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Novel Class of Ordered Separation Axioms using Limit Points

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Abstract: The paper aims to introduce a novel class of separation axioms on topological ordered spaces, namely T_{c_i}-ordered spaces $(i = 0, \frac{1}{2}, 1, 1\frac{1}{2}, 2)$. They are defined by utilizing the notion of limit points of a set. With the aid of some examples, we scrutinize the relationships between them as well as their relationships with strong T_i-ordered and T_i-spaces. Also, we investigate the interrelations between some of the initiated ordered separation axioms and some topological notions such as continuous topological ordered spaces and disconnected spaces. Furthermore, we verify that these ordered separation axioms are preserved under ordered embedding homeomorphism mappings and give a sufficient condition to be hereditary properties. Eventually, we demonstrate that the product of T_{c_i}-ordered spaces is also T_{c_i}-ordered for each $i \neq 2$.

Keywords: Hereditary property, Limit point, T_{c_i}-ordered space, Product spaces

1 Introduction

In 1965, Nachbin [1] initiated the concept of topological ordered spaces by adding a partial order relation to the structure of a topological space. Both topology and partial order relation are defined as independent from each other. However, the interaction between them occurs in the case of defining some concepts using some characterizations of topology and partial order relation such as increasing or decreasing open (closed) sets and the smallest (largest) element of some T_i-spaces. Nachbin [1] proved some results concerning T_i-ordered spaces $(i = 2,4)$ and compactness, which generalized well-known theorems for topological spaces.

McCartan [2], in 1968, introduced T_i-separation axioms $(i = 0,1,2,3,4)$ in topological ordered spaces and verified some results concerning T_i-ordered spaces $(i = 2,3)$ and local compactness. In 1971, he [3] also defined the concepts of continuity, anti-continuity, bicontinuity for topological preordered spaces and obtained interesting properties for them. We draw attention to the coincidence of defining T_i-spaces using open sets or neighborhoods. However, this matter is different in the case of defining them on ordered setting. In [4], the authors defined a concept of order-connectedness and presented four techniques of defining the order-continuous maps between topological ordered spaces. Burgess and Fitzpatrick [5] investigated the consequences of T_i-ordered spaces under the conditions of convexity, continuity, anticontinuity and bicontinuity of a topological ordered spaces.

Arya and Gupta [6], in 1991, employed semiopen sets to study new ordered separation axioms, namely semi T_1-ordered and semi T_2-ordered spaces. Similarly, Leela and Balasubramanian [7] introduced βT_1-ordered and semi βT_2-ordered spaces using β-open sets. Shanthi and Rajesh [8], in 2018, investigated ωT_i-ordered $(i = 1,2)$ and ω-regularly ordered spaces. Some studies of ordered separation axioms were conducted by replacing a partial order relation by an arbitrary binary relation; see, for example [9] and [10].

In 2002, Kumar [11] introduced the concept of continuous, open, closed and homeomorphism mappings between topological ordered spaces. Recently, we [12] have generalized them on the content of supra topological ordered spaces. Later on, the authors of [13], [14], [15], [16], [17] and [18] employed the notions of supra β-open, supra semiopen, supra preopen, supra b-open, supra α-open and supra R-open sets and increasing (decreasing,

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balancing) sets in introducing various kinds of ordered mappings. El-Shafei et al. [19] utilized the notion of monotone supra open sets instead of monotone supra neighborhoods to initiate some ordered separation axioms in supra topological ordered spaces.

The authors of [20] and [21] discussed main properties of completely regular ordered and strictly completely regular ordered spaces. Künzi and Richmond [22], in 2005, gave an explicit construction of the T_0-ordered reflection of an ordered topological space and described topological ordered spaces whose T_0-ordered reflection is T_1-ordered. Lazaar and Mhemdi [23], in 2014, explained some features of T_0-ordered reflection. In 2020, Al-shami [24] presented the concept of sum of the ordered spaces and defined some kinds of ordered additive properties.

In this regard, studies concerning the concepts of topological ordered and supra topological ordered spaces on soft setting have increased in recent years; see, for example [25], [26], [27], [28], [29], [30], [31] and [32].

We organize this paper, as follows; after this introduction, we recall some basic notions and properties of of partially ordered set and topological ordered space in Section 2. Section 3 is the main section of this work and it is devoted to introduce the concepts of T_{c_i}-ordered spaces, where $i = 0, \frac{1}{2}, 1, 1\frac{1}{2}, 2$. In general, we investigate several properties of them and illustrate the relationships between them with the help of examples. Also, we prove that T_{c_2}-ordered spaces is disconnected, and we point out the equivalence between ST_2-ordered and $T_{1\frac{1}{2}}$-ordered spaces if the limit points of the universal set is itself. Moreover, we establish some results related to hereditary properties and product spaces. We present the conclusions and future research in Section 4.

2 Preliminaries

This section involves the main definitions and properties of partial order relations and topological ordered spaces that will be needed in the sequels. They were given in [1], [33] and [34].

Definition 2.1. A binary relation $\preceq$ is called a partial order relation if it is reflexive, anti-symmetric and transitive. The usual partial order relation on the set of real numbers $\mathbb{R}$ is defined as follows $\preceq = \{(a,b) : a \leq b, a,b \in \mathbb{R}\}$.

From now on, the diagonal relation on any non-empty set X, given by $\{(a,a) : a \in X\}$, shall be shortly denoted by $\triangle$.

