Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/090409

Importance of RG Transform and Its Various Applications

Abdel Maksoud Soliman[1,2], *Kamal Raslan*[3] *and Ahmed Abdallah*[4,*]

[1]Department of Mathematics, Faculty of Science, Arish University, Egypt
[2]Academy of Scientific Research and Technology (ASRT), Cairo, Egypt
[3]Department of Mathematics, Faculty of Science, Al-Azhar University, Cairo, Egypt
[4]Department of Basic Science, Higher Technological Institute, Tenth of Ramadan, Egypt

Received: 21 Feb. 2022, Revised: 22 Mar. 2022, Accepted: 23 Apr. 2022.
Published online: 1 Oct. 2023

Abstract: In this study, certain properties and applications of the Ramadan Group (RG) transform are discussed. First, some new properties are presented. Lastly, we briefly explained some well-known applications.

Keywords: General integral transforms, Laplace transform, fractional derivatives and integrals.

1 Introduction

Transforms have become more widely used in solving many linear fractional differential equations and modified transforms for solving non-linear equations [1]. Among these transforms, the RG transform, which is one of the most important of these transforms, was shown previously in our previous work [2, 3, 4, 5]. RG transform allows to apply it in a wide range as explained in the present paper. To illustrate the accuracy, simplicity, efficiency and applicability of RG transform to solve fractional differential equations and integral equations, we divide the paper into seven sections as follows. Firstly, we mention some important properties of RG transform. Then, we apply it to famous equations in various specialists (fields) as we clarify later on.

2 Different Properties for RG Transform

This section is aimed at the diverse RG transform properties. We review basic rules and properties. We deduce more from it, too. Also, we apply RG transform to partial fraction function and singular integral equation (Abel's integral).
Inverse of its derivative K(s, v) (u(t) divided by t)

$$RG[v(t)] = \int_0^{+\infty} e^{-st} v(ut)dt = K(s,u);\, s,u > 0. \tag{1}$$

Integrate the two sides of the previous equation (1) in terms of w from s to 1

$$\int_s^{+\infty} K(w,u)dw = \int_s^{+\infty}\int_0^{+\infty} v(ut)e^{-wt}dtdw = \int_0^{+\infty} v(ut)\left[\int_s^{+\infty} e^{-wt}dw\right]dt$$

$$= \int_0^{+\infty} v(ut)\left[\left.\frac{e^{-wt}}{-t}\right|_s^{+\infty}\right]dt = \int_0^{+\infty} v(ut)\frac{e^{-st}}{t}dt$$

$$= RG\left[\frac{v(t)}{t}\right]. \tag{2}$$

Lemma 1. Thus, *if the n-th derivative* $v^{(n)}(t)$ *of v(t), hence*

$$RG[t^n v(t)] = (-1)^n u^n K^{(n)}(s,u), \tag{3}$$

*Corresponding author e-mail: ahmedabdel31@yahoo.com

Proof
$$K^{(n)}(s,u)=\frac{d^n}{du^n}\int_0^{+\infty} e^{-st}v(ut)dt=\int_0^{+\infty} e^{-st}\partial_n\left(v(ut)\right)dt,\ \partial_n=\frac{\partial^n}{\partial u^n}$$
$$=\left(\frac{-1}{u}\right)^n\int_0^{\infty} e^{-st}u^nD^nv(ut)dt=\left(\frac{-1}{u}\right)^n RG\left[t^nv^{(n)}(t)\right],\ D^nv(ut)=\left(v(ut)\right)^{(n)}. \tag{4}$$

Lemma 2. If K(s, v) is the RG transform of v(t), then
$$RG^{-1}\left[\left(\frac{u}{s}\right)^{\alpha}K(s,u)\right]=\frac{1}{\Gamma(\alpha)}\int_0^t(t-x)^{\alpha-1}v(y)dy. \tag{5}$$

