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On the Stability Analysis and Solutions of Fractional Order Pine Wilt Disease Model

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Abstract: In this paper, we present the fractional order model (FOM) for the spread of the PWD. We show that this model possesses non-negative solutions as desired in any population dynamics. We compute the basic reproduction number $\mathscr{R}_0$ and illustrate the equilibrium points (EPs) as well as their stability of this model. We apply The Natural-Adomian decomposition method (N-ADM) and fractional Euler method (FEM) to solve this model. The results are compared with those obtained by classical Rung-Kutta (RK4) method in the case of integer order.

Keywords: Pine wilt disease, Caputo fractional derivative, Fractional calculus, Fractional Euler method, Natural-Adomian decomposition method, Stability analysis.

1 Introduction

Mathematical modeling helps understand how a disease spreads and define the relevant factors. The major causes of infectious diseases are the pathogens that are bacteria, viruses, and protozoa. The infectious diseases do not only occur among humans, but they also affect the population of plants and trees. PWD is a deadly disease of pine trees, which is caused by pine-wood nematode Bursaphelenchus xylophilus [1, 2]. Pine trees infected by PWD usually die within few months. In 1905, the epidemic of the PWD first occurred in Japan [3, 4]. [5–9] considered the classical case to introduce the PWD model. In [10], the global stability of a classic host-vector model for PWD with nonlinear incidence rate was investigated.

Fractional calculus has been an active tool for modelling various phenomena (see .e.g [11–13]) because of the memory property of fractional order derivatives that allows us to understand the behavior of an epidemic among a population. In general, there are different definitions of fractional derivatives which do not coincide. One of them is Caputo fractional derivative (CFD) which is commonly used in various applications of fractional differential equations (FDEs) (see e.g. [14, 15]). Moreover, there is a new definition of fractional derivative based on the exponential function (i.e. Caputo-Fabrizio (CF) fractional operator (see e.g. [16–18])). Nowadays, many researchers attempt model real processes using this operator of fractional derivative [19–24]. One of the main advantages of the FDEs over the models of integer-order is the realistic modelling of a phenomenon that depends not only on the instant time but also on the history of the previous time (memory effect) which can be achieved using fractional calculus. This property does not exist in the classical models.

In this paper, we introduce FOM of PWD. The main purpose of this extension (where the arbitrary order $\alpha \to 1$) is that the classical model of PWD epidemic [10] fails to carry more information about the memory of the population which affects the spread of disease [25]. Moreover, the fractional operator has a non-locality property (see e.g [26], p.38-39) that can lead to substantial changes in the behavior of solutions.

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Currently, considering FOM of PWD, as follows:

$$\begin{cases} {}_0^C D_t^{\alpha} S_h(t) = \Lambda_h - \beta\delta S_h(t) I_v(t) - \gamma_h S_h(t), \\ {}_0^C D_t^{\alpha} E_h(t) = \beta\delta S_h(t) I_v(t) - (\beta_1 + \gamma_h) E_h(t), \\ {}_0^C D_t^{\alpha} I_h(t) = \beta_1 E_h(t) - \gamma_h I_h(t), \\ {}_0^C D_t^{\alpha} S_v(t) = \Lambda_v - \beta_2 S_v(t) I_h(t) - \gamma_v S_v(t), \\ {}_0^C D_t^{\alpha} I_v(t) = \beta_2 S_v(t) I_h(t) - \gamma_v I_v(t), \end{cases} \quad (1)$$

subject to the following initial conditions (ICs) (see [10])

$$S_h(0) = 300, E_h(0) = 30, I_h(0) = 20, S_v(0) = 65, I_v(0) = 20, \quad (2)$$

where, we assume N_h is the population of host pine trees, $S_h(t), E_h(t), I_h(t)$ are the susceptible, exposed and infected pine trees at any time t, respectively, so $N_h = S_h + E_h + I_h$. Also, N_v is the total vector (beetles) population consisting of adult beetles at any time t, $S_v(t), I_v(t)$ are the susceptible beetles that do not have pinewood nematode and the infected vector beetles that have the ability to carry pinewood nematode at any time t, respectively, so $N_v = S_v + I_v$. The meaning of parameters for the proposed model (1) are presented in the following table.

Table 1: Interpretation and values of the parameters in the system (1).

Parameter	Description	Value	Reference
Λ_h	The recruitment rate of the host pine population	0.009041	[8, 10]
Λ_v	A constant emergence rate of the vector pine sawyer beetle	0.002691	[8, 10]
γ_v	The natural death rate of vector population	0.011764	[10, 27]
γ_h	The natural death rate of host population	0.0000301	[10, 28]
β	The number of contacts during maturation feeding period	0.2	[10, 29]
δ	The rate in which infected beetles transmit nematode by contact	0.00166	[10, 30]
β_1	The transfer rates between the exposed and the infectious	0.057142	[10]
β_2	The rate in which the adult beetles have pinewood nematode when it escapes from dead trees	0.00305	[10, 31]

2 Preliminaries

Here, we introduce some basic definitions and notations to complete this research (see [13, 32–34]).

