theta_1+1)(\tau-t_0)+\alpha\theta_0\theta_2)(\tau-t_0)^{\alpha-1}}{\Gamma(\alpha+1)-\gamma_2((\theta_1+1)(\tau-t_0)+\alpha\theta_0\theta_2)(\tau-t_0)^{\alpha-1}}\\&+\frac{\gamma_3(\theta_1+1)(\tau-t_0)+\alpha\theta_0\theta_2(\tau-t_0)^{\alpha-1}}{\Gamma(\alpha+1)-\gamma_2((\theta_1+1)(\tau-t_0)+\alpha\theta_0\theta_2)(\tau-t_0)^{\alpha-1}}\left\|{}^c\mathscr{D}^{\beta}x_\lambda\right\|,\end{aligned}$$

and

$$\begin{aligned}\left\|{}^c\mathscr{D}^{\beta}x_\lambda\right\| \leq\ & \frac{\kappa_2(\tau-t_0)^{1-\beta}\Gamma(\alpha-\beta+1)}{\Gamma(2-\beta)\left(\Gamma(\alpha-\beta+1)-\gamma_3(1+\theta_2)(\tau-t_0)^{\alpha-\beta}\right)}\\&+\frac{\gamma_1(1+\theta_2)(\tau-t_0)^{\alpha-\beta}}{\left(\Gamma(\alpha-\beta+1)-\gamma_3(1+\theta_2)(\tau-t_0)^{\alpha-\beta}\right)}\\&+\frac{\gamma_2(1+\theta_2)(\tau-t_0)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)-\gamma_3(1+\theta_2)(\tau-t_0)^{\alpha-\beta}}\|x_\lambda\|.\end{aligned}$$

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We deduce.

$$
\begin{aligned}
\|x_\lambda\| \leq & \frac{\zeta^{-1}(\kappa_1+\theta_0\kappa_2)\Gamma(\alpha+1)}{\Gamma(\alpha+1)-\gamma_2((\theta_1+1)(\tau-t_0)+\alpha\theta_0\theta_2)(\tau-t_0)^{\alpha-1}} \\
& +\frac{\zeta^{-1}\gamma_1((\theta_1+1)(\tau-t_0)+\alpha\theta_0\theta_2)(\tau-t_0)^{\alpha-1}}{\Gamma(\alpha+1)-\gamma_2((\theta_1+1)(\tau-t_0)+\alpha\theta_0\theta_2)(\tau-t_0)^{\alpha-1}} \\
& +\frac{\zeta^{-1}\gamma_3\left((\theta_1+1)(\tau-t_0)+\alpha\theta_0\theta_2(\tau-t_0)^{\alpha-1}\right)}{\Gamma(\alpha+1)-\gamma_2((\theta_1+1)(\tau-t_0)+\alpha\theta_0\theta_2)(\tau-t_0)^{\alpha-1}} \\
& \times\frac{\zeta^{-1}\kappa_2(\tau-t_0)^{1-\beta}\Gamma(\alpha-\beta+1)}{\Gamma(2-\beta)\left(\Gamma(\alpha-\beta+1)-\gamma_3(1+\theta_2)(\tau-t_0)^{\alpha-\beta}\right)} \\
& +\frac{\zeta^{-1}\gamma_3\left((\theta_1+1)(\tau-t_0)+\alpha\theta_0\theta_2(\tau-t_0)^{\alpha-1}\right)}{\Gamma(\alpha+1)-\gamma_2((\theta_1+1)(\tau-t_0)+\alpha\theta_0\theta_2)(\tau-t_0)^{\alpha-1}} \\
& \times\frac{\zeta^{-1}\gamma_1(1+\theta_2)(\tau-t_0)^{\alpha-\beta}}{\left(\Gamma(\alpha-\beta+1)-\gamma_3(1+\theta_2)(\tau-t_0)^{\alpha-\beta}\right)},
\end{aligned}
$$

where $\zeta^{-1}=1-\frac{\gamma_3(\theta_1+1)(\tau-t_0)+\alpha\theta_0\theta_2(\tau-t_0)^{\alpha-1}}{\gamma_2(1+\theta_2)(\tau-t_0)^{\alpha-\beta}}\frac{\Gamma(\alpha+1)-\gamma_2((\theta_1+1)(\tau-t_0)+\alpha\theta_0\theta_2)(\tau-t_0)^{\alpha-1}}{\Gamma(\alpha-\beta+1)-\gamma_3(1+\theta_2)(\tau-t_0)^{\alpha-\beta}}$. The right side of the above inequality is a bound of any solution of system (10) which is independent of the value λ.

Hence, Schafaer's theorem now can be applied to yield the existence of at least one solution $x(t)$ satisfying the operator fixed point equation $\mathscr{F}x=x,\ x\in\mathscr{C}$, which means that the system (1) has at least one solution.

4 Uniqueness result

The simplest way to get the uniqueness of the solution of the problem (1) is investigating the validity of Banach fixed point theorem. To use this theorem, we have to assume the Lipschitz condition on the nonlinear term in the right-side of equation (1). Hence, we assume that $\gamma_1=0$ in the assumption (H1) and we use the next alternative assumption.

(H3)There exist positive real numbers γ_2,γ_3 such that

$$
\begin{aligned}
&\left\|A(t)[x(t)-y(t)]+[\phi(t,x(t),{}^c\mathscr{D}^\beta x(t))-\phi(t,y(t),{}^c\mathscr{D}^\beta y(t))]\right\| \\
&\leq \gamma_2\|x(t)-y(t)\|+\gamma_3\|{}^c\mathscr{D}^\beta x(t)-{}^c\mathscr{D}^\beta y(t)\|.
\end{aligned}
$$

From (7), we have

$$
\begin{aligned}
\|{}^c\mathscr{D}^\beta x(t)-{}^c\mathscr{D}^\beta y(t)\| \leq & I^{\alpha-\beta}\left\|A(t)[x(t)-y(t)]+[\phi(t,x(t),{}^c\mathscr{D}^\beta x(t))-\phi(t,y(t),{}^c\mathscr{D}^\beta y(t))]\right\| \\
& +\theta_2 I^{\alpha-\beta}\|A(\tau)[x(\tau)-y(\tau)] \\
& +[\phi(\tau,x(\tau),{}^c\mathscr{D}^\beta x(\tau))-\phi(\tau,y(\tau),{}^c\mathscr{D}^\beta y(\tau))]\Big\| \\
\leq & \left((\theta_2+1)\frac{(\tau-t_0)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)\gamma_2\|x-y\| \\
& +\left((\theta_2+1)\frac{(\tau-t_0)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)\gamma_3\|{}^c\mathscr{D}^\beta x-{}^c\mathscr{D}^\beta y\|,
\end{aligned}
$$

which implies that

$$
\|{}^c\mathscr{D}^\beta x-{}^c\mathscr{D}^\beta y\| \leq \frac{(\theta_2+1)(\tau-t_0)^{\alpha-\beta}\gamma_2}{\Gamma(\alpha-\beta+1)-\gamma_3(\theta_2+1)(\tau-t_0)^{\alpha-\beta}}\|x-y\|. \tag{11}
$$

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Theorem 2. *Assume that (H2) and (H3) are satisfied. Then the system (1) has a unique solution on J provided that $\Delta < 1$, where*

$$\Delta := \gamma_2(\tau-t_0)^{\alpha-1}\max\left\{\left[\frac{(\alpha\theta_0\theta_2+(\theta_1+1)(\tau-t_0))}{\Gamma(\alpha+1)} + \frac{\gamma_3(\alpha\theta_0\theta_2+(\theta_1+1)(\tau-t_0))(\theta_2+1)(\tau-t_0)^{\alpha-\beta}}{\Gamma(\alpha)\left(\Gamma(\alpha-\beta+1)-\gamma_3(\theta_2+1)(\tau-t_0)^{\alpha-\beta}\right)}\right], \frac{(\theta_2+1)(\tau-t_0)^{1-\beta}}{\Gamma(\alpha-\beta+1)-\gamma_3(\theta_2+1)(\tau-t_0)^{\alpha-\beta}}\right\}. \tag{12}$$