Proof. Suppose that w(t) = I^{α} v(t), then D^{α} w(t) = v(t) so that
$$RG\left[v(t)\right]=RG\left[D^{\alpha}w(t)\right]=\left(\frac{s}{u}\right)^{\alpha}RG\left[w(t)\right]-\sum_{k=0}^{m-1}\frac{s^{\alpha-k-1}}{u^{\alpha-k}}v^{(m)}(0).\quad \text{Whenever } 0<\alpha<1, \tag{6}$$

$$RG\left[v(t)\right]=\left(\frac{s}{u}\right)^{\alpha}RG\left[w(t)\right], \tag{7}$$
subsequently,
$$w(t)=RG^{-1}\left[\left(\frac{u}{s}\right)^{\alpha}K(s,u)\right]=I^{\alpha}v(t)=\frac{\int_0^t\frac{v(x)}{(t-y)^{1-\alpha}}dy.}{\Gamma(\alpha)} \tag{8}$$
The following theorem is very important to show that RG transform is as a semi-analytic method.

Theorem 1. *If the time variable delay -fractional differential equation expressed as*
$$D_t^{\alpha}v(t)=g(t)+v(t-\tau(t)),\ 0<\alpha<1 \tag{9}$$
$$v(0)=v_0,\forall t\in I_m=\left[\tau_m,\ 0\right],v(t)=(v_1,v_2,...,v_n)\in\sim^n,$$
D_t^{α} stands for Caputo fractional order derivative, t indicates that the differentiation is performed with respect to t.
Then this equation is equivalent to
$$v(t)=I_t^{\alpha}\left(g(t)+v(s-\tau(s))\right), \tag{10}$$
Proof. Let C(T, $\sim^n$) is a Banach space of continuous functions $\phi:T\to\sim^n$; $\|\phi\|=\sup_{\phi\in T}\left|\phi(\theta)\right|$.

Taking RG transform on both sides of equation (9)
$$\left(\frac{s}{u}\right)^{\alpha}K(s,u)-\frac{1}{s}\left(\frac{s}{u}\right)^{\alpha}v_0=RG\left[g(t)\right]+RG\left[v(t-\tau(t)\right]. \tag{11}$$
Consequently,
$$K(s,u)=\frac{v_0}{s}+\frac{u}{\Gamma(\alpha)}\left(\frac{\Gamma(\alpha)u^{\alpha-1}}{s^{\alpha}}RG\left[g(t)\right]\right)=\frac{\Gamma(\alpha)u^{\alpha-1}}{s^{\alpha}}+RG\left[v(t-\tau(t)\right]. \tag{12}$$
But $RG\left[t^{\alpha-1}\right]=\frac{\Gamma(\alpha)u^{\alpha-1}}{s^{\alpha}}$,
$$K(s,u)=\frac{v_0}{s}+\frac{u\left(RG\left[t^{\alpha-1}\right]RG\left[g(t)\right]\right)}{\Gamma(\alpha)}=\frac{uRG\left[v(t-\tau(t)\right]RG\left[t^{\alpha-1}\right]}{\Gamma(\alpha)}. \tag{13}$$

This leads to $v(t) = I_t^{\alpha}\left(g(t)+v(s-\tau(s))\right).$ (14)

3 Applications for RG transform

3.1. RG Transform for Partial Fraction

RG used to solve partial fraction problems as illustrated in the example below.

Ex 1 Use inverse of RG transform

$$K(s,v)=\frac{6s^2-sv-v^2}{s^3-sv^2}. \tag{15}$$

Solution. Factorize the denominator in the simplest form

$$K(s,v)=\frac{6s^2-sv-v^2}{s(s-v)(s+v)}. \tag{16}$$

By partial fractions, we get

$$K(s,v)=\frac{A}{s}+\frac{B}{(s-v)}+\frac{C}{(s+v)}. \tag{17}$$

From the inverse of RG transform,

$$K(s,v)=A+Be^{t}+Ce^{-t}. \tag{18}$$

To calculate the constants A, B and C, setting s = 0, s = v and s = −v respectively A = 1, B = 2 and C = 3, i.e.

$$K(s,v)=1+2e^{t}+3e^{-t}. \tag{19}$$

3.2. RG transform for Abel's Integral Equation

It is considered one of the famous singular integral equations.