Definition 2.2. A map $f : (X, \preceq_1) \to (Y, \preceq_2)$ is called order embedding provided that $a \preceq_1 b \Longleftrightarrow f(a) \preceq_2 f(b)$ for each $a, b \in X$.

Definition 2.3. Let A be a subset of a topological space (X, τ) and $x \in X$. Then, x is called a limit point of A, denoted by $x \in A'$, if $A \bigcap U \subseteq \{x\}$ for every open set containing x. Otherwise, $x \notin A'$.

Definition 2.4. A topological space (X, τ) is called connected provided that X can not be expressed as a union of two disjoint non-empty open (closed) sets.

Definition 2.5. Any property which when satisfied by a topological space is also satisfied by every subspace of this topology is called a hereditary property.

Definition 2.6. A map $\prod_j : X \to X_{\alpha_j}$, where $X = \prod_{\alpha \in \Lambda} X_\alpha$ which is defined as $\prod_j(x) = x_j$ is called the projection map of X into the α_j th coordinate.

Definition 2.7. Let B be a subset of a partially ordered set $(X, \preceq)$ and $x \in X$. Then,

(i)$i(x) = \{a \in X : x \preceq a\}$ and $d(x) = \{a \in X : a \preceq x\}$.
(ii)$i(B) = \bigcup\{i(b) : b \in B\}$ and $d(B) = \bigcup\{d(b) : b \in B\}$.
(iii)A set B is called increasing (resp. decreasing) if $B = i(B)$(resp. $B = d(B)$).

Definition 2.8. A triple $(X, \tau, \preceq)$ is said to be a topological ordered space, where $(X, \preceq)$ is a partially ordered set and (X, τ) is a topological space.

Henceforth, $(X, \tau, \preceq)$ and $(Y, \theta, \preceq)$ denote topological ordered spaces.

Definition 2.9. Let A be a subset of $(X, \tau, \preceq)$. We define a topological ordered subspace $(A, \tau_A, \preceq_A)$ of $(X, \tau, \preceq)$ as follows $\tau_A = \{A \bigcap G : \text{ for each } G \in \tau\}$ and $\preceq_A = \preceq \bigcap A \times A$.

Definition 2.10. Given a family of topological ordered spaces $\{(X_\alpha, \tau_\alpha, \preceq_\alpha) : \alpha \in \Lambda\}$. The product ordered space $(X, \tau, \preceq)$ of this family is defined by $X = \prod X_\alpha, \tau$ is a coarsest topology on X with respect to which all the projections maps $\prod : X \to X_\alpha$ are continuous and $\preceq = \{(a,b) : a,b \in X \text{ such that } (a_\alpha, b_\alpha) \in \preceq_\alpha \text{ for every } \alpha \in \Lambda\}$.

Definition 2.11. $(X, \tau, \preceq)$ is said to be:

(i)Lower strong T_1-ordered (briefly, lower ST_1-ordered) if for each $a \npreceq b$ in X, there exists an increasing open set G containing a such that b belongs to G^c.
(i)Upper strong T_1-ordered (briefly, upper ST_1-ordered) if for each $a \npreceq b$ in X, there exists a decreasing open set G containing b such that a belongs to G^c.
(iii)Strong T_0-ordered (briefly, ST_0-ordered) if it is lower ST_1-ordered or upper ST_1-ordered.
(iv)Strong T_1-ordered (briefly, ST_1-ordered) if it is lower ST_1-ordered and upper ST_1-ordered.

(v)Strong T_2-ordered (briefly, ST_2-ordered) if for each $a \not\preceq b$ in X, there exist disjoint open sets G_1 and G_2 containing a and b, respectively, such that G_1 is increasing and G_2 is decreasing.

3 Novel separation axioms in topological ordered spaces

In this section, we introduce the concepts of T_{c_i}-ordered spaces $(i = 0, \frac{1}{2}, 1, 1\frac{1}{2}, 2)$ as a new family of separation axioms on topological ordered spaces. We demonstrate the relationships between them and discuss main properties with the help of examples.

Definition 3.1. $(X, \tau, \preceq)$ is called T_{c_0}-ordered if for every $a \not\preceq b$ in X, there exists an increasing open set G containing a such that $b \notin G$ or a decreasing open set H containing b such that $a \notin H$.

The two examples below illustrate the existence and uniqueness of T_{c_0}-ordered.

Example 3.2. Let $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$ be a topology on $X = \{a,b,c\}$ and $\preceq = \triangle \bigcup \{(a,c), (c,b)(a,b)\}$ be a partial order relation on X.
Since $c \not\preceq a$, then $a \in \{a\} = d(a)$, $c \notin \{a\}$.
Since $b \not\preceq c$, then $b \in \{b\} = i(b)$, $c \notin \{b\}$.
Since $b \not\preceq a$, then $b \in \{b\} = i(b)$, $a \notin \{b\}$.
Hence, $(X, \tau, \preceq)$ is a T_{c_0}-ordered space.

Example 3.3. Let $\tau = \{\emptyset, X, \{a\}, \{a,b\}\}$ be a topology on $X = \{a,b,c\}$ and $\preceq = \triangle \bigcup \{(a,c), (c,b)(a,b)\}$ be a partial order relation on X. Now, $b \not\preceq c$. Since the only increasing open set containing b and the only decreasing open set containing c is X, then $(X, \tau, \preceq)$ is not a T_{c_0}-ordered space.

Proposition 3.4 Every ST_0-ordered space $(X, \tau, \preceq)$ is T_{c_0}-ordered.

Proof. Straightforward.