Definition 1 For a given function $f : \mathbb{R}^+ \to \mathbb{R}$ and $\alpha > 0$, then the Riemann-Liouville fractional integral is defined by

$${}_aI_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau,$$

where $\Gamma(\cdot)$ is the Euler gamma function defined by

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt \quad (Re(z) > 0).$$

Definition 2 Let $f(\cdot)$ be absolutely continuous functions on $[a,b]$ and $n-1 < \alpha \le n$ where $n \in \mathbb{N}$. Then, the Caputo fractional derivative (CFD) is defined by

$${}_a^C D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau.$$

The definitions of Natural transform and Laplace transform of Caputo's derivative and Mittag-Leffler function in two arguments are given, as follows:

Definition 3 Over the set of functions

$$\mathscr{A} = \{ f(t) : \exists M, \tau_1, \tau_2 > 0, |f(t)| < M e^{|t|/\tau_j}, \text{ if } t \in (-1)^j \times [0, \infty) \}.$$

The Natural transform of the time function $f(t)$ (denoted as $R(s,u)$) is defined by

$$\mathscr{N}\{f(t)\} = R(s,u) = \int_0^{\infty} f(ut)\, e^{-st} dt, \; u > 0, \; s > 0.$$

Theorem 1 The Natural transform of the CFD is defined as:-

$$\mathscr{N}\{{}^C D^{\alpha} f(t)\} = \frac{s^{\alpha}}{u^{\alpha}} R(s,u) - \sum_{k=0}^{n-1} \frac{s^{\alpha-(k+1)}}{u^{\alpha-k}} f^{(k)}(0).$$

Definition 4 If $\mathscr{L}\{f(t)\}$ is the Laplace transform of the function $f(t)$, then the Laplace transform of the CFD is defined as:

$$\mathscr{L}\{{}^C D^{\alpha} f(t)\} = s^{\alpha} F(s) - \sum_{i=0}^{n-1} s^{\alpha-i-1} f^{(i)}(0), (n-1 < \alpha \le n); n \in \mathbb{N}.$$

Definition 5 For $x \in \mathbb{R}$, the Mittag-Leffler function $E_{l,m}(x)$ is defined by

$$E_{l,m}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(ln+m)}, \; l > 0, \; m > 0.$$

Then, the Laplace transform of the function $t^{m-1} E_{l,m}(\pm \lambda t^l)$ is defined, as follows:

$$\mathscr{L}[t^{m-1} E_{l,m}(\pm \lambda t^l)] = \frac{s^{l-m}}{s^l \mp \lambda}.$$

3 Non-negative solutions

Let $\Phi = \{(S_h, E_h, I_h, S_v, I_v) \in \mathbb{R}_+^5 \,|\, 0 \le N_h \le \frac{\Lambda_h}{\gamma_h}, 0 \le N_v \le \frac{\Lambda_v}{\gamma_v}\}$. Now, we attempt to prove the positivity invariant of Φ .

Theorem 2 The FOM (1) has a unique solution at $t \ge 0$ and remains in Φ.

Proof. From the Theorem 3.1 and Remark 3.2 in [35], the solution on $(0, \infty)$ is existent and unique. For the positivity invariant (non-negative solutions) of Φ as well as from the forth and last equation for the FOM (1), we have

$${}_0^C D_t^{\alpha} N_v(t) + \gamma_v N_v(t) = \Lambda_v. \quad (3)$$

Adding the first, second and third equations for the FOM (1), we have the differential equations of $N_h(t)$, as follows:

$${}_0^C D_t^{\alpha} N_h(t) + \gamma_h N_h(t) = \Lambda_h, \quad (4)$$

In [13], applying Laplace transform method and considering ICs equal to zero, the above-mentioned equations have the following solutions

$$N_v(t) = \Lambda_v\, t^{\alpha} E_{\alpha,\alpha+1}(-\gamma_v\, t^{\alpha}) \; \ge 0,$$

$N_h(t) = \Lambda_h\, t^{\alpha} E_{\alpha,\alpha+1}(-\gamma_h\, t^{\alpha}) \geq 0,$

where $0 < \alpha < 1$, all the parameters $\Lambda_v, \gamma_v, \Lambda_v, \gamma_h$ are positive values and $E_{l,m}(x)$ is the two-parameter Mittag-Leffler function (see **Definition 5**). Mittag-Leffler function is an entire function, so $E_{\alpha,\alpha+1}(-\gamma_v\, t^{\alpha})$ and $E_{\alpha,\alpha+1}(-\gamma_h\, t^{\alpha})$ are bounded for all $t > 0$. Therefore, when $t \to \infty$, the total dynamics of pinewood trees and beetles approaches $(N_v(t), N_h(t)) \to (\frac{\Lambda_v}{\gamma_v}, \frac{\Lambda_h}{\gamma_h})$. Then, the feasible region Φ for model (1) is positivity invariant.

4 Local asymptotic stability of the equilibria

To evaluate the EPs of the FOM (1), let

$${}_0^C D_t^{\alpha} S_h = 0,\ {}_0^C D_t^{\alpha} E_h = 0,\ {}_0^C D_t^{\alpha} I_h = 0, \tag{5}$$

$${}_0^C D_t^{\alpha} S_v = 0,\ {}_0^C D_t^{\alpha} I_v = 0, \tag{6}$$

by solving Eqs. (5)-(6), then we have the following two EPs.