Proof. Let $x,y\in\mathscr{C}$. For each $t\in J$, using (11), we have

$$\begin{aligned}
\|\mathscr{F}x(t)-\mathscr{F}y(t)\| &\le \theta_1 I^{\alpha}\left\|A(\tau)[x(\tau)-y(\tau)]+[\phi(\tau,x(\tau),{}^c\mathscr{D}^{\beta}x(\tau))-\phi(\tau,y(\tau),{}^c\mathscr{D}^{\beta}y(\tau))]\right\| \\
&+ I^{\alpha}\left\|A(t)[x(t)-y(t)]+[\phi(t,x(t),{}^c\mathscr{D}^{\beta}x(t))-\phi(t,y(t),{}^c\mathscr{D}^{\beta}y(t))]\right\| \\
&+ \theta_0\theta_2 I^{\alpha-1}\left\|A(\tau)[x(\tau)-y(\tau)]+[\phi(\tau,x(\tau),{}^c\mathscr{D}^{\beta}x(\tau))-\phi(\tau,y(\tau),{}^c\mathscr{D}^{\beta}y(\tau))]\right\| \\
&\le \theta_1 I^{\alpha}\left(\gamma_2\|x(\tau)-y(\tau)\|+\gamma_3\|{}^c\mathscr{D}^{\beta}x(\tau)-{}^c\mathscr{D}^{\beta}y(\tau)\|\right) \\
&+ I^{\alpha}\left(\gamma_2\|x(t)-y(t)\|+\gamma_3\|{}^c\mathscr{D}^{\beta}x(t)-{}^c\mathscr{D}^{\beta}y(t)\|\right) \\
&+ \theta_0\theta_2 I^{\alpha-1}\left(\gamma_2\|x(\tau)-y(\tau)\|+\gamma_3\|{}^c\mathscr{D}^{\beta}x(\tau)-{}^c\mathscr{D}^{\beta}y(\tau)\|\right) \\
&\le \theta_1\frac{(\tau-t_0)^{\alpha}}{\Gamma(\alpha+1)}\left(\gamma_2\|x-y\|+\gamma_3\|{}^c\mathscr{D}^{\beta}x-{}^c\mathscr{D}^{\beta}y\|\right) \\
&+ \frac{(\tau-t_0)^{\alpha}}{\Gamma(\alpha+1)}\left(\gamma_2\|x-y\|+\gamma_3\|{}^c\mathscr{D}^{\beta}x-{}^c\mathscr{D}^{\beta}y\|\right) \\
&+ \theta_0\theta_2\frac{(\tau-t_0)^{\alpha-1}}{\Gamma(\alpha)}\left(\gamma_2\|x-y\|+\gamma_3\|{}^c\mathscr{D}^{\beta}x-{}^c\mathscr{D}^{\beta}y\|\right) \\
&= \frac{\gamma_2(\tau-t_0)^{\alpha-1}}{\Gamma(\alpha+1)}(\alpha\theta_0\theta_2+(\theta_1+1)(\tau-t_0))\|x-y\| \\
&+ \frac{\gamma_3(\tau-t_0)^{\alpha-1}}{\Gamma(\alpha)}(\alpha\theta_0\theta_2+(\theta_1+1)(\tau-t_0))\|{}^c\mathscr{D}^{\beta}x-{}^c\mathscr{D}^{\beta}y\| \\
&\le \left[\frac{\gamma_2(\tau-t_0)^{\alpha-1}}{\Gamma(\alpha+1)}(\alpha\theta_0\theta_2+(\theta_1+1)(\tau-t_0)) + \frac{\frac{\gamma_2\gamma_3}{\Gamma(\alpha)}(\alpha\theta_0\theta_2+(\theta_1+1)(\tau-t_0))(\theta_2+1)(\tau-t_0)^{2\alpha-\beta-1}}{\Gamma(\alpha-\beta+1)-\gamma_3(\theta_2+1)(\tau-t_0)^{\alpha-\beta}}\right]\|x-y\|.
\end{aligned}$$

Similarly, using (11), we have

$$\begin{aligned}
\|{}^c\mathscr{D}^{\beta}\mathscr{F}x(t)-{}^c\mathscr{D}^{\beta}\mathscr{F}y(t)\| &\le I^{\alpha-\beta}\left\|A(t)[x(t)-y(t)]+[\phi(t,x(t),{}^c\mathscr{D}^{\beta}x(t))-\phi(t,y(t),{}^c\mathscr{D}^{\beta}y(t))]\right\| \\
&+ \theta_2 I^{\alpha-\beta}\left\|A(\tau)[x(\tau)-y(\tau)]+[\phi(\tau,x(\tau),{}^c\mathscr{D}^{\beta}x(\tau))-\phi(\tau,y(\tau),{}^c\mathscr{D}^{\beta}y(\tau))]\right\| \\
&\le \frac{\gamma_2(\theta_2+1)(\tau-t_0)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\|x-y\| \\
&+ \frac{\gamma_3(\theta_2+1)(\tau-t_0)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\frac{(\theta_2+1)(\tau-t_0)^{\alpha-\beta}\gamma_2}{\Gamma(\alpha-\beta+1)-\gamma_3(\theta_2+1)(\tau-t_0)^{\alpha-\beta}}\|x-y\| \\
&\le \frac{\gamma_2(\theta_2+1)(\tau-t_0)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)-\gamma_3(\theta_2+1)(\tau-t_0)^{\alpha-\beta}}\|x-y\|.
\end{aligned}$$

Therefore, we get

$$\|\mathscr{F}x-\mathscr{F}y\|_{\mathscr{C}} \le \Delta\|x-y\|,$$

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where Δ as defined in (12). Thus, $\mathscr{F}$ is contraction. Hence, Banach's fixed point theorem guarantees a fixed point which is the unique solution of the system (1). The proof is completed.

5 Priori bound of solution

In this section we investigate the existence of a priori bounds for the solutions of system (1). The current results provide geometric insight into the potential solutions of system (1) by providing an estimate on their size and location without having an explicit knowledge of solutions. In the previous sections, we proved that system (1) has at least one solution, hence, it can be assumed metaphorically in this section that system (1) has a set of solutions. In accordance with Lemma 1, we use the following notations

$$\begin{aligned} c = {} & \kappa_1 + \theta_0\kappa_2 + \frac{\gamma_1\left(\alpha\theta_0\theta_2 + (\theta_1+1)(\tau-t_0)\right)(\tau-t_0)^{\alpha-1}}{\Gamma(\alpha+1)} \\ & + \frac{\gamma_1\gamma_3(1+\theta_1)(1+\theta_2)(\tau-t_0)^{2\alpha-\beta}}{\Gamma(\alpha+1)\left(\Gamma(\alpha-\beta+1)-\gamma_3(1+\theta_2)(\tau-t_0)^{\alpha-\beta}\right)} \\ & + \frac{\kappa_2\gamma_3(1+\theta_1)\Gamma(\alpha-\beta+1)(\tau-t_0)^{\alpha-\beta+1}}{\Gamma(\alpha+1)\Gamma(2-\beta)\left(\Gamma(\alpha-\beta+1)-\gamma_3(1+\theta_2)(\tau-t_0)^{\alpha-\beta}\right)} \\ & + \frac{\theta_0\theta_2\gamma_1\gamma_3(1+\theta_2)(\tau-t_0)^{2\alpha-\beta-1}}{\Gamma(\alpha)\left(\Gamma(\alpha-\beta+1)-\gamma_3(1+\theta_2)(\tau-t_0)^{\alpha-\beta}\right)} \\ & + \frac{\theta_0\theta_2\gamma_3\kappa_2\Gamma(\alpha-\beta+1)(\tau-t_0)^{\alpha-\beta}}{\Gamma(\alpha)\Gamma(2-\beta)\left(\Gamma(\alpha-\beta+1)-\gamma_3(1+\theta_2)(\tau-t_0)^{\alpha-\beta}\right)}, \end{aligned}$$

$$\begin{aligned} a(t,s) = {} & \gamma_2(t-s)^{\alpha-1}\left[\frac{1}{\Gamma(\alpha)} + \frac{\gamma_3(1+\theta_1)\Gamma(\alpha-\beta+1)(t-s)^{\alpha-\beta}}{\Gamma(2\alpha-\beta)\left(\Gamma(\alpha-\beta+1)-\gamma_3(1+\theta_2)(\tau-t_0)^{\alpha-\beta}\right)}\right. \\ & \left. + \frac{\theta_0\theta_2\gamma_3\Gamma(\alpha-\beta+1)(t-s)^{\alpha-\beta-1}}{\Gamma(2\alpha-\beta-1)\left(\Gamma(\alpha-\beta+1)-\gamma_3(1+\theta_2)(\tau-t_0)^{\alpha-\beta}\right)}\right], \end{aligned}$$

$$\begin{aligned} b(t,s) = {} & \gamma_2(\tau-s)^{\alpha-2}\left[\frac{\theta_1(t-s)}{\Gamma(\alpha)} + \frac{\theta_0\theta_2}{\Gamma(\alpha-1)}\right. \\ & \left. + \frac{\theta_2\gamma_3\Gamma(\alpha-\beta+1)\left(\theta_0\theta_2(2\alpha-\beta-1)+(1+\theta_1)(\tau-s)\right)(\tau-s)^{\alpha-\beta}}{\Gamma(2\alpha-\beta)\left(\Gamma(\alpha-\beta+1)-\gamma_3(1+\theta_2)(\tau-t_0)^{\alpha-\beta}\right)}\right], \end{aligned}$$

and

$$p(t) = \int_{t_0}^{\tau} b(t,s)\exp\left(\int_{t_0}^{s} a(s,\tau)\,d\tau\right)ds.$$

Theorem 3. *If (H1) and (H2) hold, then, for $t \in J$, all solutions x of system (1) satisfying the priori bound*

$$\|x(t)\| \le \frac{c}{1-p(t)}\exp\left(\int_{t_0}^{t} a(t,s)\,ds\right), \; t \in J.$$

provided that $p(t) < 1$, $t \in J$.