The converse of the above proposition fails as shown by the following example.

Example 3.5. Let $(X, \tau, \preceq)$ be the same as in Example 3.2. Since $c \not\preceq a$ and there does not exist an increasing open set G containing c such that $a \notin G$, then $(X, \tau, \preceq)$ is not lower ST_0 -ordered. Since $b \not\preceq c$ and there does not exist a decreasing open set H containing c such that $b \notin H$, then $(X, \tau, \preceq)$ is not upper ST_0-ordered. Hence, it is not ST_0-ordered. In contrast, it is a T_{c_0}-ordered space.

Definition 3.6. $(X, \tau, \preceq)$ is called monotone provided that the sets $i(G)$ and $d(G)$ are open for every open subset G of X.

Theorem 3.7. If $(X, \tau, \preceq)$ is monotone, then $(X, \tau, \preceq)$ is T_{c_0}-ordered if and only if $\overline{d(a)} \neq \overline{d(b)}$ or $\overline{i(a)} \neq \overline{i(b)}$ for every $a \neq b$.

Proof. *Necessity:* Suppose that $(X, \tau, \preceq)$ is a T_{c_0}-ordered space and let $a \not\preceq b$. Then, we have the following two cases:

(i)Either there exists an increasing open set G containing a such that $b \notin G$. Then, $\overline{d(b)} \subseteq G^c$. Hence, $a \notin \overline{d(b)}$. Obviously, $a \in \overline{d(a)}$. Thus, $\overline{d(a)} \neq \overline{d(b)}$.
(ii)Or there exists a decreasing open set H containing b such that $a \notin H$. We can similarly obtain $\overline{i(a)} \neq \overline{i(b)}$.

Sufficiency: Suppose that $\overline{d(a)} \neq \overline{d(b)}$ and let $a \not\preceq b$. Then, there exists $x \in X$ such that $x \in \overline{d(a)}$ and $x \notin \overline{d(b)}$ or $x \in \overline{d(b)}$ and $x \notin \overline{d(a)}$. Say, $x \in \overline{d(a)}$ and $x \notin \overline{d(b)}$. Therefore, there exists an open set G containing x such that $G \bigcap d(b) = \emptyset$. This implies that $i(G) \bigcap \{b\} = \emptyset$. Since $(X, \tau, \preceq)$ is monotone, then $i(G)$ is an open set. Since $i(G) \bigcap d(a) \neq \emptyset$, then $a \in i(G)$. Thus, $(X, \tau, \preceq)$ is a T_{c_0}-ordered space.

Definition 3.8. $(X, \tau, \preceq)$ is called $T_{c_{\frac{1}{2}}}$-ordered if for every $a \not\preceq b$ in X, there exists an increasing open set G containing a such that b is a limit point of G^c or a decreasing open set H containing b such that a is a limit point of H^c.

The two examples below illustrate the existence and uniqueness of $T_{c_{\frac{1}{2}}}$-ordered.

Example 3.9. Consider $\tau = \{\emptyset, \mathbb{R}, E_x = (-\infty, x) : x \in \mathbb{R}\}$ is the left hand topology on the set of real numbers $\mathbb{R}$ and $\preceq$ is the usual partial order relation on $\mathbb{R}$. Let $a \not\preceq b$. Then, $a - b > 0$. Therefore, $G = (-\infty, a)$ is a decreasing open set containing b such that a is a limit point of G^c. Thus, $(\mathbb{R}, \tau, \preceq)$ is a $T_{c_{\frac{1}{2}}}$-ordered space.

Example 3.10. The topological ordered space defined in Example 3.2 is not $T_{c_{\frac{1}{2}}}$-ordered because $b \not\preceq a$, and there does not exist an increasing open set G containing b such that a is a limit point of G^c or a decreasing open set H containing a such that b is a limit point of H^c.

Proposition 3.11. Let $(X, \tau, \preceq)$ be a $T_{c_{\frac{1}{2}}}$-ordered space. Then, we have the following results:

(i)If $|X|$ is even, then $|X'| \geq \frac{1}{2}|X|$.
(ii)If $|X|$ is odd, then $|X'| \geq \frac{1}{2}|X - 1|$.
(iii)If $|X|$ is infinite, then $|X'| = |X|$.

Proof.

(i)Let $X = \{x_1, x_2, ..., x_{2n-1}, x_{2n}\}$. For x_1 and x_2, we have $x_1 \not\preceq x_2$ or $x_2 \not\preceq x_1$. Say, $x_1 \not\preceq x_2$. By hypothesis, there exists an increasing open set G containing x_1 such that x_2 is a limit point of G^c or a decreasing open set H containing x_2 such that x_1 is a limit point of H^c. Thus, $x_1 \in X'$ or $x_2 \in X'$. Similarly, we do that for the next two elements x_3 and x_4. This implies that $|X'|$ contains at least $\frac{1}{2}|X|$ points.

(ii)Following similar arguments in (i), the result (ii) holds.

(iii)According to (i) and (ii), we have $|X'| \geq \frac{1}{2}|X|$ or $|X'| \geq \frac{1}{2}|X-1|$. Since X is infinite, then $|X| = \frac{1}{2}|X| = \frac{1}{2}|X-1|$. Hence, the desired result is proved.

Corollary 3.12. If $(X,\tau,\preceq)$ is a $T_{c_{\frac{1}{2}}}$-ordered space, then $X' \neq \emptyset$.