1. *A disease-free equilibrium (DFE) solution:*

The DFE of the model (1) is

$$E_0 = (S_h^{eq}, E_h^{eq}, I_h^{eq}, S_v^{eq}, I_v^{eq}) = \left(\frac{\Lambda_h}{\gamma_h}, 0, 0, \frac{\Lambda_v}{\gamma_v}, 0\right).$$

To derive the basic reproduction number $\mathscr{R}_0$ of system (1) using the concept of next generation matrix [36, 37]. Following [36], $\mathscr{R}_0$ can be computed as

$$\mathscr{R}_0 = \rho(FV^{-1}),$$

where ρ is the spectral radius and the necessary matrices F, V are given by

$$F = \begin{bmatrix} 0 & \beta_1 & 0 & 0 & 0 \\ 0 & 0 & \frac{\beta_2 \Lambda_v}{\gamma_v} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\beta \delta \Lambda_h}{\gamma_h} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$V = \begin{bmatrix} 0 & 0 & \gamma_h & 0 & 0 \\ 0 & 0 & 0 & & \gamma_v \\ 0 & \beta_1 + \gamma_h & 0 & 0 & 0 \\ 0 & 0 & \frac{\beta_2 \Lambda_v}{\gamma_v} & \gamma_v & 0 \\ \gamma_h & 0 & 0 & 0 & \frac{\beta \delta \Lambda_h}{\gamma_h} \end{bmatrix}.$$

Thus,

$$\mathscr{R}_0 = \frac{\beta \beta_1 \beta_2 \delta \Lambda_h \Lambda_v}{(\beta_1 + \gamma_h)\gamma_h^2 \gamma_v^2}. \tag{7}$$

2. *An endemic equilibrium solution:*

$$E^* = (S_h^*, E_h^*, I_h^*, S_v^*, I_v^*),$$

where,

$$\begin{cases} S_h^* = \frac{\Lambda_h}{\beta \delta I_v^* + \gamma_h}, \\ E_h^* = \frac{\Lambda_h \beta \delta I_v^*}{(\beta_1 + \gamma_h)(\beta \delta I_v^* + \gamma_h)} \\ I_h^* = \frac{\Lambda_h \beta \beta_1 \delta}{\gamma_h(\beta_1 + \gamma_h)(\beta \delta I_v^* + \gamma_h)}, \\ S_v^* = \frac{\Lambda_v \gamma_h (\beta_1 + \gamma_h I_v^*)(\beta \delta I_v^* + \gamma_h)}{\beta_1 \beta_2 \Lambda_h \beta \delta I_v^* + \gamma_v \gamma_h (\beta_1 + \gamma_h)(\beta \delta I_v^* + \gamma_h)}, \\ I_v^* = \frac{\Lambda_v \gamma_h (\mathscr{R}_0 - 1)}{\gamma_h \gamma_v \mathscr{R}_0 + \beta \delta \Lambda_v}. \end{cases} \tag{8}$$

Now, we can summarize the existence of E_0 and E^* by the following theorem:

Theorem 3 A DFE E_0 of the FOM (1) always exists, and there exists a unique positive endemic equilibrium E^* represented in (8), whenever $\mathscr{R}_0 > 1$.

We state the theorems of the stability of the EPs E_0 and E^*, as follows:

Theorem 4 The DFE E_0 of the FOM (1) is locally asymptotically stable, if $\mathscr{R}_0 \leq 1$, and unstabel if $\mathscr{R}_0 > 1$.

Proof. The Jacobian matrix of the system (1) at E_0 is

$$J_{E_0} = \begin{bmatrix} -\gamma_h & 0 & 0 & 0 & \frac{-\beta \delta \Lambda_h}{\gamma_h} \\ 0 & -(\beta_1 + \gamma_h) & 0 & 0 & \frac{\beta \delta \Lambda_h}{\gamma_h} \\ 0 & \beta_1 & -\gamma_h & 0 & 0 \\ 0 & 0 & \frac{-\beta_2 \Lambda_v}{\gamma_v} & -\gamma_v & 0 \\ 0 & 0 & \frac{\beta_2 \Lambda_v}{\gamma_v} & 0 & -\gamma_v \end{bmatrix}.$$

Calculate the eigenvalues of J_{E_0},

$$\lambda_1 = -\gamma_h < 0,\ \lambda_2 = -\gamma_v < 0,$$

and the remaining eigenvalues are given by

$$\lambda^3 + A_1\, \lambda^2 + A_2\, \lambda + A_3 = 0, \tag{9}$$

where

$$A_1 = 2\gamma_h + \gamma_v + \beta_1 > 0,$$

$$A_2 = \gamma_h \gamma_v + (\gamma_h + \gamma_v)(\beta_1 + \gamma_h) > 0$$

$$A_3 = \gamma_h \gamma_v (\beta_1 + \gamma_h)(1 - \mathscr{R}_0) \text{ is always positive when } \mathscr{R}_0 < 1. \tag{10}$$

Remark 1 Whenever the coefficients A_1, A_2 and A_3 have the same signal (positive), the eigenvalues (roots) have negative real part (see e.g. [38]).

Thus, the DFE is

1.locally asymptotically stable, if $A_3 > 0$ (i.e $\mathscr{R}_0 < 1$). In case $A_3 = 0$ (i.e $\mathscr{R}_0 = 1$), the DFE is locally stable, where $\lambda_3 = 0$ and the eigenvalue λ_4, λ_5 satisfy this condition $|\arg(\lambda_i)| > \frac{\alpha\pi}{2}; \forall i = 4,5$ (see e.g. [39,40]).
2.Unstable, if $A_3 < 0 \Leftrightarrow \mathscr{R}_0 > 1$.