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Proof. Let x be a solution of system (1) on J. If x is a zero solution, then the result is obvious. Hence, we assume that x is a nonzero vector function. In accordance with Lemma 2, we have

$$\begin{aligned}x(t) =& (M_1+N_1)^{-1}b_1\\&+\left[(t-t_0)I_n-(\tau-t_0)(M_1+N_1)^{-1}N_1\right](M_2+N_2)^{-1}b_2\\&-(M_1+N_1)^{-1}N_1I^{\alpha}\left(Ax(\tau)+\phi\left(\tau,x(\tau),{}^{c}\mathscr{D}^{\beta}x(\tau)\right)\right)\\&+I^{\alpha}\left(Ax(t)+\phi\left(t,x(t),{}^{c}\mathscr{D}^{\beta}x(t)\right)\right)\\&+\left[(\tau-t_0)(M_1+N_1)^{-1}N_1-(t-t_0)I_n\right](M_2+N_2)^{-1}N_2\\&\times I^{\alpha-1}\left(Ax(\tau)+\phi\left(\tau,x(\tau),{}^{c}\mathscr{D}^{\beta}x(\tau)\right)\right).\end{aligned}$$

Then,

$$\begin{aligned}\|x(t)\| \le& \kappa_1+\theta_0\kappa_2+\theta_1\, I^{\alpha}\left\|Ax(\tau)+\phi\left(\tau,x(\tau),{}^{c}\mathscr{D}^{\beta}x(\tau)\right)\right\|\\&+\ I^{\alpha}\left\|Ax(t)+\phi\left(t,x(t),{}^{c}\mathscr{D}^{\beta}x(t)\right)\right\|\\&+\ \theta_0\theta_2I^{\alpha-1}\left\|Ax(\tau)+\phi\left(\tau,x(\tau),{}^{c}\mathscr{D}^{\beta}x(\tau)\right)\right\|,\end{aligned}$$

and

$$\begin{aligned}\left\|{}^{c}\mathscr{D}^{\beta}x(t)\right\| \le& I^{1-\beta}\kappa_2+I^{\alpha-\beta}\left\|A(t)x(t)+\phi\left(t,x(t),{}^{c}\mathscr{D}^{\beta}x(t)\right)\right\|\\&+\ \theta_2I^{\alpha-\beta}\left(A(\tau)x(\tau)+\phi\left(\tau,x(\tau),{}^{c}\mathscr{D}^{\beta}x(\tau)\right)\right).\end{aligned}$$

Condition (9) implies

$$\begin{aligned}\|x(t)\| \le& \kappa_1+\theta_0\kappa_2+\theta_1I^{\alpha}\left(\gamma_1+\gamma_2\|x(\tau)\|+\gamma_3\left\|{}^{c}\mathscr{D}^{\beta}x(\tau)\right\|\right)\\&+\ I^{\alpha}\left(\gamma_1+\gamma_2\|x(t)\|+\gamma_3{}^{c}\mathscr{D}^{\beta}\|x(t)\|\right)\\&+\ \theta_0\theta_2I^{\alpha-1}\left(\gamma_1+\gamma_2\|x(\tau)\|+\gamma_3\left\|{}^{c}\mathscr{D}^{\beta}x(\tau)\right\|\right)\\=&\ \kappa_1+\theta_0\kappa_2\\&+\ \theta_1\left(\frac{\gamma_1(\tau-t_0)^{\alpha}}{\Gamma(\alpha+1)}+\gamma_2I^{\alpha}\|x(\tau)\|+\gamma_3I^{\alpha}\left\|{}^{c}\mathscr{D}^{\beta}x(\tau)\right\|\right)\\&+\ \frac{\gamma_1(t-t_0)^{\alpha}}{\Gamma(\alpha+1)}+\gamma_2I^{\alpha}\|x(t)\|+\gamma_3I^{\alpha}\left\|{}^{c}\mathscr{D}^{\beta}x(t)\right\|\\&+\ \theta_0\theta_2\left(\frac{\gamma_1(\tau-t_0)^{\alpha-1}}{\Gamma(\alpha)}+\gamma_2I^{\alpha-1}\|x(\tau)\|+\gamma_3I^{\alpha-1}\,{}^{c}\mathscr{D}^{\beta}\|x(\tau)\|\right),\end{aligned}$$

and

$$\begin{aligned}\left\|{}^{c}\mathscr{D}^{\beta}x(t)\right\| \le& \frac{\kappa_2(\tau-t_0)^{1-\beta}}{\Gamma(2-\beta)}+\frac{\gamma_1(\tau-t_0)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+\frac{\theta_2\gamma_1(\tau-t_0)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\\&+\ \gamma_2I^{\alpha-\beta}\|x(t)\|+\theta_2\gamma_2I^{\alpha-\beta}\|x(\tau)\|\\&+\ \gamma_3I^{\alpha-\beta}\left\|{}^{c}\mathscr{D}^{\beta}x(t)\right\|+\theta_2\gamma_3I^{\alpha-\beta}\left\|{}^{c}\mathscr{D}^{\beta}x(\tau)\right\|.\end{aligned}$$

This inequality can be reduced to

$$\begin{aligned}\left\|{}^{c}\mathscr{D}^{\beta}x\right\| \le& \frac{\kappa_2\Gamma(\alpha-\beta+1)(\tau-t_0)^{1-\beta}}{\Gamma(2-\beta)\left(\Gamma(\alpha-\beta+1)-\gamma_3(1+\theta_2)(\tau-t_0)^{\alpha-\beta}\right)}\\&+\frac{\gamma_1\Gamma(\alpha-\beta+1)(\tau-t_0)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)\left(\Gamma(\alpha-\beta+1)-\gamma_3(1+\theta_2)(\tau-t_0)^{\alpha-\beta}\right)}\\&+\frac{\theta_2\gamma_1\Gamma(\alpha-\beta+1)(\tau-t_0)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)\left(\Gamma(\alpha-\beta+1)-\gamma_3(1+\theta_2)(\tau-t_0)^{\alpha-\beta}\right)}\\&+\frac{\gamma_2\Gamma(\alpha-\beta+1)}{\left(\Gamma(\alpha-\beta+1)-\gamma_3(1+\theta_2)(\tau-t_0)^{\alpha-\beta}\right)}I^{\alpha-\beta}\|x(t)\|\\&+\frac{\theta_2\gamma_2\Gamma(\alpha-\beta+1)}{\left(\Gamma(\alpha-\beta+1)-\gamma_3(1+\theta_2)(\tau-t_0)^{\alpha-\beta}\right)}I^{\alpha-\beta}\|x(\tau)\|.\end{aligned}$$