Ex 2. Evaluate the integral by RG transform

$$\int_0^y (y-t)^{\frac{-1}{2}}\, v(t)\,dt = 1+t+t^2. \tag{20}$$

Solution

Upon applying the RG transform

$$uRG\left[v(t)\right]RG\left[t^{\frac{-1}{2}}\right]=\frac{1}{s}+\frac{u}{s^2}+\frac{2u^2}{s^3}. \tag{21}$$

$$\begin{aligned} RG\left[v(t)\right] &= \sqrt{\frac{s}{\pi u}}\left(\frac{1}{s}+\frac{u}{s^2}+\frac{2u^2}{s^3}\right) \\ &= \frac{1}{\sqrt{\pi}}\left(\sqrt{\frac{1}{su}}+\frac{1}{s}\sqrt{\frac{u}{s}}+\frac{2u}{s^2}\sqrt{\frac{u}{s}}\right) \\ &= \frac{1}{\pi}\left(\sqrt{\frac{\pi}{us}}+\frac{2}{2s}\sqrt{\frac{\pi u}{s}}+\frac{8(3)u}{3(4)s^2}\sqrt{\frac{\pi u}{s}}\right), \end{aligned} \tag{22}$$

in account of the inverse of RG transform, $RG\left[v(t)\right]=\frac{\sqrt{t}}{\pi}\left(\frac{1}{t}+2+\frac{8}{3}t\right).$ (23)

Lemma 1. The RG transform of the Bluge function $e^{-\frac{(t-m)^2}{2}}$ is taken the form

$$RG\left[e^{-\frac{(t-m)^2}{2}}\right]=\frac{e^{-\frac{m^2}{2}}}{s}\left[1+m\left(\frac{v}{s}\right)+(m^2-1)\left(\frac{v}{s}\right)^2+(m^3-3)\left(\frac{v}{s}\right)^3+O\left(\frac{v}{s}\right)^4\right]. \quad (24)$$

Proof. From the exponential functions properties, we find

$$\begin{aligned} e^{-\frac{(t-m)^2}{2}} &= e^{-\frac{m^2}{2}}\left[e^{tm-\frac{t^2}{2}}\right] \\ &= \frac{e^{-\frac{m^2}{2}}}{s}\left[1+mt+(\frac{m^2}{2}-\frac{1}{2})t^2+(\frac{m^3}{6}-\frac{1}{2})\left(\frac{v}{s}\right)^3+Ot^4\right]. \end{aligned} \quad (25)$$

It follows from RG transform in [3, 5, 6]

$$RG\left[e^{-\frac{(t-m)^2}{2}}\right]=e^{-\frac{m^2}{2}}\left[\frac{1}{s}+\frac{mv}{s^2}+\left(\frac{m^2}{2}-\frac{1}{2}\right)\frac{2v^2}{s^3}+\left(\frac{m^3}{6}-\frac{1}{2}\right)\frac{6v^3}{s^4}+...\right]$$

$$RG\left[e^{-\frac{(t-m)^2}{2}}\right]=\frac{e^{-\frac{m^2}{2}}}{s}\left[1+m\left(\frac{v}{s}\right)+(m^2-1)\left(\frac{v}{s}\right)^2+(m^3-3)\left(\frac{v}{s}\right)^3+O\left(\frac{v}{s}\right)^4\right]$$

3.3. Bagley-Torvik equation

This equation studied the viscoelastically damped structures. It investigated the material behavior. It is possible to describe the motion of physical systems by Bagley-Torvik equation. One of the famous applications of this equations is viscous fluid [7, 8].

The general form is denoted as

$$AD_1^{\alpha}v(t)+BD_2^{\alpha}v(t)+Cv(t)=g(t),\ A\neq 0,\ \alpha_1>\alpha_2, t\in(0,+\infty) \quad (26)$$

with the initial condition $u(0)=a, u'(t)=b, A, B, C\in$ ~ .