We now state the theorem of local stability for E^*.
Theorem 5 Whenever $\mathscr{R}_0 > 1$, the endemic equilibrium E^* of the system (1) is locally asymptotically stable.
Proof. The Jacobian matrix of the system (1) at E^* is

$$J_{E^*} = \begin{bmatrix} -\beta\delta I_v^* - \gamma_h & 0 & 0 & 0 & -\beta\delta S_h^* \\ \beta\delta I_v^* & -(\beta_1+\gamma_h) & 0 & 0 & \beta\delta S_h^* \\ 0 & (\beta_1 & -\gamma_h & 0 & 0 \\ 0 & 0 & -\beta_2 S_v^* & -\beta_2 I_h^* - \gamma_v & 0 \\ 0 & 0 & \beta_2 S_v^* & \beta_2 I_h^* & -\gamma_v \end{bmatrix}.$$

Then, we obtain the characteristic polynomial of J_{E^*}, as follows:

$$\chi^5 + K_1\chi^4 + K_2\chi^3 + K_3\chi^2 + K_4\chi + K_5 = 0, \tag{11}$$

where

$$\begin{aligned} K_1 &= \mathcal{F}_1 + \mathcal{F}_2 + \gamma_h + \gamma_v, \\ K_2 &= \mathcal{F}_1(\mathcal{F}_2 + \gamma_h + \gamma_v) + \mathcal{F}_2(\gamma_h + \gamma_v + \gamma_h\gamma_v) + \mathcal{F}_3, \\ K_3 &= \mathcal{F}_3(\mathcal{F}_2 + \gamma_h + \gamma_v) + \mathcal{F}_1(\mathcal{F}_2\gamma_h + \mathcal{F}_2\gamma_v + \gamma_h\gamma_v) + \mathcal{F}_2\gamma_h\gamma_v, \\ K_4 &= \mathcal{F}_1\gamma_h\gamma_v + \mathcal{F}_3(\mathcal{F}_2\gamma_h + \mathcal{F}_2\gamma_v + \gamma_h\gamma_v) + \beta\beta_1\beta_2\delta S_h^* S_v^* \\ &\quad \times (1 + \beta\delta I_v^*), \\ K_5 &= \mathcal{F}_3\mathcal{F}_2\gamma_h\gamma_v + \beta\beta_1\beta_2\delta\gamma_v S_h^* S_v^* (1 + \beta\delta I_v^*), \end{aligned} \tag{12}$$

and

$$\begin{aligned} \mathcal{F}_1 &= \beta\delta I_v^* + \beta_1 + 2\gamma_h, \\ \mathcal{F}_2 &= \beta_2 I_h^* + \gamma_v, \\ \mathcal{F}_3 &= (\beta\delta I_v^* + \gamma_h)(\beta_1 + \gamma_h). \end{aligned}$$

Clearly, if the coefficients K_i, for $i = 1,2,3,4,5$ are positive and satisfy the Routh–Hurwitz conditions $K_1K_2K_3 > K_3^2 + K_1^2K_4$ and $(K_1K_4 - K_5)(K_1K_2K_3 - K_3^2 - K_1^2K_4) > K_5(K_1K_2 - K_3)^2 + K_1K_5^2$ when $\mathscr{R}_0 > 1$, then the Eq. (11) will give five negative eigenvalues. Thus it follows from Routh–Hurtwiz criteria that the FOM (1) at E^* is locally asymptotically stable, whenever $\mathscr{R}_0 > 1$.

5 The Natural-Adomian Decomposition Method

Adomian decomposition method is a powerful analytical method that has been used to solve linear and nonlinear functional equations of several types. The ADM was first introduced by Adomian in 1980's (see e.g [41,42]). The N-ADM demonstrates how the Natural transform maybe combined with the ADM to obtain analytic approximate solution of nonlinear differential equations. We now consider the FOM (1) subject to the ICs in Eq. (2). Then, $S_h\ I_v$ and $S_v\ I_h$ are the nonlinear terms in this model. Applying the Natural transform to both sides of Eq. (1) and

$$\begin{cases} \mathscr{N}\{{}_0^C D_t^\alpha S_h(t)\} = \mathscr{N}\{\Lambda_h - \beta\delta S_h(t)I_v(t) - \gamma_h S_h(t)\}, \\ \mathscr{N}\{{}_0^C D_t^\alpha E_h(t)\} = \mathscr{N}\{\beta\delta S_h(t)I_v(t) - (\beta_1+\gamma_h)E_h(t)\}, \\ \mathscr{N}\{{}_0^C D_t^\alpha I_h(t)\} = \mathscr{N}\{\beta_1 E_h(t) - \gamma_h I_h(t)\}, \\ \mathscr{N}\{{}_0^C D_t^\alpha S_v(t)\} = \mathscr{N}\{\Lambda_v - \beta_2 S_v(t)I_h(t) - \gamma_v S_v(t)\}, \\ \mathscr{N}\{{}_0^C D_t^\alpha I_v(t)\} = \mathscr{N}\{\beta_2 S_v(t)I_h(t) - \gamma_v I_v(t)\}, \end{cases} \tag{13}$$

