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An Efficient Three-Level Explicit Time-Split Approach for Solving Two-Dimensional Heat Conduction Equation

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Abstract: The present paper considers an efficient time-split approach (i.e. a three-level explicit time-split MacCormack scheme) for solving two-dimensional heat conduction equations. Computational cost reduces the two-level explicit MacCormack and splitting. Both stability and convergence of the method are deeply analyzed in $L^{\infty}(0,T;L^2)$-norm under a suitable time step restriction. Numerical experiments suggest that the new algorithm is fast, second order accurate in time and fourth order convergent in space. This shows effectiveness of the numerical scheme and improves some well-known results in literature.

Keywords: A three-level explicit timesplit MacCormack method, Explicit MacCormack scheme, 2D heat equations, locally one-dimensional operators (splitting), Stability and convergence. **AMS Subject Classification (MSC). 35K05, 35A35, 65N35**.

1 Introduction

Most real world problems are related to multi-physics, multi-component and multi-time scale in nature [1, 3]. Some engineering applications of such problems include various heat conduction models [2, 26]. For instance: heat exchangers, mathematical finance (this model is obtained by transforming the Black-Schole equation into the heat one), various biological and chemical systems including diffusion and transportation problems. Numerical simulations of these problems are indispensable and challenging. This paper considers the following two-dimensional heat conduction model subject to the initial and boundary conditions,

$$\frac{\partial u}{\partial t}-a\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=0,\quad (x,y)\in\Omega,\quad t\in(0,T]; \quad (1)$$

$$u(x,y,0)=u_0(x,y),\quad (x,y)\in\overline{\Omega}; \quad (2)$$

$$u(x,y,t)=\varphi(x,y,t),\quad (x,y)\in\partial\Omega,\quad t\in(0,T]; \quad (3)$$

where a is the thermal diffusivity, $\Omega=(0,1)\times(0,1)$ and $\partial\Omega$ is the boundary of Ω. We assume that the initial condition u_0 and the boundary condition φ are assumed to be regular enough and satisfy the condition $\varphi(x,y,0)=u_0(x,y)$, for any $(x,y)\in\partial\Omega$. This requirement guarantees the existence and uniqueness of a smooth solution to the heat equations (1)-(3).

In [3–11], a large class of efficient numerical schemes are based on the collocation and time splitting methods (i.e. operator splitting or fractional step methods). Multiple natural splitting approaches are frequently constructed according to either physical components and subsystems, such as density, velocity, energy and pressure or physical processes, such as reaction, diffusion and convection. However, these splitting methods comprise two major disadvantages: splitting error in the composite algorithm and determination of boundary conditions for the split equations [4, 6]. In literature [11–14], splitting errors of time splitting methods have been extensively investigated. Although the results have been appreciated (stability results), the algorithms have provided unsatisfactory convergence rate (less or equal than two). High order time splitting schemes can be developed to reduce splitting errors.

MacCormack approach [15–17] has been used to solve a wide class of partial differential equations (PDEs).

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There exist both explicit and implicit versions of the method, but the explicit predates the implicit by more than a decade, and it is considered one of the milestones of computational fluid dynamics. MacCormack explicit method is popular because of its simplicity and ease of implementation. The predictor and corrector steps use forward differencing for first-order time derivatives, with alternate one-side differencing for first-order space derivatives. This is particularly advantageous for systems of equations with nonlinear advective jacobian matrices associated with one-side explicit schemes, such as Lax-Wendroff approach (for example, [17–19]). The drawback of this approach stems from its inappropriateness for solving high Reynolds numbers flows. To overcome this difficulty, MacCormack constructed a hybrid version of his scheme (i.e. MacCormack rapid solver method) [20]. This algorithm has been used for solving a large set of nonlinear PDEs [21–25]. The results suggested that the rapid solver method had a good stability condition and was much faster than numerical schemes adopted to solve both steady and unsteady flows at high to low Reynolds numbers [20].