In Example 3.2, we find that $X' = \{c\}$, but $(X,\tau,\preceq)$ is not a $T_{c_{\frac{1}{2}}}$-ordered space. Then, the converse of the above proposition and corollary need not be true in general.

Definition 3.13. $(X,\tau,\preceq)$ is called T_{c_1}-ordered if for every $a \not\preceq b$ in X, there exist an increasing open set G containing a such that $b \in G^c$ and a decreasing open set H containing b such that a is a limit point of H^c.

In the following, we present two examples: The first one satisfies a T_{c_1}-ordered space and the second does not satisfy a T_{c_1}-ordered space.

Example 3.14. Let $X = (0,1)$ and (X,τ) be a subspace of the usual topological space $(\mathbb{R},\mu)$. We define a topology and a partial order relation on a set $[0,1)$ as follows: $\tau^* = \{G^* : G^* = G \text{ or } G^* = G\bigcup\{0\},\ G \in \tau\}$ and $\preceq$ is the usual partial order relation on $[0,1)$. For each $a \in (0,1)$, we have $a \not\preceq 0$. Then, we find that:

(i)$a \in G^* = (\frac{1}{2}a, 1)$ which is an increasing open set and $0 \in (G^*)^c = [0, \frac{1}{2}a]$.

(ii)$0 \in H^* = \{0\}$ which is a decreasing open set, $a \in (H^*)^c$ and $a \in ((H^*)^c)^{'} = (0,1)$.

Therefore, $([0,1),\tau^\star,\preceq)$ is a T_{c_1}-ordered space.

Example 3.15. The topological ordered space defined in Example 3.9 is not T_{c_1}-ordered because $a \not\preceq b$, and there does not exist an increasing open set G containing a such that $b \notin G^c$.

Proposition 3.16. Every T_{c_1}-ordered space $(X,\tau,\preceq)$ is infinite.

Proof. Let $(X,\tau,\preceq)$ be a T_{c_1}-ordered space. Suppose, to the contrary, that X is finite. Then, τ is the discrete topology. Therefore, every subset of X has no limit points. However, this contradicts Definition 3.13. Hence, X must be that infinite.

Definition 3.17. $(X,\tau,\preceq)$ is called $T_{c_{1\frac{1}{2}}}$-ordered if for every $a \not\preceq b$ in X, there exist an increasing open set G containing a and a decreasing open set H containing b such that b and a are limit points of G^c and H^c, respectively.

The next example proves the existence of a $T_{c_{1\frac{1}{2}}}$-ordered space.

Example 3.18. Consider $(\mathbb{R},\mu,\preceq)$ is the usual topological ordered space. Let $a \not\preceq b$ and $a-b = 3r : r > 0$. Then, we find that:

(i)$a \in (a-r,\infty)$ which is an increasing open set and $b \in (-\infty, a-r]^{'} = (-\infty, a-r]$.

(ii)$b \in (-\infty, b+r)$ which is a decreasing open set and $a \in [b+r,\infty)^{'} = [b+r,\infty)$.

Therefore, $(\mathbb{R},\mu,\preceq)$ is a $T_{c_{1\frac{1}{2}}}$-ordered space.

The next example proves that there exists a topological ordered space which is not $T_{c_{1\frac{1}{2}}}$-ordered.

Example 3.19. Since 0 is not a limit point of any subset of the topological ordered space defined in Example 3.14, then it is not $T_{c_{1\frac{1}{2}}}$-ordered.

Theorem 3.20. If $X^{'} = X$, then every ST_2-ordered space $(X,\tau,\preceq)$ is $T_{c_{1\frac{1}{2}}}$-ordered.

Proof. Let $(X,\tau,\preceq)$ be an ST_2-ordered space. Then, for all $a,b \in X$ such that $a \not\preceq b$, there exist an increasing open set G containing a and a decreasing open set H containing b such that $H\bigcap G = \emptyset$. Since $H^c \bigcup G^c = X$, then $(H^c)^{'} \bigcup (G^c)^{'} = X$. It is clear that $\{a,b\} \subseteq X = (H^c)^{'} \bigcup (G^c)^{'}$. Since $a \in G$ and $G \bigcap G^c = \emptyset$, then $a \notin (G^c)^{'}$. Therefore, it must be that $a \in (H^c)^{'}$. Similarly, one can prove that it must be that $b \in (G^c)^{'}$. Thus, $(X,\tau,\preceq)$ is a $T_{c_{1\frac{1}{2}}}$-ordered space.

The condition of $X^{'} = X$ in the above theorem is necessary as illustrated in the following example.

Example 3.21. Let τ be the discrete topology on $X = \{a,b,c\}$ and $\preceq = \triangle \bigcup \{(a,b)\}$. It is clear that $(X,\tau,\preceq)$ is an ST_2-ordered space. However, $(X,\tau,\preceq)$ is not a $T_{c_{1\frac{1}{2}}}$-ordered space because $X^{'} = \emptyset$.

Proposition 3.22. If $(X,\tau,\preceq)$ is a $T_{c_{1\frac{1}{2}}}$-ordered space, then $X' = X$.

Proof. For each $a \in X$, there exists $b \in X$ such that $a \not\preceq b$ or $b \not\preceq a$. Say, $a \not\preceq b$. Then, there exists a decreasing open set H containing b such that a is a limit point of H^c. So a is a limit point of X. This implies that $X \subseteq X'$. Thus,

$X' = X$, as required.

The converse of the above proposition need not be true in general as shown in Example 3.9, where $\mathbb{R}' = \mathbb{R}$, but $(\mathbb{R}, \tau, \preceq)$ is not a $T_{c_{1\frac{1}{2}}}$-ordered space.