using property of the Natural transform, we get

$$\begin{cases} \frac{s^\alpha}{u^\alpha}\mathscr{N}\{S_h(t)\} - \frac{s^{\alpha-1}}{u^\alpha}S_h(0) = \mathscr{N}\{\Lambda_h - \beta\delta S_h(t)I_v(t) - \gamma_h S_h(t)\}, \\ \frac{s^\alpha}{u^\alpha}\mathscr{N}\{E_h(t)\} - \frac{s^{\alpha-1}}{u^\alpha}E_h(0) = \mathscr{N}\{\beta\delta S_h(t)I_v(t) - (\beta_1+\gamma_h)E_h(t)\}, \\ \frac{s^\alpha}{u^\alpha}\mathscr{N}\{I_h(t)\} - \frac{s^{\alpha-1}}{u^\alpha}I_h(0) = \mathscr{N}\{\beta_1 E_h(t) - \gamma_h I_h(t)\}, \\ \frac{s^\alpha}{u^\alpha}\mathscr{N}\{S_v(t)\} - \frac{s^{\alpha-1}}{u^\alpha}S_v(0) = \mathscr{N}\{\Lambda_v - \beta_2 S_v(t)I_h(t) - \gamma_v S_v(t)\}, \\ \frac{s^\alpha}{u^\alpha}\mathscr{N}\{I_v(t)\} - \frac{s^{\alpha-1}}{u^\alpha}I_v(0) = \mathscr{N}\{\beta_2 S_v(t)I_h(t) - \gamma_v I_v(t)\}, \end{cases} \tag{14}$$

using the ICs in (2)

$$\begin{cases} \mathscr{N}\{S_h(t)\} = \frac{S_h(0)}{s} + \frac{u^\alpha}{s^\alpha}\mathscr{N}\{\Lambda_h - \beta\delta S_h(t)I_v(t) - \gamma_h S_h(t)\}, \\ \mathscr{N}\{E_h(t)\} = \frac{E_h(0)}{s} + \frac{u^\alpha}{s^\alpha}\mathscr{N}\{\beta\delta S_h(t)I_v(t) - (\beta_1+\gamma_h)E_h(t)\}, \\ \mathscr{N}\{I_h(t)\} = \frac{I_h(0)}{s} + \frac{u^\alpha}{s^\alpha}\mathscr{N}\{\beta_1 E_h(t) - \gamma_h I_h(t)\}, \\ \mathscr{N}\{S_v(t)\} = \frac{S_v(0)}{s} + \frac{u^\alpha}{s^\alpha}\mathscr{N}\{\Lambda_v - \beta_2 S_v(t)I_h(t) - \gamma_v S_v(t)\}, \\ \mathscr{N}\{I_v(t)\} = \frac{I_v(0)}{s} + \frac{u^\alpha}{s^\alpha}\mathscr{N}\{\beta_2 S_v(t)I_h(t) - \gamma_v I_v(t)\}. \end{cases} \tag{15}$$

Assuming that the solutions $S_h(t), I_h(t), S_v(t), I_v(t)$ in the form of infinite series are given by:

$$S_h(t) = \sum_{n=0}^{\infty} S_h^{(n)}(t),\ E_h(t) = \sum_{n=0}^{\infty} E_h^{(n)}(t), \tag{16}$$

$$I_h(t) = \sum_{n=0}^{\infty} I_h^{(n)}(t),\ S_v(t) = \sum_{n=0}^{\infty} S_v^{(n)}(t), I_v(t) = \sum_{n=0}^{\infty} I_v^{(n)}(t),$$

and the non-linear terms involved in the model S_hI_v, S_vI_h are decomposed by Adomain polynomials as

$$S_h(t)I_v(t) = \sum_{n=0}^{\infty} A_n(t),\ S_v(t)I_h(t) = \sum_{n=0}^{\infty} A_n^*(t), \tag{17}$$

where A_n, A_n^* are Adomian Polynomials defined as

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\sum_{j=0}^{n}\lambda^j S_h^{(j)}\sum_{j=0}^{n}\lambda^j I_v^{(j)}\right]\Bigg|_{\lambda=0},$$
$$A_n^* = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\sum_{(j)=0}^{n}\lambda^j S_v^{(j)}\sum_{j=0}^{n}\lambda^j I_h^{(j)}\right]\Bigg|_{\lambda=0}. \quad (18)$$

Substituting from Eqs. (16), (17) and (18) into (15), then the results are obtained as follows:

$$\begin{cases} \mathcal{N}\{S_h^{(0)}\} = \frac{S_h(0)}{s}, \\ \mathcal{N}\{E_h^{(0)}\} = \frac{E_h(0)}{s}, \\ \mathcal{N}\{I_h^{(0)}\} = \frac{I_h(0)}{s}, \\ \mathcal{N}\{S_v^{(0)}\} = \frac{S_v(0)}{s}, \\ \mathcal{N}\{I_v^{(0)}\} = \frac{I_v(0)}{s}, \end{cases} \quad (19)$$

$$\begin{cases} \mathcal{N}\{S_h^{(1)}\} = \frac{u^\alpha}{s^\alpha}\mathcal{N}\{\Lambda_h - \beta\delta A_0 - \gamma_h S_h^{(0)}(t)\}, \\ \mathcal{N}\{E_h^{(1)}\} = \frac{u^\alpha}{s^\alpha}\mathcal{N}\{\beta\delta A_0 - (\beta_1+\gamma_h)E_h^{(0)}(t)\}, \\ \mathcal{N}\{I_h^{(1)}\} = \frac{u^\alpha}{s^\alpha}\mathcal{N}\{\beta_1 E_h^{(0)}(t) - \gamma_h I_h^{(0)}(t)\}, \\ \mathcal{N}\{S_v^{(1)}\} = \frac{u^\alpha}{s^\alpha}\mathcal{N}\{\Lambda_v - \beta_2 A_0^* - \gamma_v S_v^{(0)}(t)\}, \\ \mathcal{N}\{I_v^{(1)}\} = \frac{u^\alpha}{s^\alpha}\mathcal{N}\{\beta_2 A_0^* - \gamma_v I_v^{(0)}(t)\}, \end{cases} \quad (20)$$