Therefore,

$$
\begin{aligned}
\|x(t)\| \leq\ & \kappa_1+\theta_0\kappa_2+\frac{\theta_1\gamma_1(\tau-t_0)^{\alpha}}{\Gamma(\alpha+1)}+\frac{\theta_0\theta_2\gamma_1(\tau-t_0)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\gamma_1(t-t_0)^{\alpha}}{\Gamma(\alpha+1)}\\
&+\frac{\theta_2\gamma_1\gamma_3(\tau-t_0)^{2\alpha-\beta}}{\Gamma(\alpha+1)\left(\Gamma(\alpha-\beta+1)-\gamma_3(1+\theta_2)(\tau-t_0)^{\alpha-\beta}\right)}\\
&+\frac{\gamma_3\kappa_2\Gamma(\alpha-\beta+1)(\tau-t_0)^{\alpha-\beta+1}}{\Gamma(\alpha+1)\Gamma(2-\beta)\left(\Gamma(\alpha-\beta+1)-\gamma_3(1+\theta_2)(\tau-t_0)^{\alpha-\beta}\right)}\\
&+\frac{\gamma_1\gamma_3(\tau-t_0)^{2\alpha-\beta}}{\Gamma(\alpha+1)\left(\Gamma(\alpha-\beta+1)-\gamma_3(1+\theta_2)(\tau-t_0)^{\alpha-\beta}\right)}\\
&+\frac{\theta_1\gamma_3\kappa_2\Gamma(\alpha-\beta+1)(\tau-t_0)^{\alpha-\beta+1}}{\Gamma(\alpha+1)\Gamma(2-\beta)\left(\Gamma(\alpha-\beta+1)-\gamma_3(1+\theta_2)(\tau-t_0)^{\alpha-\beta}\right)}\\
&+\frac{\theta_1\gamma_3\gamma_1(\tau-t_0)^{2\alpha-\beta}}{\Gamma(\alpha+1)\left(\Gamma(\alpha-\beta+1)-\gamma_3(1+\theta_2)(\tau-t_0)^{\alpha-\beta}\right)}\\
&+\frac{\theta_1\theta_2\gamma_1\gamma_3(\tau-t_0)^{2\alpha-\beta}}{\Gamma(\alpha+1)\left(\Gamma(\alpha-\beta+1)-\gamma_3(1+\theta_2)(\tau-t_0)^{\alpha-\beta}\right)}\\
&+\frac{\theta_0\theta_2\gamma_3\kappa_2\Gamma(\alpha-\beta+1)(\tau-t_0)^{\alpha-\beta}}{\Gamma(\alpha)\Gamma(2-\beta)\left(\Gamma(\alpha-\beta+1)-\gamma_3(1+\theta_2)(\tau-t_0)^{\alpha-\beta}\right)}\\
&+\frac{\theta_0\theta_2\gamma_3\gamma_1(\tau-t_0)^{2\alpha-\beta-1}}{\Gamma(\alpha)\left(\Gamma(\alpha-\beta+1)-\gamma_3(1+\theta_2)(\tau-t_0)^{\alpha-\beta}\right)}\\
&+\frac{\theta_0\theta_2\gamma_3\theta_2\gamma_1(\tau-t_0)^{2\alpha-\beta-1}}{\Gamma(\alpha)\left(\Gamma(\alpha-\beta+1)-\gamma_3(1+\theta_2)(\tau-t_0)^{\alpha-\beta}\right)}\\
&+\gamma_2 I^{\alpha}\|x(t)\|\\
&+\frac{\gamma_2\gamma_3(1+\theta_1)\Gamma(\alpha-\beta+1)}{\left(\Gamma(\alpha-\beta+1)-\gamma_3(1+\theta_2)(\tau-t_0)^{\alpha-\beta}\right)}I^{2\alpha-\beta}\|x(t)\|\\
&+\frac{\theta_0\theta_2\gamma_3\gamma_2\Gamma(\alpha-\beta+1)}{\left(\Gamma(\alpha-\beta+1)-\gamma_3(1+\theta_2)(\tau-t_0)^{\alpha-\beta}\right)}I^{2\alpha-\beta-1}\|x(t)\|\\
&+\theta_1\gamma_2 I^{\alpha}\|x(\tau)\|+\theta_0\theta_2\gamma_2 I^{\alpha-1}\|x(\tau)\|\\
&+\frac{\theta_2\gamma_2\gamma_3(1+\theta_1)\Gamma(\alpha-\beta+1)}{\left(\Gamma(\alpha-\beta+1)-\gamma_3(1+\theta_2)(\tau-t_0)^{\alpha-\beta}\right)}I^{2\alpha-\beta}\|x(\tau)\|\\
&+\frac{\theta_0\theta_2\gamma_3\theta_2\gamma_2\Gamma(\alpha-\beta+1)}{\left(\Gamma(\alpha-\beta+1)-\gamma_3(1+\theta_2)(\tau-t_0)^{\alpha-\beta}\right)}I^{2\alpha-\beta-1}\|x(\tau)\|,
\end{aligned}
$$

This can be rewritten in the form

$$
\|x(t)\| \leq c+\int_{t_0}^{t} a(t,s)\|x(s)\|\,ds+\int_{t_0}^{\tau} b(t,s)\|x(s)\|\,ds,\ t\in J.
$$

Using Lemma 1, we deduce the result. This finishes the proof.

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6 Dependence of parameters

For ϕ Lipschitz in the second and third variables, the solution's dependence on the order of the differential operator, the boundary values, and the nonlinear term ϕ are discussed in this section. To proceed further, the following condition is set.

(H4)For any $\varepsilon > 0$, there exists positive constants ς_1 and ς_2 such that

$$\left\| \frac{\Gamma(\alpha-\varepsilon)(t-s)^{\varepsilon}}{\Gamma(\alpha)} g(x(s)) - g(x_\varepsilon(s)) \right\| \le \varsigma_1 \| g(x(s)) - g(x_\varepsilon(s)) \|,$$
$$\left\| \frac{\Gamma(\alpha-\varepsilon-1)(t-s)^{\varepsilon}}{\Gamma(\alpha-1)} f g(x(s)) - g(x_\varepsilon(s)) \right\| \le \varsigma_2 \| g(x(s)) - g(x_\varepsilon(s)) \|.$$

for any $s \in J$, where $g(x(s)) = Ax(s) + \phi\left(s, x(s), {}^c\mathscr{D}^\beta x(s)\right)$. Moreover, we assume that

$$\gamma_2(\tau-t_0)^{\alpha-\varepsilon}\left(1+\theta_1+\frac{\theta_0\theta_2(\alpha-\varepsilon)}{\tau-t_0}\right) < \Gamma(\alpha-\varepsilon+1),$$
$$\gamma_3(1+\theta_2)(\tau-t_0)^{\alpha-\varepsilon-\beta} < \Gamma(\alpha-\varepsilon-\beta+1).$$

Theorem 4. *Let (H1) and (H4) be hold. If x, and x_ε are the respective solutions of the problems (1) and*

$$\begin{cases} {}^c\mathscr{D}^{\alpha-\varepsilon} x(t) = A(t)x(t) + \phi\left(t, x(t), {}^c\mathscr{D}^\beta x(t)\right), \varepsilon > 0, t \in (t_0, \tau), \\ M_1 x(t_0) + N_1 x(\tau) = b_1, M_2 x'(t_0) + N_2 x'(\tau) = b_2, \end{cases} \tag{13}$$

where $1 < \alpha - \varepsilon < \alpha < 2$, then there exists a constant $k_\varepsilon > 0$ such that

$$\|x - x_\varepsilon\|_{\mathscr{C}} \le k_\varepsilon. \tag{14}$$

Proof. By Lemma 2, we can obtain

$$\begin{aligned} x_\varepsilon(t) = {} & (M_1+N_1)^{-1} b_1 \\ & + \left[(t-t_0)I_n - (\tau-t_0)(M_1+N_1)^{-1} N_1\right](M_2+N_2)^{-1} b_2 \\ & - (M_1+N_1)^{-1} N_1 I^{\alpha-\varepsilon}\left(A(\tau)x_\varepsilon(\tau) + \phi\left(\tau, x_\varepsilon(\tau), {}^c\mathscr{D}^\beta x_\varepsilon(\tau)\right)\right) \\ & + I^{\alpha-\varepsilon}\left(A(t)x_\varepsilon(t) + \phi\left(t, x_\varepsilon(t), {}^c\mathscr{D}^\beta x_\varepsilon(t)\right)\right) \\ & + \left[(\tau-t_0)(M_1+N_1)^{-1} N_1 - (t-t_0)I_n\right](M_2+N_2)^{-1} N_2 \\ & \times I^{\alpha-\varepsilon-1}\left(A(\tau)x_\varepsilon(\tau) + \phi\left(\tau, x_\varepsilon(\tau), {}^c\mathscr{D}^\beta x_\varepsilon(\tau)\right)\right), \end{aligned} \tag{15}$$