The information gained from both MacCormack and MacCormack rapid solver methods empowers us to analyze a three-level explicit time-split MacCormack approach for solving equations (1)-(3). Recently, the authors [26] have applied a method of fundamental solutions to initial-value problem (1)-(3). Although the numerical experiments performed in their work have suggested that a method of fundamental solutions was so efficient, the convergence rate of the algorithm had not been provided, which is a fundamental tool to appreciate accuracy of a numerical scheme. Another drawback of this technique stems from the fact that numerical evidences have been discussed for large values of the mesh grid $h \in [0.5, 4]$. In fact, the numerical solution obtained using this method reduces for $h \in [0, 0.5)$ (see [26], page 9, Example 1). The time-split approach we study for the model (1)-(3) is new, a three-level explicit predictor-corrector scheme, second order accurate in time and fourth order convergent in space under the time step restriction: $\frac{2ak}{h^2} \leq 1$. In addition, and it is motivated by: (a) the time step restriction (in fact, lots of explicit schemes for solving equation (1)-(3), are stable under the well-known condition of Courant-Friedrich-Lewy: $\frac{4ak}{h^2} \leq 1$); (b) the explicit MacCormack approach and (c) the form of the splitting. Based on the numerical examples presented by the two methods (see tests 2, 3, section 5 of this paper and examples 1, 2, in [26], pages 9-12), the graphs show that both schemes seem to converge with the same degree of accuracy. Unfortunately, the only information provided by the figures is insufficient to make this conclusion. The above-mentioned discussion demonstrates that it is easy to observe that the three-level explicit time-split MacCormack approach is superior to a method of fundamental solutions when solving the two-dimensional heat conduction (1)-(3). An explicit time-split MacCormack algorithm [10, 27–32] "splits" the original MacCormack scheme into a sequence of one-dimensional operations, so it achieves a good stability condition. More precisely, splitting allows to advance the solution in each direction with the maximum allowable time step. This is particularly advantageous if the allowable time steps Δt_x and Δt_y are much different because of differences in the mesh spacings Δx and Δy. To explain this method, we will make use of the 1D difference operators $L_x(\Delta t_x)$ and $L_y(\Delta t_y)$. Setting $u_{ij}^n = u(x_i, y_j, t^n)$, the $L_x(\Delta t_x)$ operator applied to u_{ij}^n,

$$u_{ij}^* = L_x(\Delta t_x) u_{ij}^n, \tag{4}$$

is by definition equivalent to the two-step predictor-corrector MacCormack formulation. The $L_y(\Delta t_y)$ operator is defined in a similar manner, i.e.

$$v_{ij}^* = L_y(\Delta t_y) u_{ij}^n. \tag{5}$$

These expressions make use of a dummy time index, which is denoted by the asterisk. Now, letting $\Delta t_x = \Delta t$ and $\Delta t_y = \frac{\Delta t}{2m}$, where m is a positive integer, a second order accurate scheme can be constructed by applying the L_x and L_y operators to u_{ij}^n, as follows:

$$u_{ij}^{n+1} = \left[L_y\left(\frac{\Delta t}{2m}\right)\right]^m L_x(\Delta t) \left[L_y\left(\frac{\Delta t}{2m}\right)\right]^m u_{ij}^n.$$

This sequence is quite useful for the case $\Delta y << \Delta x$.

To construct this method, we consider the 1D difference operators $L_x(\Delta t_x)$ and $L_y(\Delta t_y)$ defined by equations (4) and (5), respectively. Following the approach presented in ([33], page 231), a second-order accurate scheme can be constructed by applying the L_x and L_y operators to u_{ij}^n in the following manner:

$$u_{ij}^{n+1} = L_y(k/2) L_x(k) L_y(k/2) u_{ij}^n. \tag{6}$$

Using these tools, we can provide a three-level explicit time-split MacCormack method for solving the initial-boundary value problem (1)-(3). Putting $\Delta t_x = k$, $\Delta t_y = \frac{k}{2}$ and $\Delta x = \Delta y := h$, it comes from equations (4), (5) and (6) that

$$u_{ij}^* = L_y(k/2) u_{ij}^n,$$
$$u_{ij}^{**} = L_x(k) u_{ij}^* = L_x(k) L_y(k/2) u_{ij}^n, \textit{ and } u_{ij}^{n+1} = L_y(k/2) u_{ij}^{**}. \tag{7}$$

In the following, we should find explicit expressions of equations $u_{ij}^* = L_y(k/2) u_{ij}^n$ and $u_{ij}^{**} = L_x(k) u_{ij}^*$. This will help giving an explicit formula of the equation $u_{ij}^{n+1} = L_y(k/2) u_{ij}^{**}$, which represents a "one-step time-split MacCormack algorithm". For simplicity, we use both notations: $u)_{ij}^n = u_{ij}^n$ and $[u+v]_{ij}^n = u_{ij}^n + v_{ij}^n$.