Ex 3. By using RG transform, solve Bagley-Torvik equation

$$D^{1.5}u(t)+u(t)=t^2-t+\frac{2t^{0.5}}{\Gamma(1.5)}, \quad (27)$$

$v(0)=0,\ v'(t)=-1..$

Solution

Applying the RG transform, we obtain

$$\left(\frac{s}{u}\right)^{1.5}K(s,u)-\frac{s^{\frac{1}{2}}}{u^{\frac{3}{2}}}v(0)-\frac{s^{\frac{-1}{2}}}{u^{\frac{1}{2}}}v'(0)+K(s,u)=\frac{2u^2}{s^3}-\frac{u}{s^2}+\frac{2u^{0.5}}{s^{1.5}}. \quad (28)$$

This is also equivalent to,

$$\left[\left(\frac{s}{u}\right)^{1.5}+1\right]K(s,u)=\frac{2u^2}{s^3}-\frac{u}{s^2}+\frac{2u^{0.5}}{s^{1.5}}+\frac{1}{u^{0.5}s^{0.5}}, \tag{29}$$

we concluded that

$$K(s,u)=\frac{2u^2}{s^3}-\frac{u}{s^2}. \tag{30}$$

At the end, performing the inverse of RG transform

$$v(t)=t^2-t. \tag{31}$$

Ex 4. Let the Bagley-Torvik equation is denoted as

$$D^2v(t)+D^{\frac{3}{2}}v(t)+v(t)=1+t, \tag{32}$$

$v(0)=v^{/}(t)=1.$ Evaluate the exact solution by RG transform.

Solution

$$\left[\left(\frac{s}{u}\right)^{1.5}+\left(\frac{s}{u}\right)^2+1\right]K(s,u)=\frac{su^2+u^3+us^2+u^{0.5}s^{2.5}+u^{1.5}s^{1.5}}{u^2s^2}. \tag{33}$$

This leads to,

$$K(s,u)=\frac{u+s}{s^2}, \tag{34}$$

from the inverse of RG transform, we have

$$u(t)=1+t. \tag{35}$$

3.4. Harmonic vibration equation

Ex 5. By using RG transform, solve the general harmonic vibration equation [8, 9]

$$D^{\alpha}v(t)+\lambda^2v(t)=c,\ \alpha\in[1,2] \tag{36}$$

$v(0)=A, v^{/}(t)=B.$

Solution

Applying the RG transform on (36), we get

$$\left(\frac{s}{u}\right)^{\alpha}K(s,u)-\sum_{k=0}^{n}\frac{s^{\alpha-k-1}}{u^{\alpha-k}}v^{(k)}(0)+\lambda^2K(s,u)=\frac{\frac{c}{u}}{\frac{s}{u}}, \tag{37}$$

thus,

$$\left[\left(\frac{s}{u}\right)^{\alpha}+\lambda^2\right]K(s,u)=\frac{1}{u}\left[\frac{as^{\alpha-1}}{u^{\alpha-1}}+\frac{bs^{\alpha-2}}{u^{\alpha-2}}+\frac{c}{\frac{s}{u}}\right], \tag{38}$$

using the inverse of RG transform, we find

$$v(t) = aE_{\alpha,1}\left(-\lambda^2 t^\alpha\right) + bE_{\alpha,2}\left(-\lambda^2 t^\alpha\right) + \frac{c}{\lambda^2}\left[1 - E_\alpha\left(-\lambda^2 t^\alpha\right)\right]. \tag{39}$$

3.5. Equation of Fractional Lane Emdern

Numerous phenomena in astrophysics and mathematical physics can be formulated through the use of this equation [10].

$$D^\alpha y(t) + \frac{2}{t} y_t(t) + F(y) = 0,\ \alpha \in (0,1]$$

(40)

$v(0) = a,\ v^{/}(t) = b,\ a, b$ are constants.