Definition 3.23. $(X, \tau, \preceq)$ is T_{c_2}-ordered if for every $a \npreceq b$ in X, there exist disjoint an increasing open set G containing a and a decreasing open set H containing b such that $a \in (H^c)'$, $b \in (G^c)'$ and $(H^c)' \bigcap (G^c)' = \emptyset$.

Now, we present an example which satisfies a T_{c_2}-ordered space.

Example 3.24. Consider $(\mathbb{R}, \tau, \preceq)$ is a topological ordered space, where τ is the upper limit topology on $\mathbb{R}$ and $\preceq$ is the usual partial order relation on $\mathbb{R}$. Let $a, b \in \mathbb{R}$ such that $a \npreceq b$. Then, we find that:

(i) $a \in (b, \infty)$ which is an increasing open set and $b \in (-\infty, b]' = (-\infty, b]$.
(ii) $b \in (-\infty, b]$ which is a decreasing open set and $a \in (b, \infty)' = (b, \infty)$.
(iii) $(b, \infty) \bigcap (-\infty, b] = \emptyset$.

Thus, $(\mathbb{R}, \tau, \preceq)$ is a T_{c_2}-ordered space.

Theorem 3.25. Every T_{c_2}-ordered space $(X, \tau, \preceq)$ is disconnected.

Proof. Let $(X, \tau, \preceq)$ be a T_{c_2}-ordered space. Then, for each $a \npreceq b$ in X, there exist an increasing open set G containing a and a decreasing open set H containing b such that

$$H \bigcap G = \emptyset, \ and \tag{1}$$

$$(H^c)' \bigcap (G^c)' = \emptyset \tag{2}$$

$$From (1), \ we \ find \ that \ : H^c \bigcup G^c = X \tag{3}$$

$$From (3), \ we \ find \ that \ : (H^c)' \bigcup (G^c)' = X \tag{4}$$

From (2) and (4), we infer that X is disconnected.

We introduce an example which does not satisfy T_{c_2}-ordered space.

Example 3.26. The topological ordered space defined in Example 3.18 is not T_{c_2}-ordered as illustrated in the following. Suppose, to the contrary, that $(\mathbb{R}, \mu, \preceq)$ is a T_{c_2}-ordered space. Then, for every $a \npreceq b$, there exist an increasing open set G containing a and a decreasing open set H containing b such that

$$H \bigcap G = \emptyset, \ and \tag{5}$$

$$(H^c)' \bigcap (G^c)' = \emptyset \tag{6}$$

$$By (5), \ we \ obtain \ H^c \bigcup G^c = \mathbb{R} \Rightarrow (H^c)' \bigcup (G^c)' = \mathbb{R} \tag{7}$$

From (6) and (7), we infer that $\mathbb{R}$ is disconnected. Nevertheless, this contradicts the well-known fact of the connectedness of the usual topological space. Hence, $(\mathbb{R}, \mu, \preceq)$ is not a T_{c_2}-ordered space.

One can deduce several characterizations and properties of the T_{c_i}-ordered spaces. Some of them are listed in the following results.

Proposition 3.27. Every T_{c_i}-ordered space is a $T_{c_{i-\frac{1}{2}}}$-ordered space for $i = \frac{1}{2}, 1, 1\frac{1}{2}, 2$.

Proof. Straightforward.

Fig. 1 illustrates that the converse of the above proposition need not be true in general. Also, it summarizes the implications between T_{c_i}-ordered spaces and strong T_i-ordered and T_i-spaces $(i = 0, 1, 2)$.

Proposition 3.28. Every T_{c_i}-ordered space $(X, \tau, \preceq)$ is a T_i-space for $i = 0, 1, 2$.

Proof. We prove the proposition in case $i = 0$ and the other cases follow similar lines.
For all $a \neq b$, we have $a \npreceq b$ or $b \npreceq a$. Suppose that $a \npreceq b$. Since $(X, \tau, \preceq)$ is a T_{c_0}-ordered space, then there exists an increasing open set G containing a such that b belongs to G^c or a decreasing open set H containing b such that a belongs to H^c. Therefore, $(X, \tau, \preceq)$ is a T_0-space.

Corollary 3.29. If $(X, \tau, \preceq)$ is a T_{c_0}-ordered space, then $\overline{\{a\}} \neq \overline{\{b\}}$ for each $a \neq b$.

Corollary 3.30. If $(X, \tau, \preceq)$ is a T_{c_1}-ordered space, then every singleton subset of X is closed.

Corollary 3.31. If $(X, \tau, \preceq)$ is a T_{c_1}-ordered space, then every finite subset of X has no limit points.

Corollary 3.32. If $(X, \tau, \preceq)$ is a T_{c_2}-ordered space, then $\{a\} = \bigcap \{F_i : \ F_i$ is a closed neighborhood of $a\}$ for each $a \in X$.

The converse of Proposition 3.28 need not be true as explained in the following:

(i) The topological ordered space defined in Example 3.3 is a T_0-space, but it is not a T_{c_0}-ordered space.
(ii) The topological ordered space defined in Example 3.21 is a T_2-space, but it is not a T_{c_1}-ordered space.