The present paper aims to find an efficient solution to the initial value problem (1)-(3), using a three-level explicit time-split MacCormack approach. More precisely, we focus on the following four items:

1. Detailed description of the three-level explicit time-split MacCormack scheme for solving the two-dimensional heat equations (1)-(3);
2. Analysis of the stability of the method;
3. Error estimates of the numerical scheme;
4. A large set of numerical evidences which provide the convergence rate, asserts the theoretical results and shows effectiveness of the method.

Items 1, 2 and 3 represent our original contributions since as far as we know that no piece of literature solves the 2D heat conduction equations (1)-(3) using a three-level explicit time-split MacCormack scheme.

This paper is organized, as follows: In section 2, we present some notations as well as function spaces. We also provide a detailed description of a three-level explicit time-split MacCormack method for solving problem (1)-(3). In Section 3, we explore stability of numerical scheme under the time-step restriction provided in Section 1. Section 4 analyzes the error estimates and the convergence of the method. A wide set of numerical examples which consider the convergence rate of the new algorithm and confirm the theoretical result (on the stability) are presented and critically discussed in Section 5. Conclusion and further research are presented in Section 6.

2 Overview of the three-level explicit time-split MacCormack scheme

This section considers a full description of a three-level explicit time-split MacCormack method applied to two-dimensional heat equations (1)-(3).

Let N and M be two positive integers. Set $k := \Delta t = \frac{T}{N}$; $h := \Delta x = \Delta y = \frac{1}{M}$, be the time step and mesh size, respectively. Put $t^n = kn$, $t^* = (n+r)k$, $t^{**} = (n+s)k$, where $0 < r < s < 1$, so $t^* \in (t^n, t^{n+1})$, $t^{**} \in (t^*, t^{n+1})$; $n = 0, 1, 2, ..., N-1$; $x_i = ih$; $y_j = jh$; $0 \leq i, j \leq M$. Also, let $\Omega_k = \{t^n, 0 \leq n \leq N\}$; $\overline{\Omega}_h = \{(x_i, y_j), 0 \leq i, j \leq M\}$; $\Omega_h = \overline{\Omega}_h \cap \Omega$ and $\partial\Omega_h = \overline{\Omega}_h \cap \partial\Omega$.

Consider $\mathscr{U}_h = \{u^n_{ij}, n = 0, 1, ..., N;\ i, j = 0, 1, 2, ..., M\}$ be the space of grid functions defined on $\Omega_h \times \Omega_k$. Let

$$\delta_t u^*_{ij} = \frac{u^*_{ij} - u^n_{ij}}{k/2};\ \delta_t u^{**}_{ij} = \frac{u^{**}_{ij} - u^*_{ij}}{k};\ \delta_t u^{n+1}_{ij} = \frac{u^{n+1}_{ij} - u^{**}_{ij}}{k/2};$$

$$\delta_x u^n_{i+\frac{1}{2},j} = \frac{u^n_{i+1,j} - u^n_{ij}}{h};\ \delta_y u^n_{i,j+\frac{1}{2}} = \frac{u^n_{i,j+1} - u^n_{ij}}{h};$$

$$\delta^2_x u^n_{ij} = \frac{\delta_x u^n_{i+\frac{1}{2},j} - \delta_x u^n_{i-\frac{1}{2},j}}{h};\ \delta^2_y u^n_{ij} = \frac{\delta_y u^n_{i,j+\frac{1}{2}} - \delta_y u^n_{i,j-\frac{1}{2}}}{h}. \quad (8)$$

Using this, we define the following norms and scalar products.

$$\|u^n\|_{L^2} = h\left(\sum_{i,j=1}^{M-1} |u^n_{ij}|^2\right)^{\frac{1}{2}};$$

$$\|\delta_x u^n\|_{L^2} = h\left(\sum_{j=1}^{M-1}\sum_{i=0}^{M-1} |\delta_x u^n_{i+\frac{1}{2},j}|^2\right)^{\frac{1}{2}};$$

$$\|\delta_y u^n\|_{L^2} = h\left(\sum_{j=0}^{M-1}\sum_{i=1}^{M-1} |\delta_y u^n_{i,j+\frac{1}{2}}|^2\right)^{\frac{1}{2}};$$

$$\|\delta^2_\lambda u^n\|_{L^2} = h\left(\sum_{i,j=1}^{M-1} |\delta^2_\lambda u^n_{ij}|^2\right)^{\frac{1}{2}}, \quad (9)$$

where $\lambda = x$ or y. Furthermore, the scalar products are defined as

$$(u^n, v^n) = h^2 \sum_{i,j=1}^{M-1} u^n_{ij} v^n_{ij};$$

$$< \delta_x u^n, \delta_x v^n >_x = h^2 \sum_{j=1}^{M-1}\sum_{i=0}^{M-1} \delta_x u^n_{i+\frac{1}{2},j} \delta_x v^n_{i+\frac{1}{2},j};$$

and

$$< \delta_y u^n, \delta_y v^n >_y = h^2 \sum_{j=0}^{M-1}\sum_{i=1}^{M-1} \delta_y u^n_{i,j+\frac{1}{2}} \delta_y v^n_{i,j+\frac{1}{2}}. \quad (10)$$