which is the solution of (13). Then

$$\begin{aligned} \|x(t) - x_\varepsilon(t)\| \le {} & \theta_1 \left\| I^\alpha\left(A(\tau)x(\tau) + \phi\left(\tau, x(\tau), {}^c\mathscr{D}^\beta x(\tau)\right)\right) \right. \\ & \left. - I^{\alpha-\varepsilon}\left(A(\tau)x_\varepsilon(\tau) + \phi\left(\tau, x_\varepsilon(\tau), {}^c\mathscr{D}^\beta x_\varepsilon(\tau)\right)\right) \right\| \\ & + \left\| I^\alpha\left(A(t)x(t) + \phi\left(t, x(t), {}^c\mathscr{D}^\beta x(t)\right)\right) \right. \\ & \left. - I^{\alpha-\varepsilon}\left(A(t)x_\varepsilon(t) + \phi\left(t, x_\varepsilon(t), {}^c\mathscr{D}^\beta x_\varepsilon(t)\right)\right) \right\| \\ & + \theta_0\theta_2 \left\| I^{\alpha-1}\left(A(\tau)x(\tau) + \phi\left(\tau, x(\tau), {}^c\mathscr{D}^\beta x(\tau)\right)\right) \right. \\ & \left. - I^{\alpha-\varepsilon-1}\left(A(\tau)x_\varepsilon(\tau) + \phi\left(\tau, x_\varepsilon(\tau), {}^c\mathscr{D}^\beta x_\varepsilon(\tau)\right)\right) \right\|. \end{aligned}$$

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Using the identity (5), we get

$$\begin{aligned}\|x(t)-x_\varepsilon(t)\| \leq & \frac{\theta_1\varsigma_1}{\Gamma(\alpha-\varepsilon)}\int_{t_0}^{\tau}(\tau-s)^{\alpha-\varepsilon-1}\|A(s)(x(s)-x_\varepsilon(s))\\ &+\phi\left(s,x(s),{}^c\mathscr{D}^\beta x(s)\right)-\phi\left(s,x_\varepsilon(s),{}^c\mathscr{D}^\beta x_\varepsilon(s)\right)\Big\|ds\\ &+\frac{1}{\Gamma(\alpha-\varepsilon)}\int_{t_0}^{t}(t-s)^{\alpha-\varepsilon-1}\|A(s)(x(s)-x_\varepsilon(s))\\ &+\phi\left(s,x(s),{}^c\mathscr{D}^\beta x(s)\right)-\phi\left(s,x_\varepsilon(s),{}^c\mathscr{D}^\beta x_\varepsilon(s)\right)\Big\|ds\\ &+\frac{\theta_0\theta_2\varsigma_2}{\Gamma(\alpha-\varepsilon-1)}\int_{t_0}^{\tau}(\tau-s)^{\alpha-\varepsilon-2}\|A(s)(x(s)-x_\varepsilon(s))\\ &+\phi\left(s,x(s),{}^c\mathscr{D}^\beta x(s)\right)-\phi\left(s,x_\varepsilon(s),{}^c\mathscr{D}^\beta x_\varepsilon(s)\right)\Big\|ds.\end{aligned}$$

The hypothesis (H1) implies that

$$\begin{aligned}\|x(t)-x_\varepsilon(t)\| \leq & \frac{(\tau-t_0)^{\alpha-\varepsilon-1}}{\Gamma(\alpha-\varepsilon+1)}\left(\theta_0\theta_2\varsigma_2(\alpha-\varepsilon)+(1+\theta_1\varsigma_1)(\tau-t_0)\right)\\ &\times\left(\gamma_1+\gamma_2\|x-x_\varepsilon\|+\gamma_3\left\|{}^c\mathscr{D}^\beta x-{}^c\mathscr{D}^\beta x_\varepsilon\right\|\right).\end{aligned}$$

Hence

$$\|x-x_\varepsilon\| \leq \frac{(\tau-t_0)^{\alpha-\varepsilon-1}\left(\theta_0\theta_2\varsigma_2(\alpha-\varepsilon)+(1+\theta_1\varsigma_1)(\tau-t_0)\right)\left(\gamma_1+\gamma_3\left\|{}^c\mathscr{D}^\beta x-{}^c\mathscr{D}^\beta x_\varepsilon\right\|\right)}{\Gamma(\alpha-\varepsilon+1)-\gamma_2(\tau-t_0)^{\alpha-\varepsilon-1}\left(\theta_0\theta_2\varsigma_2(\alpha-\varepsilon)+(1+\theta_1\varsigma_1)(\tau-t_0)\right)}. \tag{16}$$

By virtue of (15) and (7), we have

$$\begin{aligned}{}^c\mathscr{D}^\beta x_\varepsilon(t) = & I^{1-\beta}(M_2+N_2)^{-1}b_2\\ &+I^{\alpha-\varepsilon-\beta}\left(A(t)x_\varepsilon(t)+\phi\left(t,x_\varepsilon(t),{}^c\mathscr{D}^\beta x_\varepsilon(t)\right)\right)\\ &-(M_2+N_2)^{-1}N_2I^{\alpha-\varepsilon-\beta}\left(A(\tau)x_\varepsilon(\tau)+\phi\left(\tau,x_\varepsilon(\tau),{}^c\mathscr{D}^\beta x_\varepsilon(\tau)\right)\right).\end{aligned}$$

Following similar steps as the above, we get

$$\begin{aligned}\left\|{}^c\mathscr{D}^\beta x-{}^c\mathscr{D}^\beta x_\varepsilon\right\| \leq & I^{\alpha-\beta}\left(A(t)(x(t)-x_\varepsilon(t))+\phi\left(t,x(t),{}^c\mathscr{D}^\beta x(t)\right)-\phi\left(t,x_\varepsilon(t),{}^c\mathscr{D}^\beta x_\varepsilon(t)\right)\right)\\ &+\theta_2I^{\alpha-\beta}\left(A(\tau)(x(\tau)-x_\varepsilon(\tau))+\phi\left(\tau,x(\tau),{}^c\mathscr{D}^\beta x(\tau)\right)-\phi\left(\tau,x_\varepsilon(\tau),{}^c\mathscr{D}^\beta x_\varepsilon(\tau)\right)\right)\\ \leq & \frac{(\tau-t_0)^{\alpha-\varepsilon-\beta}(1+\theta_2)(\gamma_1+\gamma_2\|x-x_\varepsilon\|)}{\Gamma(\alpha-\varepsilon-\beta+1)-\gamma_3(1+\theta_2)(\tau-t_0)^{\alpha-\varepsilon-\beta}}.\end{aligned} \tag{17}$$

Using the inequalities (17) and (16), and simplifying lead to

$$\|x-x_\varepsilon\| \leq \frac{\gamma_1\Gamma(\alpha-\varepsilon-\beta+1)(1+\theta_1+\theta_0\theta_2)(\tau-t_0)^{\alpha-\varepsilon}}{\varkappa},$$

and

$$\left\|{}^c\mathscr{D}^\beta x-{}^c\mathscr{D}^\beta x_\varepsilon\right\| \leq \frac{\gamma_1\Gamma(\alpha-\varepsilon+1)(1+\theta_2)(\tau-t_0)^{\alpha-\varepsilon-\beta}}{\varkappa},$$

where

$$\begin{aligned}\varkappa = & \Gamma(\alpha-\varepsilon+1)\Gamma(\alpha-\varepsilon-\beta+1)-\gamma_3\Gamma(\alpha-\varepsilon+1)(1+\theta_2)(\tau-t_0)^{\alpha-\varepsilon-\beta}\\ &-\gamma_2\Gamma(\alpha-\varepsilon-\beta+1)(1+\theta_1+\theta_0\theta_2)(\tau-t_0)^{\alpha-\varepsilon}.\end{aligned}$$

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Put

$$k_{\varepsilon}=\frac{\gamma_1(\tau-t_0)^{\alpha-\varepsilon}}{\varkappa}\max\{\Gamma(\alpha-\varepsilon+1)(1+\theta_2)(\tau-t_0)^{-\beta},\Gamma(\alpha-\varepsilon-\beta+1)(1+\theta_1+\theta_0\theta_2)\},$$

we get (14). This finishes the proof.