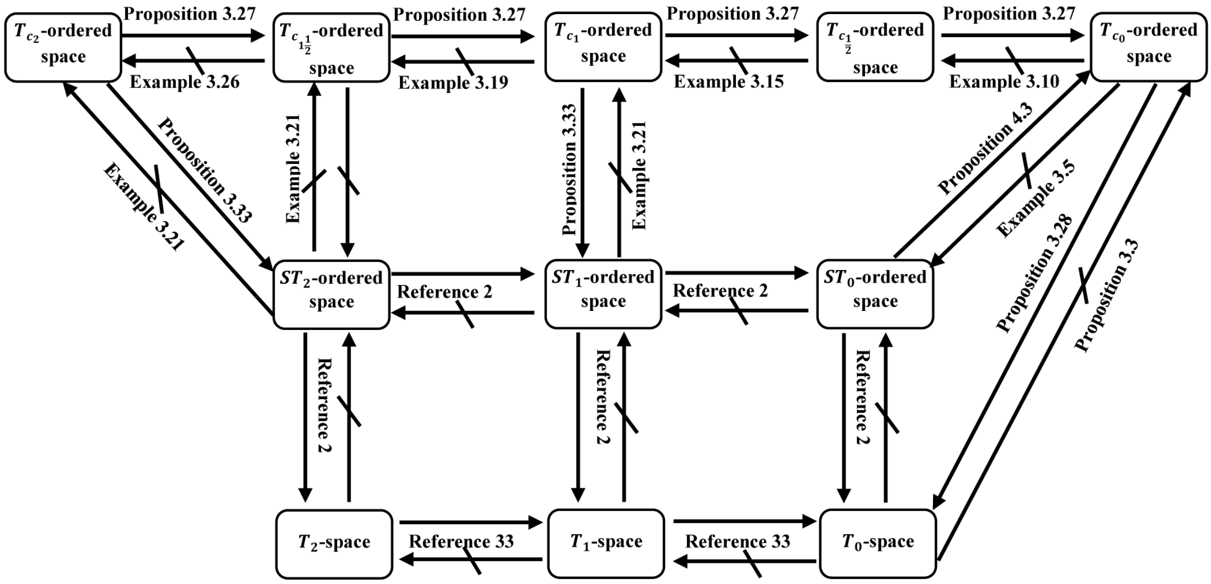

Fig. 1: The relationships among T_{c_i}-ordered spaces, T_i-ordered and T_i-spaces

Proposition 3.33. Every T_{c_i}-ordered space $(X,\tau,\preceq)$ is an ST_i-ordered space for $i=1,2$.

Proof. We only prove the proposition in the case of $i=1$ and the other case can be made similarly.
For all $a \neq b$, either $a \not\preceq b$ or $b \not\preceq a$. Suppose that $a \not\preceq b$. Since $(X,\tau,\preceq)$ is a T_{c_1}-ordered space, then there exist an increasing open set G containing a such that b belongs to G^c and a decreasing open set H containing b such that a is a limit point of H^c. Since H^c is closed, then $a \in H^c$. Hence, $(X,\tau,\preceq)$ is an ST_1-ordered space.

Corollary 3.34. If $(X,\tau,\preceq)$ is a T_{c_1}-ordered space, then $i(a)$ and $d(a)$ are closed sets for all $a \in X$.

Corollary 3.35. If $(X,\tau,\preceq)$ is a T_{c_2}-ordered space, then the graph of the partial order relation $\preceq$ is a closed subset of the product space $X \times X$.

The converse of the Proposition 3.33 need not be true as shown in Example 3.21.

Theorem 3.36. Let $f:(X,\tau,\preceq_1) \to (Y,\theta,\preceq_2)$ be an ordered embedding homeomorphism map. Then, $(X,\tau,\preceq_1)$ is T_{c_i}-ordered space if and only if $(Y,\theta,\preceq_2)$ is T_{c_i}-ordered space, for each $i=0,\frac{1}{2},1,1\frac{1}{2},2$.

Proof. We prove the theorem when $i=2$ and the other can be made similarly.
To prove the necessary part, suppose that $(X,\tau,\preceq)$ is a T_{c_2}-ordered space and let $x,y \in Y$ such that $x \not\preceq_2 y$. Then, there exist $a,b \in X$ such that $a=f^{-1}(x)$ and $b=f^{-1}(y)$. Since f is an ordered embedding map, then $a \not\preceq_1 b$. Thus, by hypothesis, there exist disjoint an increasing open set G containing a and a decreasing open set H containing b such that $a \in (H^c)'$, $b \in (G^c)'$ and $(H^c)' \bigcap (G^c)' = \emptyset$. Now, $x \in f(G)$ which is an increasing open set and $y \in f(H)$ which is a decreasing open set. Obviously, $f(G) \bigcap f(H) = \emptyset$. Since f is a homeomorphism map, then $y \in f((G^c)') = ((f(G))^c)'$, $x \in f((H^c)') = ((f(H))^c)'$ and $((f(G))^c)' \bigcap ((f(H))^c)' = \emptyset$. Hence, $(Y,\theta,\preceq_2)$ is a T_{c_2}-ordered space.
The proof of the sufficient part is made similarly.

Lemma 3.37. If U is an increasing (resp. a decreasing) subset of $(X,\tau,\preceq)$, then $U \bigcap A$ is an increasing (resp. a decreasing) subset of a topological ordered subspace $(A,\tau_A,\preceq_A)$.