Ex 6. In view of RG transform, solve fractional Lane Emdern equation

$$D^\alpha v(t) + \frac{2}{t} v^{/}(t) = (4t^2 + 6)v,\ \alpha \in (0,1] \tag{41}$$

$v(0) = 0,\ v^{/}(t) = 0$.

Solution Eq.(41) is the same as

$$tD^\alpha v(t) + 2v^{/}(t) = (4t^3 + 6t)v,\ \alpha \in (0,1]. \tag{42}$$

Again, performing the RG transform on (41), we obtain

$$u\frac{d^\alpha}{du^\alpha} K(s,u) + 2\left[\left(\frac{s}{u}\right) - \sum_{k=0}^{n} v^{(k)}(0)\right] = RG\left[A_n\right], \tag{43}$$

that is, upon using the inverse of RG transform, we arrive at

$$v_0(t) = 1,\ v_1 = t^\alpha + \frac{t^{2\alpha}}{5},\ v_2 = \frac{3t^{2\alpha}}{10} + \frac{13t^{3\alpha}}{105} + \frac{t^{4\alpha}}{90}, \dots \tag{44}$$

Finally,

$$v = v_0 + v_1 + v_2 + \dots + v_n = 1 + t^\alpha + \frac{t^{2\alpha}}{2!} + \frac{t^{3\alpha}}{3!} + \dots = e^{t^\alpha}. \tag{45}$$

3.6. Fokker-Planck equation

The Fokker-Planck equation describes the time evolution “change over time” of the probability density of the brownian particle [8]. This equation appears on various fields as physics, fluid, statistical mechanics, biology, circuit theory, etc.

Ex 7. Let the equation of fractional Fokker-Planck equation is denoted as

$$D_t^\gamma z(y,t) = \frac{\partial z}{\partial y} + \frac{\partial^2 z}{\partial y^2}, \tag{46}$$

subject to z(y, 0) = y.

Solution Passing the RG transform through (46)

$$RG\left[z(y,t)\right]\left(\frac{s}{u}\right)^\gamma - \sum_{k=0}^{n} \frac{s^{\gamma-k-1}}{u^{\gamma-k}} z^{(k)}(y,t) = RG\left[\frac{\partial z}{\partial y}\right] + RG\left[\frac{\partial^2 z}{\partial y^2}\right], \tag{47}$$

as 0 < $0 < \gamma \le 1$ and therefore m = 1.

$$RG\left[z(y,t)\right]\left(\frac{s}{u}\right)^\gamma = \sum_{k=0}^{n} \frac{s^{\gamma-1}}{u^\gamma} y + RG\left[\frac{\partial z}{\partial y}\right] + RG\left[\frac{\partial^2 z}{\partial y^2}\right], \tag{48}$$

thus yielding

$$z(y,t) = y + RG^{-1}\left[\left(\frac{u}{s}\right)^{\gamma} RG\left[\frac{\partial z}{\partial y}\right] + \left(\frac{u}{s}\right)^{\gamma} RG\left[\frac{\partial^2 z}{\partial y^2}\right]\right]. \quad (49)$$

Hence, thanks to Adomain decomposition method

$$z(y,t) = \sum_{k=0}^{\infty} z_k, \quad (50)$$

$$z_0(y,t) = z(y,0) = y, \quad (51)$$

$$z_1(y,t) = RG^{-1}\left[\left(\frac{u}{s}\right)^{\gamma} RG\left[\frac{\partial z_0}{\partial y}\right] + \left(\frac{u}{s}\right)^{\gamma} RG\left[\frac{\partial^2 z_0}{\partial y^2}\right]\right]$$

$$= \frac{1}{\Gamma(\gamma+1)} RG^{-1}\left[\left(\frac{u}{s}\right)^{\gamma} \frac{\Gamma(\gamma+1)}{S}\right] \quad (52)$$