The dependence of parameters on the right-hand side of equation (1) is investigated in the next result. For this purpose, we use the following notations

$$k_1=\Gamma(\alpha-\beta+1)(\alpha\theta_0\theta_2+(1+\theta_1)(\tau-t_0))(\tau-t_0)^{\alpha-1},$$
$$k_2=\Gamma(\alpha-\beta+1)\Gamma(\alpha+1)-\gamma_3\Gamma(\alpha+1)(1+\theta_2)(\tau-t_0)^{\alpha-\beta}-\gamma_2k_1,$$
$$k_3=\Gamma(\alpha+1)(1+\theta_2)(\tau-t_0)^{\alpha-\beta}.$$

Theorem 5. *Suppose that (H1) hold. Let x, and x_{ε} be the respective solutions of the problems (1) and*

$$\begin{cases} {}^c\mathscr{D}^{\alpha}x_{\varepsilon}(t)=A(t)x(t)+\phi\left(t,x(t),{}^c\mathscr{D}^{\beta}x(t)\right)+\varepsilon h_{\varepsilon}(t),h_{\varepsilon}\in C([t_0,\tau],\mathbb{R}),t\in(t_0,\tau),\\ M_1x(t_0)+N_1x(\tau)=b_1,M_2x'(t_0)+N_2x'(\tau)=b_2,\end{cases}\tag{18}$$

where $\varepsilon>0$. Then $\|x-x_{\varepsilon}\|_{\mathscr{C}}\leq(\varepsilon\|h_{\varepsilon}\|+\gamma_1)\frac{\max\{k_1,k_3\}}{k_2}$, whenever $k_2>0$.

Proof. In view of Lemma 2, we have

$$\begin{aligned}x_{\varepsilon}(t)=&(M_1+N_1)^{-1}b_1\\&+\left[(t-t_0)I_n-(\tau-t_0)(M_1+N_1)^{-1}N_1\right](M_2+N_2)^{-1}b_2\\&-(M_1+N_1)^{-1}N_1I^{\alpha}\left(A(\tau)x_{\varepsilon}(\tau)+\varepsilon h_{\varepsilon}(\tau)+\phi\left(\tau,x_{\varepsilon}(\tau),{}^c\mathscr{D}^{\beta}x_{\varepsilon}(\tau)\right)\right)\\&+I^{\alpha}\left(A(t)x_{\varepsilon}(t)+\varepsilon h_{\varepsilon}(t)+\phi\left(t,x_{\varepsilon}(t),{}^c\mathscr{D}^{\beta}x_{\varepsilon}(t)\right)\right)\\&+\left[(\tau-t_0)(M_1+N_1)^{-1}N_1-(t-t_0)I_n\right](M_2+N_2)^{-1}N_2\\&\times I^{\alpha-1}\left(A(\tau)x_{\varepsilon}(\tau)+\varepsilon h_{\varepsilon}(\tau)+\phi\left(\tau,x_{\varepsilon}(\tau),{}^c\mathscr{D}^{\beta}x_{\varepsilon}(\tau)\right)\right).\end{aligned}$$

Then

$$\begin{aligned}\|x(t)-x_{\varepsilon}(t)\|\leq&\ \varepsilon\theta_1I^{\alpha}|h_{\varepsilon}(\tau)|+\theta_1I^{\alpha}\Big\|\left(A(\tau)x(\tau)+\phi\left(\tau,x(\tau),{}^c\mathscr{D}^{\beta}x(\tau)\right)\right)\\&-\left(A(\tau)x_{\varepsilon}(\tau)+\phi\left(\tau,x_{\varepsilon}(\tau),{}^c\mathscr{D}^{\beta}x_{\varepsilon}(\tau)\right)\right)\Big\|\\&+\varepsilon I^{\alpha}|h_{\varepsilon}(t)|+I^{\alpha}\Big\|\left(A(t)x(t)+\phi\left(t,x(t),{}^c\mathscr{D}^{\beta}x(t)\right)\right)\\&-\left(A(t)x_{\varepsilon}(t)+\phi\left(t,x_{\varepsilon}(t),{}^c\mathscr{D}^{\beta}x_{\varepsilon}(t)\right)\right)\Big\|\\&+\varepsilon\theta_0\theta_2I^{\alpha-1}|h_{\varepsilon}(\tau)|+\theta_0\theta_2I^{\alpha-1}\Big\|\left(A(\tau)x(\tau)+\phi\left(\tau,x(\tau),{}^c\mathscr{D}^{\beta}x(\tau)\right)\right)\\&-\left(A(\tau)x_{\varepsilon}(\tau)+\phi\left(\tau,x_{\varepsilon}(\tau),{}^c\mathscr{D}^{\beta}x_{\varepsilon}(\tau)\right)\right)\Big\|.\end{aligned}$$

The hypothesis (H1) implies that

$$\begin{aligned}\|x(t)-x_{\varepsilon}(t)\|\leq&\ \varepsilon(\alpha\theta_0\theta_2+(1+\theta_1)(\tau-t_0))\frac{\|h_{\varepsilon}\|(\tau-t_0)^{\alpha-1}}{\Gamma(\alpha+1)}\\&+\frac{(\tau-t_0)^{\alpha-1}}{\Gamma(\alpha+1)}(\alpha\theta_0\theta_2+(1+\theta_1)(\tau-t_0))\\&\times\left(\gamma_1+\gamma_2\|x-x_{\varepsilon}\|+\gamma_3\left\|{}^c\mathscr{D}^{\beta}x-{}^c\mathscr{D}^{\beta}x_{\varepsilon}\right\|\right).\end{aligned}$$

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Consequently

$$\|x-x_\varepsilon\| \leq \frac{\varepsilon\left(\alpha\theta_0\theta_2+(1+\theta_1)(\tau-t_0)\right)\|h_\varepsilon\|(\tau-t_0)^{\alpha-1}}{\left(\Gamma(\alpha+1)-\gamma_2(\tau-t_0)^{\alpha-1}\left(\alpha\theta_0\theta_2+(1+\theta_1)(\tau-t_0)\right)\right)} + \frac{\gamma_1(\tau-t_0)^{\alpha-1}\left(\alpha\theta_0\theta_2+(1+\theta_1)(\tau-t_0)\right)}{\left(\Gamma(\alpha+1)-\gamma_2(\tau-t_0)^{\alpha-1}\left(\alpha\theta_0\theta_2+(1+\theta_1)(\tau-t_0)\right)\right)} + \frac{\gamma_3(\tau-t_0)^{\alpha-1}\left(\alpha\theta_0\theta_2+(1+\theta_1)(\tau-t_0)\right)\left\|{}^c\mathscr{D}^\beta x-{}^c\mathscr{D}^\beta x_\varepsilon\right\|}{\left(\Gamma(\alpha+1)-\gamma_2(\tau-t_0)^{\alpha-1}\left(\alpha\theta_0\theta_2+(1+\theta_1)(\tau-t_0)\right)\right)}.$$

From (7), we have

$$\left\|{}^c\mathscr{D}^\beta x-{}^c\mathscr{D}^\beta x_\varepsilon\right\| \leq \frac{(\varepsilon\|h_\varepsilon\|+\gamma_1)(1+\theta_2)(\tau-t_0)^{\alpha-\beta}}{\left(\Gamma(\alpha-\beta+1)-\gamma_3(1+\theta_2)(\tau-t_0)^{\alpha-\beta}\right)} + \frac{\gamma_2(1+\theta_2)(\tau-t_0)^{\alpha-\beta}\|x-x_\varepsilon\|}{\left(\Gamma(\alpha-\beta+1)-\gamma_3(1+\theta_2)(\tau-t_0)^{\alpha-\beta}\right)}.$$

Therefore

$$\|x-x_\varepsilon\| \leq \frac{\varepsilon\|h_\varepsilon\|k_1}{k_2}+\frac{\gamma_1 k_1}{k_2},$$

and

$$\left\|{}^c\mathscr{D}^\beta x-{}^c\mathscr{D}^\beta x_\varepsilon\right\| \leq \frac{\varepsilon\|h_\varepsilon\|k_3}{k_2}+\frac{\gamma_1 k_3}{k_2}.$$

It is obvious that $\|x-x_\varepsilon\|_{\mathscr{C}} \leq \max\{\frac{\varepsilon\|h_\varepsilon\|k_1}{k_2}+\frac{\gamma_1 k_1}{k_2}, \frac{\varepsilon\|h_\varepsilon\|k_3}{k_2}+\frac{\gamma_1 k_3}{k_2}\}$, hence the result follows. This completes the proof.