Proof. Let U be an increasing subset of $(X,\tau,\preceq)$. In a topological ordered subspace $(A,\tau_A,\preceq_A)$, let $a \in i_{\preceq_A}(U \bigcap A)$. Since $i_{\preceq_A}(U \bigcap A) \subseteq i_{\preceq_A}(U) \bigcap i_{\preceq_A}(A) \subseteq U \bigcap A$, then $a \in U \bigcap A$. So, $i_{\preceq_A}(U \bigcap A) = U \bigcap A$. Thus, $U \bigcap A$ is an increasing set in $(A,\tau_A,\preceq_A)$.
The proof is similar when U is a decreasing set.

Theorem 3.38. The property of being a T_{c_0}-ordered space is hereditary.

Proof. Let $(A,\tau_A,\preceq_A)$ be a topological ordered subspace of a T_{c_0}-ordered space $(X,\tau,\preceq)$. For each $a,b \in A \subseteq X$ such that $a \not\preceq_A b$, we find that $a \not\preceq b$. So by hypothesis, there exists an increasing open set G containing a such that b belongs to G^c or a decreasing open set H containing b such that a belongs to H^c. Thus, $a \in U = G \bigcap A$ which is an increasing open subset of $(A, \tau_A, \preceq_A)$ and $b \in V = H \bigcap A$ which is a decreasing open subset of $(A, \tau_A, \preceq_A)$. Obviously, $b \notin U$ and $a \notin V$. Hence, $(A, \tau_A, \preceq_A)$ is a T_{c_0}-ordered space.

Theorem 3.39. Every open ordered subspace of a T_{c_i}-ordered space is a T_{c_i}-ordered space, for $i = \frac{1}{2}, 1, 1\frac{1}{2}, 2$.

Proof. We only prove the theorem case $i = 2$ and the other cases can be made similarly.
Let $(A, \tau_A, \preceq_A)$ be an open topological ordered subspace of a T_{c_2}-ordered space $(X, \tau, \preceq)$. Then, for each $a, b \in A$ such that $a \npreceq_A b$, we obtain that $a \npreceq b$. So by hypothesis, there exist disjoint an increasing open set U containing a and a decreasing open set V containing b such that $a \in (V^c)'$, $b \in (U^c)'$ and $(V^c)' \bigcap (U^c)' = \emptyset$. Obviously, $a \in G_A = U \bigcap A$ which is an increasing open subset of $(A, \tau_A, \preceq_A)$ and $b \in G_A^c = U^c \bigcap A$. Assume that $b \notin (G_A^c)' = (U^c \bigcap A)' \bigcap A$. Then, $b \notin (U^c \bigcap A)'$. Therefore, there exists $D \in \tau$ such that $b \in D$ and $D \bigcap U^c \bigcap A \subseteq \{b\}$. However, this contradicts that $b \in (U^c)'$. Thus, $b \in (G_A^c)'$. Similarly, $b \in H_A = V \bigcap A$ which is a decreasing open subset of $(A, \tau_A, \preceq_A)$ and $a \in (H_A^c)'$. We can observe that $G_A \bigcap H_A = \emptyset$ and $(G_A^c)' \bigcap (H_A^c)' = \emptyset$. Hence, $(A, \tau_A, \preceq_A)$ is a T_{c_2}-ordered space.

The next four examples illustrate the necessity of an open condition in the above theorem.

Example 3.40. Consider $(A, \tau_A, \preceq_A)$ is a topological ordered subspace of the topological ordered space given in Example 3.9, where $A = \{1, 2, 3\}$. Then, $\tau_A = \{\emptyset, A, \{1\}, \{1, 2\}\}$ and $\preceq_A = \triangle_A \bigcup \{(1, 2), (2, 3), (1, 3)\}$. Since $2 \npreceq 1$ and there does not exist a decreasing open subset G of $(A, \tau_A, \preceq_A)$ containing $\{1\}$ such $2 \notin (G^c)'$, then $(A, \tau_A, \preceq_A)$ is not a $T_{c_{\frac{1}{2}}}$-ordered space.

Example 3.41. Consider $(A, \tau_A, \preceq_A)$ is a topological ordered subspace of the topological ordered space given in Example 3.14, where $A = \{\frac{1}{4}, \frac{1}{2}\}$. Then, τ_A is the discrete topology on A and $\preceq_A = \triangle_A \bigcup \{(\frac{1}{4}, \frac{1}{2})\}$. Since $(A, \tau_A, \preceq_A)$ is a finite Hausdorff space, then any subset of $(A, \tau_A, \preceq_A)$ has no limit points. Hence, $(A, \tau_A, \preceq_A)$ is not a T_{c_1}-ordered space.

Example 3.42. Consider $(A, \tau_A, \preceq_A)$ is a topological ordered subspace of the topological ordered space given in Example 3.18, where $A = \{0\} \bigcup \{\frac{1}{n} : n \in \mathbb{N}\}$. Then, τ_A is the discrete topology on A and $\preceq_A$ is the usual partial order relation on A. Since the limit points of any subset of A is empty, then $(A, \tau_A, \preceq_A)$ is not a $T_{c1\frac{1}{2}}$-ordered space.

Example 3.43. Consider $(A, \tau_A, \preceq_A)$ is a topological ordered subspace of the topological ordered space given in Example 3.24, where $A = \{\frac{1}{n} : n \in \mathbb{N}\}$. Then, τ_A is the discrete topology on A and $\preceq_A$ is the usual partial order relation on A. Since any subset of $(A, \tau_A, \preceq_A)$ has no a limit points, then $(A, \tau_A, \preceq_A)$ is not a T_{c_2}-ordered space.