$$\begin{cases} \mathcal{N}\{S_h^{(2)}\} = \frac{u^\alpha}{s^\alpha}\mathcal{N}\{\Lambda_h - \beta\delta A_1 - \gamma_h S_h^{(1)}(t)\}, \\ \mathcal{N}\{E_h^{(2)}\} = \frac{u^\alpha}{s^\alpha}\mathcal{N}\{\beta\delta A_1 - (\beta_1+\gamma_h)E_h^{(1)}(t)\}, \\ \mathcal{N}\{I_h^{(2)}\} = \frac{u^\alpha}{s^\alpha}\mathcal{N}\{\beta_1 E_h^{(1)}(t) - \gamma_h I_h^{(1)}(t)\}, \\ \mathcal{N}\{S_v^{(2)}\} = \frac{u^\alpha}{s^\alpha}\mathcal{N}\{\Lambda_v - \beta_2 A_1^* - \gamma_v S_v^{(1)}(t)\}, \\ \mathcal{N}\{I_v^{(2)}\} = \frac{u^\alpha}{s^\alpha}\mathcal{N}\{\beta_2 A_1^* - \gamma_v I_v^{(1)}(t)\}, \end{cases} \quad (21)$$

$$\vdots$$

$$\begin{cases} \mathcal{N}\{S_h^{(n+1)}\} = \frac{u^\alpha}{s^\alpha}\mathcal{N}\{\Lambda_h - \beta\delta A_n - \gamma_h S_h^{(n)}(t)\}, \\ \mathcal{N}\{E_h^{(n+1)}\} = \frac{u^\alpha}{s^\alpha}\mathcal{N}\{\beta\delta A_n - (\beta_1+\gamma_h)E_h^{(n)}(t)\}, \\ \mathcal{N}\{I_h^{(n+1)}\} = \frac{u^\alpha}{s^\alpha}\mathcal{N}\{\beta_1 E_h^{(n)}(t) - \gamma_h I_h^{(n)}(t)\}, \\ \mathcal{N}\{S_v^{(n+1)}\} = \frac{u^\alpha}{s^\alpha}\mathcal{N}\{\Lambda_v - \beta_2 A_n^* - \gamma_v S_v^{(n)}(t)\}, \\ \mathcal{N}\{I_v^{(n+1)}\} = \frac{u^\alpha}{s^\alpha}\mathcal{N}\{\beta_2 A_n^* - \gamma_v I_v^{(n)}(t)\}. \end{cases}$$

Studying the mathematical behavior of the solutions $S_h(t)$, $E_h(t)$, $I_h(t)$, $S_v(t)$, $I_v(t)$ at different values of α is the main purpose. Applying the inverse Natural transform to both sides of Eq.(19), the values of $S_h^{(0)}$, $E_h^{(0)}$, $I_h^{(0)}$, $S_v^{(0)}$, $I_v^{(0)}$ are obtained. Substituting these values of $S_h^{(0)}$, $E_h^{(0)}$, $I_h^{(0)}$, $S_v^{(0)}$, $I_v^{(0)}$, A_0 and A_0^* into Eq.(20), the first components $S_h^{(1)}$, $E_h^{(1)}$, $I_h^{(1)}$, $S_v^{(1)}$, $I_v^{(1)}$ are obtained. The other terms of $S_h^{(2)}, S_h^{(3)}, \ldots,$ $E_h^{(2)}, E_h^{(3)}, \ldots,$ $I_h^{(2)}, I_h^{(3)}, \ldots$ $S_v^{(2)}, S_v^{(3)}, \ldots$ $I_v^{(2)}, I_v^{(3)}, \ldots$ can be computed respectively using the same way and we can write the solutions, as follows:

$$\begin{aligned} S_h(t) &= S_h^{(0)} + S_h^{(1)} + \ldots, \\ E_h(t) &= E_h^{(0)} + E_h^{(1)} + \ldots, \\ I_h(t) &= I_h^{(0)} + I_h^{(1)} + \ldots, \\ S_v(t) &= S_v^{(0)} + S_v^{(1)} + \ldots, \\ I_v(t) &= I_v^{(0)} + I_v^{(1)} + \ldots. \end{aligned} \quad (22)$$

6 Fractional Euler Method (FEM)

In [43], FEM has been introduced and numerical experiments are reported in (see e.g [44]). For instance, considering the general initial value problem

$${}_0^C D_t^\alpha y(t) = f(t, y(t)), \quad y(0) = y_0, \quad 0 < \alpha \leq 1, \quad 0 < t \leq T. \quad (23)$$

In a discrete numerical method the time interval $[0,T]$ is replaced by a discrete set of points $t_j = jh$, $h = \frac{T}{N}$, $j = 0, 1, \ldots, N$, so the solution is approximated by a sequence $\{y_j\}_{j=0,1,\ldots,N}$ such that $y_j \approx y(t_j)$. The exact solution of (23) can be written in terms of a Volterra integral equation of the second kind with a weakly singular kernel,

$$y(t) = y(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, y(s))ds. \quad (24)$$

The method is based on the approximation of the integral on the right-hand side of Eq. (24) by the product rectangle rule. This leads to the formula

$$y_k = y_0 + \frac{h^\alpha}{\Gamma(\alpha+1)}\sum_{j=0}^{k-1} b_{j,k} f(t_j, y_j), \quad (25)$$

for $k = 1, 2, \ldots, N$ where the weights

$$b_{j,k} = (k-j)^\alpha - (k-1-j)^\alpha. \quad (26)$$

In the limit case $\alpha \to 1$ the generalized one-step Adams-Bashforth method (i.e. the Fractional Euler method) reduces to the classical first-order Adams-Bashforth formula (i.e. the forward Euler method). It is an explicit method since y_k does not appear on the right-hand side of (25). As a consequence of Corollary 2.1 in [11], the error can be estimated as follows:

Theorem 6 The approximation computed by the Adams-Bashforth method satisfies the error bound

$$|y(t_j) - y_j| = O(h)$$

uniformly for all j if ${}_0^C D_t^\alpha y \in C^1[0,T]$.