To investigate small perturbation in the boundary conditions, we use the following notations

$$\begin{aligned}
h_1 &= \left(\alpha\theta_0\theta_2+(1+\theta_1)(\tau-t_0)\right)(\tau-t_0)^{\alpha-1},\\
h_2 &= \Gamma(\alpha-\beta+1)-\gamma_3(1+\theta_2)(\tau-t_0)^{\alpha-\beta},\\
h_3 &= \Gamma(2-\beta)\Gamma(\alpha+1)[\vartheta_1+\theta_0\vartheta_2],\\
h_4 &= h_2h_3+\gamma_3\vartheta_2h_1\Gamma(\alpha-\beta+1)(\tau-t_0)^{1-\beta},\\
h_5 &= \vartheta_2\Gamma(\alpha-\beta+1)\left(\Gamma(\alpha+1)-h_1\gamma_2\right)(\tau-t_0)^{1-\beta}+\gamma_2h_3(1+\theta_2)(\tau-t_0)^{\alpha-\beta},\\
h_6 &= h_2\Gamma(\alpha+1)-\gamma_2h_1\Gamma(\alpha-\beta+1).
\end{aligned}$$

Theorem 6. *Suppose that (H1) hold. Let x, and x_ϖ be the respective solutions of the problems (1) and*

$$\begin{cases} {}^c\mathscr{D}^\alpha x(t)=A(t)x(t)+\phi\left(t,x(t),{}^c\mathscr{D}^\beta x(t)\right), t\in(t_0,\tau),\\ M_1x(t_0)+N_1x(\tau)=b_1+\varpi, M_2x'(t_0)+N_2x'(\tau)=b_2+\varpi, \end{cases}$$

where $\varpi\in\mathbb{R}^n$. Then

$$\|x-x_\varepsilon\|_{\mathscr{C}} \leq \frac{\|\varpi\|\max\{h_4,h_5\}}{h_6\Gamma(2-\beta)} + \frac{\gamma_1\max\left\{h_1\Gamma(\alpha-\beta+1),(1+\theta_2)\Gamma(\alpha+1)(\tau-t_0)^{\alpha-\beta}\right\}}{h_6}.$$

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Proof. From Lemma 2, we have

$$
\begin{aligned}
x_{\varpi}(t) = & \left(M_1+N_1\right)^{-1}\left(b_1+\varpi\right) \\
& +\left[(t-t_0)I_n-(\tau-t_0)\left(M_1+N_1\right)^{-1}N_1\right]\left(M_2+N_2\right)^{-1}\left(b_2+\varpi\right) \\
& -\left(M_1+N_1\right)^{-1}N_1 I^{\alpha}\left(A(\tau)x_{\varpi}(\tau)+\phi\left(\tau, x_{\varpi}(\tau), {}^{c}\mathscr{D}^{\beta}x_{\varpi}(\tau)\right)\right) \\
& +I^{\alpha}\left(A(t)x_{\varpi}(t)+\phi\left(t, x_{\varpi}(t), {}^{c}\mathscr{D}^{\beta}x_{\varpi}(t)\right)\right) \\
& +\left[(\tau-t_0)\left(M_1+N_1\right)^{-1}N_1-(t-t_0)I_n\right]\left(M_2+N_2\right)^{-1}N_2 \\
& \times I^{\alpha-1}\left(A(\tau)x_{\varpi}(\tau)+\phi\left(\tau, x_{\varpi}(\tau), {}^{c}\mathscr{D}^{\beta}x_{\varpi}(\tau)\right)\right).
\end{aligned}
$$

Then

$$
\begin{aligned}
\|x-x_{\varpi}\| \leq & \|\varpi\|\left(\vartheta_1+\theta_0\vartheta_2\right) \\
& +\left(\alpha\theta_0\theta_2+\left(1+\theta_1\right)(\tau-t_0)\right)\frac{(\tau-t_0)^{\alpha-1}}{\Gamma(\alpha+1)} \\
& \times\left(\gamma_1+\gamma_2\|x-x_{\varpi}\|+\gamma_3\left\|{}^{c}\mathscr{D}^{\beta}x-{}^{c}\mathscr{D}^{\beta}x_{\varpi}\right\|\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\left\|{}^{c}\mathscr{D}^{\beta}x-{}^{c}\mathscr{D}^{\beta}x_{\varepsilon}\right\| \leq & \frac{\vartheta_2\|\varpi\|(\tau-t_0)^{1-\beta}}{\Gamma(2-\beta)}+\frac{\left(1+\theta_2\right)(\tau-t_0)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \\
& \times\left(\gamma_1+\gamma_2\|x-x_{\varpi}\|+\gamma_3\left\|{}^{c}\mathscr{D}^{\beta}x-{}^{c}\mathscr{D}^{\beta}x_{\varpi}\right\|\right).
\end{aligned}
$$

The above two estimates lead to

$$
\begin{aligned}
\|x-x_{\varpi}\| \leq & \frac{\|\varpi\|\Gamma(\alpha+1)\left(\vartheta_1+\theta_0\vartheta_2\right)}{\left(\Gamma(\alpha+1)-\gamma_2\left(\alpha\theta_0\theta_2+\left(1+\theta_1\right)(\tau-t_0)\right)(\tau-t_0)^{\alpha-1}\right)} \\
& +\frac{\gamma_1\left(\alpha\theta_0\theta_2+\left(1+\theta_1\right)(\tau-t_0)\right)(\tau-t_0)^{\alpha-1}}{\left(\Gamma(\alpha+1)-\gamma_2\left(\alpha\theta_0\theta_2+\left(1+\theta_1\right)(\tau-t_0)\right)(\tau-t_0)^{\alpha-1}\right)} \\
& +\frac{\gamma_3\left\|{}^{c}\mathscr{D}^{\beta}x-{}^{c}\mathscr{D}^{\beta}x_{\varpi}\right\|\left(\alpha\theta_0\theta_2+\left(1+\theta_1\right)(\tau-t_0)\right)(\tau-t_0)^{\alpha-1}}{\left(\Gamma(\alpha+1)-\gamma_2\left(\alpha\theta_0\theta_2+\left(1+\theta_1\right)(\tau-t_0)\right)(\tau-t_0)^{\alpha-1}\right)},
\end{aligned}
$$

and

$$
\begin{aligned}
\left\|{}^{c}\mathscr{D}^{\beta}x-{}^{c}\mathscr{D}^{\beta}x_{\varepsilon}\right\| \leq & \frac{\vartheta_2\|\varpi\|\Gamma(\alpha-\beta+1)(\tau-t_0)^{1-\beta}}{\Gamma(2-\beta)\left(\Gamma(\alpha-\beta+1)-\gamma_3\left(1+\theta_2\right)(\tau-t_0)^{\alpha-\beta}\right)} \\
& +\frac{\gamma_1\left(1+\theta_2\right)(\tau-t_0)^{\alpha-\beta}}{\left(\Gamma(\alpha-\beta+1)-\gamma_3\left(1+\theta_2\right)(\tau-t_0)^{\alpha-\beta}\right)} \\
& +\frac{\gamma_2\|x-x_{\varpi}\|\left(1+\theta_2\right)(\tau-t_0)^{\alpha-\beta}}{\left(\Gamma(\alpha-\beta+1)-\gamma_3\left(1+\theta_2\right)(\tau-t_0)^{\alpha-\beta}\right)}.
\end{aligned}
$$

These two estimates can be solved to get

$$
\begin{aligned}
& \|x-x_{\varpi}\| \leq \|\varpi\| \\
& \times\left(\frac{h_2h_3+\gamma_3\vartheta_2h_1\Gamma(\alpha-\beta+1)(\tau-t_0)^{1-\beta}}{\Gamma(2-\beta)\left(h_2\Gamma(\alpha+1)-\gamma_2h_1\Gamma(\alpha-\beta+1)\right)}\right) \\
& +\frac{\gamma_1h_1\Gamma(\alpha-\beta+1)}{h_2\Gamma(\alpha+1)-\gamma_2h_1\Gamma(\alpha-\beta+1)},
\end{aligned}
$$

and

$$\begin{aligned}&\left\|{}^{c}\mathscr{D}^{\beta}x-{}^{c}\mathscr{D}^{\beta}x_{\varepsilon}\right\|\\ \leq\ &\|\varpi\|\\ &\times\left(\frac{\vartheta_{2}\Gamma(\alpha-\beta+1)(\Gamma(\alpha+1)-h_{1}\gamma_{2})(\tau-t_{0})^{1-\beta}+\gamma_{2}h_{3}(1+\theta_{2})(\tau-t_{0})^{\alpha-\beta}}{\Gamma(2-\beta)(h_{2}\Gamma(\alpha+1)-\gamma_{2}h_{1}\Gamma(\alpha-\beta+1))}\right)\\ &+\left(\frac{\gamma_{1}(1+\theta_{2})\Gamma(\alpha+1)(\tau-t_{0})^{\alpha-\beta}}{h_{2}\Gamma(\alpha+1)-\gamma_{2}h_{1}\Gamma(\alpha-\beta+1)}\right).\end{aligned}$$

The two estimates can be maximized to get

$$\begin{aligned}\|x-x_{\varepsilon}\|_{\mathscr{C}}\leq\ &\frac{\|\varpi\|\max\{h_{4},h_{5}\}}{h_{6}\Gamma(2-\beta)}\\ &+\frac{\gamma_{1}\max\left\{h_{1}\Gamma(\alpha-\beta+1),(1+\theta_{2})\Gamma(\alpha+1)(\tau-t_{0})^{\alpha-\beta}\right\}}{h_{6}}.\end{aligned}$$

This finishes the proof.