$$= \frac{t^{\gamma}}{\Gamma(\gamma+1)}$$

$$z_2(y,t) = \ldots = z_k(y,t) = \ldots = 0. \quad (53)$$

$$z(y,t) = y + \frac{t^{\gamma}}{\Gamma(\gamma+1)}. \quad (54)$$

if we put $\gamma = 1$, hence the solution can be shortened to

$$z(y,t) = y + t. \quad (55)$$

3.7. Klein-Gordon equation

This equation is distinguished by describing many phenomena as well as its many uses in (plasma, relativistic) physics, nonlinear optics and quantum field theory [8]. The general form is taken as

$$D_t^{\alpha} v(y,t) + h(u) = g(y,t) + v_{yy}, \quad (56)$$

Example 8. Let the equation of fractional Klein-Gordon is expressed as [4, 5, 8, 11].

$$D_t^{\alpha} z(y,t) + z^2 = y^2 t^2 + z_{yy}, \quad (57)$$

Subject to $z(y,0) = 0, z_t(y,0) = y.$

Solution Passing the RG transform for (57)

$$RG[z(y,t)]\left(\frac{s}{u}\right)^{\alpha} - \sum_{k=0}^{n} \frac{s^{\alpha-k-1}}{u^{\alpha-k}} v^{(k)}(y,t) = 2!\frac{y^2 u^2}{s^3} + RG\left[\frac{\partial^2 z}{\partial y^2} - z^2\right], \quad (58)$$

$$K(s,u) = RG\left(\frac{2x^2 u^{2+\alpha}}{s^{3+\alpha}}\right) + RG\left(\frac{yu}{s^2}\right) + \left(\frac{u}{s}\right)^{\alpha} RG\left[\frac{\partial^2 z}{\partial y^2} - z^2\right], \quad (59)$$

by Adomain decomposition method,

$$z(y,t) = \frac{2y^2 t^{2+\alpha}}{\Gamma(3+\alpha)} + yt + RG^{-1}[A_m], \quad (60)$$

where

$A_m = \left(\frac{u}{s}\right)^{\alpha} RG\left[\frac{\partial^2 z}{\partial y^2} - z^2\right]$, moreover $A_1 = (z_0)_{yy} = \frac{4t^{2+\alpha}}{\Gamma(3+\alpha)}$, consequently,

$$z_0(y,t) = yt + \frac{2y^2 t^{2+\alpha}}{\Gamma(3+\alpha)}, \tag{61}$$

$$z_m(x,t) = RG^{-1}\left[A_{m-1}\right], m = 1, 2, ..., n.$$

(62)

This means to

$$z(y,t) = z_0(y,t) + z_1(y,t) + ... + yt + z_m(x,t) = yt. \tag{63}$$

This is the exact solution for (57).

3.8. Equation of Non-linear time fractional gas dynamics

General form of the non-linear time fractional gas dynamics equation is expressed as [12, 13, 14].

$$D_t^{\gamma} v(x,t) + vv_y + v(v-1) = 0, \ \gamma \in (0,1] \tag{64}$$

with $v(y,0) = e^{-y}$ and for the case $\gamma = 1$, the exact solution $v(y,t) = e^{t-y}$.

Solution Passing RG transform to (64), we get

$$RG\left[v(y,t)\right]\left(\frac{s}{u}\right)^{\gamma} - \frac{s^{\gamma-1}}{u^{\gamma}} v(y,0) = -RG\left[vv_y + v(v-1)\right], \tag{65}$$

thus yielding

$$RG\left[v(y,t)\right] = \frac{1}{s} e^{-y} - \left(\frac{u}{s}\right)^{\gamma} RG\left[vv_y + v(v-1)\right], \tag{66}$$

as a straightforward consequence of the inverse of RG transform, Hence, thanks to Adomain decomposition method

$$v_n (v_n)_y = A_n, \ v_n (v_n - 1) = B_n, \tag{67}$$

moreover,

$$A_0 = -e^{-2y}, B_0 = -e^{-2y} - e^{-y}. \tag{68}$$

However,

$$v_1(y,t) = e^{-y}\left(1 + \frac{t^{\gamma}}{\Gamma(\gamma+1)}\right), \tag{69}$$