Applying this formula (25) to the FOM (1), we can write the following approximate solutions:

$$S_h^k = S_h(0) + \frac{h^\alpha}{\Gamma(\alpha+1)} \sum_{j=0}^{k-1} b_{j,k}[\Lambda_h - \beta\delta S_h^j I_v^j - \gamma_h S_h^j],$$

$$E_h^k = E_h(0) + \frac{h^\alpha}{\Gamma(\alpha+1)} \sum_{j=0}^{k-1} b_{j,k}[\beta\delta S_h^j I_v^j - (\beta_1+\gamma_h)E_h^j],$$

$$I_h^k = I_h(0) + \frac{h^\alpha}{\Gamma(\alpha+1)} \sum_{j=0}^{k-1} b_{j,k}[\beta_1 E_h^j - \gamma_h I_h^j],$$

$$S_v^k = S_v(0) + \frac{h^\alpha}{\Gamma(\alpha+1)} \sum_{j=0}^{k-1} b_{j,k}[\Lambda_v - \beta_2 S_v^j I_h^j - \gamma_v S_v^j],$$

$$I_v^k = I_v(0) + \frac{h^\alpha}{\Gamma(\alpha+1)} \sum_{j=0}^{k-1} b_{j,k}[\beta_2 S_v^j I_h^j - \gamma_v I_v^j],$$

where $b_{j,k}$ is given by (26).

7 Numerical simulation and discussion

FEM and N-ADM are applied to solve the system (1) when $\alpha = 1$. In Tables 2-6, we compare our results with those of the classical RK4. The N-ADM gives analytic approximate solution in terms of an infinite power series. We calculated the first four terms of the N-ADM series solution for the FOM (1). Two of them are presented, as follows:

$$S_h^{(0)} = 300, \quad E_h^{(0)} = 30, \quad I_h^{(0)} = 20, \quad S_v^{(0)} = 65, \quad I_v^{(0)} = 20.$$

$$S_h^{(1)} = \left(\Lambda_h - 6000\beta\delta - 300\gamma_h\right)\frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$E_h^{(1)} = \left(6000\beta\delta - 30(\beta_1+\gamma_h)\right)\frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$I_h^{(1)} = \left(30\beta_1 - 20\gamma_h\right)\frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$S_v^{(1)} = \left(\Lambda_v - 1300\beta_2 - 65\gamma_v\right)\frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$I_v^{(1)} = \left(1300\beta_2 - 20\gamma_v\right)\frac{t^\alpha}{\Gamma(\alpha+1)}.$$

Table 2: The numerical results of $S_h(t)$

t	FEM	N-ADM	RK4
0	300	300	300
0.2	299.5944	299.5926	299.5944
0.4	299.1746	299.1709	299.1746
0.6	298.7406	298.7402	298.7406
0.8	298.2924	298.2911	298.2924

Table 3: The numerical results of $E_h(t)$

t	FEM	N-ADM	RK4
0	30	30	30
0.2	30.0622	30.0610	30.0622
0.4	30.1379	30.1343	30.1378
0.6	30.2268	30.2260	30.2268
0.8	30.3288	30.3088	30.3287

Table 4: The numerical results of $I_h(t)$

t	FEM	N-ADM	RK4
0	20	20	20
0.2	20.3431	20.3430	20.3431
0.4	20.6869	20.6867	20.6869
0.6	21.0317	21.0310	21.0317
0.8	21.3779	21.3760	21.3776

Table 5: The numerical results of $S_v(t)$

t	FEM	N-ADM	RK4
0	65	65	65
0.2	64.0548	64.0543	64.0548
0.4	63.1101	63.1069	63.1100
0.6	62.1662	62.1611	62.1662
0.8	61.2236	61.2198	61.2236

Table 6: The numerical results of $I_v(t)$

t	FEM	N-ADM	RK4
0	20	20	20
0.2	20.7460	20.7230	20.7460
0.4	21.4920	21.4422	21.4920
0.6	22.2375	22.2175	22.2376
0.8	22.9823	22.9723	22.9824