*Remark.*In case $\gamma_1=0$, then previous results easily follows due to Lipschitz condition.

7 Application

In this section, we provide a particular example for the purpose of validating and confirming the proposed results.

*Example 1.*Consider the BVP

$$\begin{aligned}{}^{c}\mathscr{D}^{\frac{7}{4}}\begin{pmatrix}x_{1}(t)\\x_{2}(t)\end{pmatrix}=0.04\begin{pmatrix}t&0\\0&t\end{pmatrix}\begin{pmatrix}x_{1}(t)\\x_{2}(t)\end{pmatrix}\\ +\begin{pmatrix}2+0.016t^{2}\sin{}^{c}\mathscr{D}^{\frac{1}{2}}x_{1}(t)\\ \frac{e^{-t}(x_{1}(t)+x_{2}(t))^{2}}{1+(x_{1}(t)+x_{2}(t))^{2}}+0.016t^{3}\sin{}^{c}\mathscr{D}^{\frac{1}{2}}x_{2}(t)\end{pmatrix},\end{aligned}$$

for $t\in[0,2]$. Clearly $\phi\left(t,x(t),{}^{c}\mathscr{D}^{\beta}x(t)\right)=\begin{pmatrix}2+0.016t^{2}\sin{}^{c}\mathscr{D}^{\frac{1}{2}}x_{1}(t)\\ \frac{e^{-t}(x_{1}(t)+x_{2}(t))^{2}}{1+(x_{1}(t)+x_{2}(t))^{2}}+0.016t^{3}\sin{}^{c}\mathscr{D}^{\frac{1}{2}}x_{2}(t)\end{pmatrix}$ is continuous and satisfies the following estimate:

$$\begin{aligned}&\left\|A(t)[x(t)-y(t)]+\left[\phi\left(t,x(t),{}^{c}\mathscr{D}^{\beta}x(t)\right)-\phi\left(t,x(t),{}^{c}\mathscr{D}^{\beta}y(t)\right)\right]\right\|\\ =&\left\|0.04\begin{pmatrix}t&0\\0&t\end{pmatrix}\left[\begin{pmatrix}x_{1}(t)\\x_{2}(t)\end{pmatrix}-\begin{pmatrix}y_{1}(t)\\y_{2}(t)\end{pmatrix}\right]\right.\\ &\left.+\begin{pmatrix}0.016t^{2}\sin{}^{c}\mathscr{D}^{\frac{1}{2}}x_{1}(t)-0.016t^{2}\sin{}^{c}\mathscr{D}^{\frac{1}{2}}y_{1}(t)\\ \frac{e^{-t}(x_{1}(t)+x_{2}(t))^{2}}{1+(x_{1}(t)+x_{2}(t))^{2}}-\frac{e^{-t}(y_{1}(t)+y_{2}(t))^{2}}{1+(y_{1}(t)+y_{2}(t))^{2}}+0.016t^{3}\sin^{c}\mathscr{D}^{\frac{1}{2}}x_{2}(t)-0.016t^{3}\sin^{c}\mathscr{D}^{\frac{1}{2}}y_{2}(t)\end{pmatrix}\right\|\\ =&\max\left\{\left|0.04t(x_{1}(t)-y_{1}(t))+0.016t^{2}\sin\left({}^{c}\mathscr{D}^{\frac{1}{2}}x_{1}(t)\right)-0.016t^{2}\sin\left({}^{c}\mathscr{D}^{\frac{1}{2}}y_{1}(t)\right)\right|,\right.\\ &\left|0.04t(x_{2}(t)-y_{2}(t))+\frac{e^{-t}(x_{1}(t)+x_{2}(t))^{2}}{1+(x_{1}(t)+x_{2}(t))^{2}}-\frac{e^{-t}(y_{1}(t)+y_{2}(t))^{2}}{1+(y_{1}(t)+y_{2}(t))^{2}}\right.\\ &\left.+0.016t^{3}\sin\left({}^{c}\mathscr{D}^{\frac{1}{2}}x_{2}(t)\right)-0.016t^{3}\sin\left({}^{c}\mathscr{D}^{\frac{1}{2}}y_{2}(t)\right)\right|\\ \leq&\ be^{-t}+2(0.04)\max\{|x_{1}(t)-y_{1}(t)|,|x_{2}(t)-y_{2}(t)|\}\\ &+0.016\max\left\{t^{2}\left|\sin\left({}^{c}\mathscr{D}^{\frac{1}{2}}x_{1}(t)\right)-\sin\left({}^{c}\mathscr{D}^{\frac{1}{2}}y_{1}(t)\right)\right|,t^{3}\left|\sin\left({}^{c}\mathscr{D}^{\frac{1}{2}}x_{2}(t)\right)-\sin\left({}^{c}\mathscr{D}^{\frac{1}{2}}y_{2}(t)\right)\right|\right\}\\ \leq&\ 1+2(0.04)\|x-y\|+8(0.016)\left\|{}^{c}\mathscr{D}^{\frac{1}{2}}x-{}^{c}\mathscr{D}^{\frac{1}{2}}y\right\|.\end{aligned}$$

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Assume that

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, N_1 = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}, x(2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$
$$M_2 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, x'(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, N_2 = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}, x'(2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Hence

$$b_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ and } b_2 = \begin{pmatrix} 1 \\ 8 \end{pmatrix}.$$

Simple calculations lead to

$$\vartheta_1 = \left\| \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{6} \end{pmatrix} \right\| = \frac{1}{2} = \vartheta_2 = \left\| \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{8} \end{pmatrix} \right\|,$$
$$\kappa_1 = \vartheta_1 \|b_1\| = \frac{1}{2}, \kappa_2 = 4, \theta_1 = 2, \theta_2 = \frac{5}{2},$$

and

$$\theta_0 = \max_{t\in[0,2]} \left\| \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} - 2\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{4}{6} \end{pmatrix} \right\| = \max_{t\in[0,2]} \left\| \begin{pmatrix} t-1 & 0 \\ 0 & t-\frac{4}{3} \end{pmatrix} \right\|$$
$$= \max_{t\in[0,2]} \max\left(|t-1|, \left| t-\frac{4}{3} \right| \right) = \frac{4}{3}.$$

Also, we have

$$\max\left\{ \frac{\gamma_2 \left((\theta_1+1)(\tau-t_0) + \alpha\theta_0\theta_2 \right)(\tau-t_0)^{\alpha-1}}{\Gamma(\alpha+1)}, \frac{\gamma_3 (1+\theta_2)(\tau-t_0)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \right\}$$
$$= \max\left\{ \frac{2(0.04)\left(4+\frac{7}{4}\frac{4}{3}\frac{5}{2}\right)2^{\frac{3}{4}}}{\Gamma\left(\frac{11}{4}\right)}, \frac{8(0.016)\left(\frac{7}{2}\right)\left(2^{\frac{5}{4}}\right)}{\Gamma\left(\frac{9}{4}\right)} \right\}$$
$$= \max\{0.836, 0.97\} = 0.97 < 1.$$

Theorem 1 implies that the BVP has at least one solution on $[0,2]$.

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8 Conclusion

In this paper, we study some qualitative properties for a type of semi linear fractional differential system. Precisely speaking, properties like the existence and uniqueness, priori bounds and the dependence on parameters (order, initial function, right-hand function) of initial and boundary value problems for fractional differential systems have been under consideration. We employ fixed point theorems and Pachpatte inequality to prove the main results. Fractional differential systems are rarely used in the literature. The main existence theorem is conducted within non–Lipscitzian condition. We believe that the current results of this paper are of great significance for relevant audience.

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