$$v_2(y,t) = e^{-y}\left(1 + \frac{t^{\gamma}}{\Gamma(\gamma+1)} + \frac{t^{2\gamma}}{\Gamma(2\gamma+1)}\right), \tag{70}$$

also,

$$v(y,t) = e^{-y}\left(1 + \frac{t^{\chi}}{\Gamma(\gamma+1)} + \frac{t^{2\gamma}}{\Gamma(2\gamma+1)} + ...\right) = e^{-y} \sum_{n=0}^{\infty} \frac{t^{n\gamma}}{\Gamma(n\gamma+1)}, \tag{71}$$

i.e.,

$$v(y,t) = e^{-y} E_{\gamma,1}(t); \ v(y,t) = v_1 + v_2 + ... + v_n. \tag{72}$$

At $\gamma = 1$, $$v(y,t) = e^{-y}\left(1 + \frac{t}{1!} + \frac{t^2}{2!} + ...\right) = e^{t-y}. \tag{73}$$

4 Conclusion

RG transform is just as important as other transforms. There are many applications that we can handle with it. And what was mentioned in our manuscript is some, not all applications. The fundamental result of this paper is to employ the RG transform for solving several problems which arise in most in most of applications. It has been exactly achieved by finding the exact solution of the mentioned problems.

Acknowledgment

The Academy of Scientific Research and Technology (ASRT) Egypt provided financial support for this project with Grant No. 6605.

References

[1] K. A. Chauhan, et al. , Existence of solutions of non-autonomous fractional differential equations with integral impulse condition, *Adv. Differ. Equ.*, **434** (2020).
[2] H. Eltayeb, et al., Modified Sumudu decomposition method for Lane- Emden-type differential equations, *J. Adv. Mech. Eng.*, **7**(2), 1–6 (2015).
[3[A. Jhangeer, et al., Lie analysis, conservation laws and travelling wave structures of nonlinear Bogoyavlenskii–Kadomtsev–Petviashvili equation, *Result. Phy*. **19**, 103492 (2020).
[4] M. A. Ramadan, The convolution of Ramadan group integral transform: theory and applications, *J. Adv. Trend. Basic Appl. Sci.* **1**(2), 191–197 (2017).
[5] M. A. Ramadan and A. K. Mesrage, Solution of partial and integro-differential equations using the convolution of Ramadan Group transform, *Asian Res. J. Math*. **11**(3), 1–15 (2018).
[6] K. R. Raslan, et al., On a new general integral transform: some properties and remarks, J. Math. Comput. Sci. **6**(1), 103–109 (2016).
[7] K. Shah, et al., Combination of integral and projected differential transform methods for time-fractional gas dynamics equations, *Ain Shams Engn. J.* **9**, 1683–1688 (2018).
[8] K. Shah, T. Singh and A. Kilicman, Combination of integral and projected differential transform methods for time-fractional gas dynamics equations, *Ain Shams Engn. J.* **9**, 1683–1688 (2018).
[9] A. S. Shaikh and K. S. Nisar, Transmission dynamics of fractional order typhoid fever model using Caputo–Fabrizio operator , *Chaos Solit. Fract.* **128**, 355-365 (2019).
[10] A. A. Soliman, et al., On some modified methods on fractional delay and nonlinear integro-differential equation, *Sound Vibr.* **55**(4), 263-279 (2021).
[11] A. A. Soliman, K. R. Raslan and A. M. Abdallah, Analysis for fractional integro-differential equation with time delay, *Italian J. Pure Appl. Math.* **46**, 989– 1007 (2021).
[12] A. A. Soliman, et al., On fractional integro-differential equation with nonlinear time varying delay, *Sound Vibr.* **56**(2), 147-163 (2022).
[13] A. A. Soliman, K. R. Raslan and A. M. Abdallah, Properties and dualities for Ramadan Group transform, *Inform. Sci. Lett.*, Accepted, (2021).
[14] G. B. Whitham, *Linear and nonlinear waves*, John Wiley New York., 1976.