In general, comparative results in Tables 2-6 showed the efficiency of the N-ADM for small time (i.e., $t << 1$) as the responses of the FOM (1) follow the results of RK4 when $\alpha = 1$. However, the dynamical equations of fractional order can no longer be solved analytically over a long time intervals. Thus numerical simulations must be used to demonstrate the theoretical results (see e.g. [44, 45]). Now, we consider relatively long time intervals and different values of α to show some numerical experiments for the FOM (1). From the values of parameters in Table 1, the approximate solutions are displayed in Figs.1-5 using the formula (25) with stepsize $h = 0.01$ and different values of fractional order $0 < \alpha \leq 1$. The numerical simulations indicate that the disease is endemic, where $\mathscr{R}_0 >> 1$, i.e. the disease will persist among the population of host pine trees (see Fig. 3). Precisely, in Fig. 1, the number of susceptible pine trees S_h steadily reduces, while the number of infected host I_h increases (see Fig. 3). In Fig.5, we can note that the number of $I_v(t)$ never goes to extinction. Furthermore, whenever $\mathscr{R}_0 > 1$, the exposed host E_h and the infected vector I_v are oscillating (see Fig. 2 and Fig. 5, respectively), but they are stable at the end of time interval. This illustrates the analytical results in Theorem 4. The advantage of the comparison between the numerical simulations of integer order (when $\alpha = 1$) and fractional order (when $\alpha = 0.98, 0.95$) is to explore the dynamical behavior of the FOM in a more sophisticated way. One can observe that PWD model (1) with fractional order derivative has more degree of freedom. For the fractional order case, the climax of $E_h(t)$ and $I_v(t)$ reduces in Fig.2 and Fig.5. Numerical results in Tables 2-6 and Figs.1-5 demonstrates that the approximate solutions continuously depend on the time-fractional derivative α.

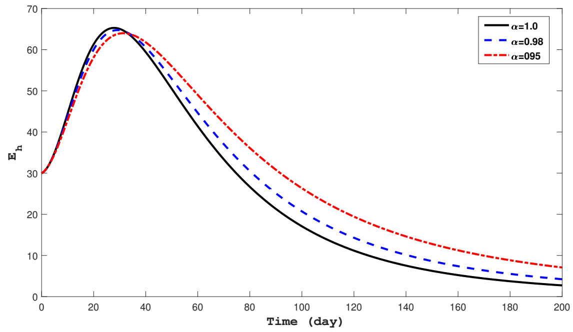

Fig. 2: Plot of the exposed host population versus time with different values of α.

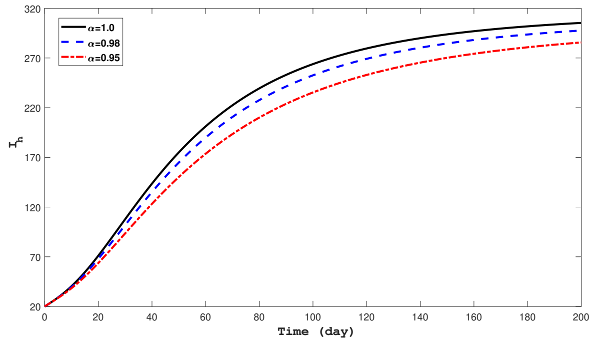

Fig. 3: Plot of the infected host population versus time with different values of α.

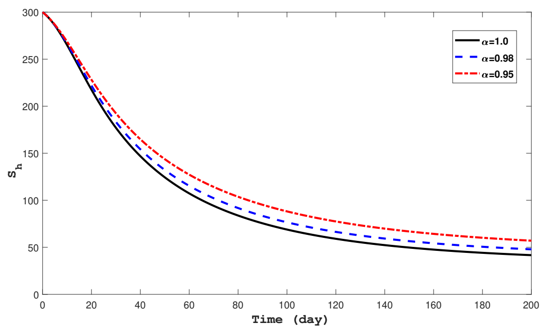

Fig. 1: Plot of the susceptible host population versus time with different values of α.

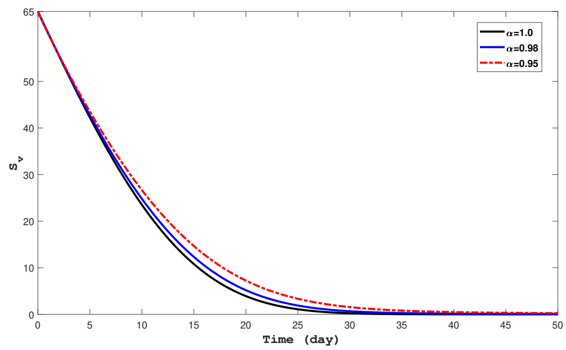

Fig. 4: Plot of the susceptible vector population versus time with different values of α.

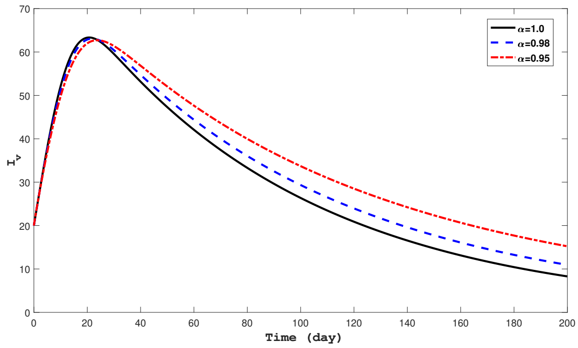

Fig. 5: Plot of the infected vector population versus time with different values of α.

8 Conclusion

In this work, we considered PWD of fractional order α (where $0 < \alpha \leq 1$) which has possessed memory. We proved the non-negative solutions of the fractional model using Laplace transform method. The general form of basic reproduction number $\mathscr{R}_0$ was computed and we showed the stability analysis of the disease free equilibria E_0 and the endemic equilibrium E^* of the proposed model on the basis of $\mathscr{R}_0$. In Tables 2-6, the solutions by N-ADM (for small time interval) demonstrated our theoretical analysis for the FOM. Moreover, our numerical simulation (Figs. 1-5) by FEM asserted persistence of the disease. Furthermore, the order of the proposed model can be defined using the collected data from naturally occurring epidemic of PWD.

Future researches can be conducted by considering the compartments of the pine trees populations and other factors such as the existence of vector using various operators of fractional derivative.

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Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this article

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