Lemma 3.44. Let $\{(X_\alpha, \tau_\alpha, \preceq_\alpha) : \alpha \in \Lambda\}$ be the collection of topological ordered spaces and $(X, \tau, \preceq)$ be their product space. If G_j is an increasing (resp. a decreasing) open subset of the coordinate space X_j, then $\prod_j^{-1}(G_j)$ is an increasing (resp. a decreasing) open subset of X, where $\prod_j^{-1}(G_j) = \{\prod X_\alpha : \alpha \neq j\} \times G_j$.

Proof. Let G_j be an increasing open subset of the coordinate space X_j. Then, $\prod_j^{-1}(G_j)$ is an open subset of X. Suppose, to the contrary, that $\prod_j^{-1}(G_j)$ is not increasing. Then, there exists $b = (b_1, b_2, \cdots, b_j, \cdots)$ such that $b \in i(\prod_j^{-1}(G_j))$ and $b \notin \prod_j^{-1}(G_j)$. Therefore, $b_j \in i(G_j)$ and $b_j \notin (G_j)$. This contradicts the increase of G_j. Hence, $\prod_j^{-1}(G_j)$ is an increasing open subset of X.
Similarly, one can prove the lemma when G_j is a decreasing open subset of the coordinate space X_j.

Now, we are in a position to prove the main theorem in this paper.

Theorem 3.45. The product of a family of T_{c_i}-ordered spaces is also a T_{c_i}-ordered space for all $i = 0, \frac{1}{2}, 1, 1\frac{1}{2}$.

Proof. We prove the theorem in the cases of $i = 0$, $i = \frac{1}{2}$. The other two cases can be made similarly.
Let $\{(X_\alpha, \tau_\alpha, \preceq_\alpha) : \alpha \in \Lambda\}$ be a family of T_{c_i}-ordered spaces and $(X, \tau, \preceq)$ be their product ordered space. Assume that $a \npreceq b$ in X, where $a = (a_1, a_2, \cdots, a_n, \cdots)$ and $b = (b_1, b_2, \cdots, b_n, \cdots) : a_j, b_j \in X_{\alpha_j}$, then there exists $\alpha_0 \in \Lambda$ such that $a_{\alpha_0} \npreceq_{\alpha_0} b_{\alpha_0}$ in $(X_{\alpha_0}, \tau_{\alpha_0}, \preceq_{\alpha_0})$.

(i)If $i = 0$, then there exists an increasing (or a decreasing) open subset $G_{\alpha_0} of X_{\alpha_0}$ containing $a_{\alpha_0}(b_{\alpha_0})$ such that $b_{\alpha_0}(a_{\alpha_0}) \in (G_{\alpha_0})^c$. Say, G_{α_0} is an increasing open set. If $\prod_{\alpha_0} : X \to X\alpha_0$ is the projection map of X onto the α_0th coordinate, then $\prod_{\alpha_0}^{-1}(G_{\alpha_0})$ is an increasing open subset of X containing a and $b \in [\prod_{\alpha_0}^{-1}(G_{\alpha_0})]^c = X_1 \times X_2 \times \cdots \times G_{\alpha_0}^c \times \cdots$.

(ii)If $i = \frac{1}{2}$, then there exists an increasing (or a decreasing) open subset G_{α_0} of X_{α_0} containing $a_{\alpha_0}(b_{\alpha_0})$ such that $b_{\alpha_0}(a_{\alpha_0}) \in (G_{\alpha_0}^c)'$. Say, G_{α_0} is an increasing open set containing a_{α_0} and $b_{\alpha_0} \in (G_{\alpha_0}^c)'$. Now, $\prod_{\alpha_0}^{-1}(G_{\alpha_0})$ is an increasing open subset of X containing a and $b \in [\prod_{\alpha_0}^{-1}(G_{\alpha_0})]^c = X_1 \times X_2 \times \cdots \times G_{\alpha_0}^c \times \cdots$. Suppose $b \notin ((\prod_{\alpha_0}^{-1} G_{\alpha_0})^c)'$. Then, there exists an open subset H of X such that $b \in H$ and $(\prod_{\alpha_0}^{-1} G_{\alpha_0})^c \bigcap H \subseteq \{b\}$.

Since $\prod_{\alpha_0}$ is an open map, then $\prod_{\alpha_0}(H)$ is an open set. Therefore, $G^c_{\alpha_0} \bigcap \prod_{\alpha_0}(H) \subseteq \{b_{\alpha_0}\}$, but this contradicts that $b_{\alpha_0} \in (G^c_{\alpha_0})'$. Thus, $b \in ((\prod^{-1}_{\alpha_0} G_{\alpha_0})^c)'$.

4 Conclusion

The concept of topological ordered spaces was formulated for the first time by Nachbin [1]. To contribute to this area, we have defined the concepts of T_{c_i}-ordered space $(i = 0, \frac{1}{2}, 1, 1\frac{1}{2}, 2)$. The idea of these concepts is separating the distinct elements using monotonic neighborhoods and their limit points. We have showed the relationships between them as well as their relationships with strong T_i-ordered spaces [2] and T_i-spaces [33] $(i = 0, 1, 2)$ with the help of convenient examples. Some results that connect between these new ordered separation axioms and some topological concepts such as hereditary and topological properties and finite product spaces were established.

Our future works will highlight generalizing these concepts on the contents of supra topology and soft topology.

Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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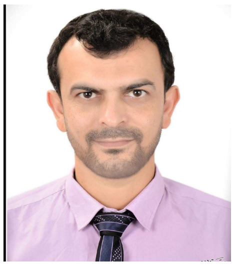

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