We recall that a three-level explicit time-split MacCormack [5, 27, 30] "splits" the original MacCormack scheme into a sequence of 1D operators, so it achieves a good stability condition. More precisely, splitting allows to advance the solution in each direction with the maximum allowable time step ([33], page 231).

Now, expanding the Taylor series about (x_i, y_j, t^n) at the predictor and corrector steps with time step $k/2$ using forward difference representations yields

$$u^{\bar{*}}_{ij} = u^n_{ij} + \frac{k}{2} u_t)^n_{ij} + \frac{k^2}{8} u_{2t})^n_{ij} + O(k^3);$$

$$u^{\bar{\bar{*}}}_{ij} = u^n_{ij} + \frac{k}{2} u_t)^{\bar{*}}_{ij} + \frac{k^2}{8} u_{2t})^{\bar{*}}_{ij} + O(k^3). \quad (11)$$

Adopting the definition of the operator $L_y(k/2)$, we can consider the equation

$$u_t - a u_{yy} = 0, \text{ which is equivalent to } u_t = a u_{yy}. \quad (12)$$

Using equation (12), we find it easy to observe that

$$u_{2t} = (a u_{yy})_t = a^2 u_{4y}.$$

This fact together with equation (11) provide

$$u^{\bar{*}}_{ij} = u^n_{ij} + \frac{ak}{2}[u_{yy}]^n_{ij} + \frac{k^2 a^2}{8}[u_{4y}]^n_{ij} + O(k^3);$$

$$u^{\bar{\bar{*}}}_{ij} = u^n_{ij} + \frac{ak}{2}[u_{yy}]^{\bar{*}}_{ij} + \frac{k^2 a^2}{8}[u_{4y}]^{\bar{*}}_{ij} + O(k^3). \quad (13)$$

The application of the Taylor series expansion about (x_i,y_j,t^n) with mesh size h using central difference representations provides

$$u^n_{yy,ij}=\delta_y^2u^n_{ij}+O(h^2);\ u^{\bar{*}}_{yy,ij}=\delta_y^2u^{\bar{*}}_{ij}+O(h^2);$$
$$u^n_{4y,ij}=\delta_y^2(\delta_y^2u^n_{ij})+O(h^2);$$
$$u^{\bar{*}}_{4y,ij}=\delta_y^2(\delta_y^2u^{\bar{*}}_{ij})+O(h^2), \qquad (14)$$

where $\delta_y^2w^l_{ij}$ is given by relation (8). Substituting equations (14) into equations (13) to obtain

$$u^{\bar{*}}_{ij}=u^n_{ij}+\frac{ak}{2}\delta_y^2u^n_{ij}+k^2\rho^n_{ij}+O(k^3+kh^2)$$
$$u^{\bar{\bar{*}}}_{ij}=u^n_{ij}+\frac{ak}{2}\delta_y^2u^{\bar{*}}_{ij}+k^2\rho^{\bar{*}}_{ij}+O(k^3+kh^2), \qquad (15)$$

where

$$\rho^{\alpha}_{ij}=\frac{a^2}{8}\delta_y^2(\delta_y^2u^{\alpha}_{ij}), \qquad (16)$$

where $\alpha=n,\bar{*}$. Substituting the first equation of (15) into both (16) and the second one, straightforward computations give

$$\rho^{\bar{*}}_{ij}=\frac{a^2}{8}\delta_y^2(\delta_y^2u^n_{ij})+O(k)$$
$$u^{\bar{\bar{*}}}_{ij}=u^n_{ij}+\frac{ak}{2}\delta_y^2u^n_{ij}+k^2(2\rho^n_{ij}+\rho^{\bar{*}}_{ij})+O(k^3+kh^2). \quad (17)$$

Taking the average of $u^{\bar{*}}_{ij}$ and $u^{\bar{\bar{*}}}_{ij}$ to get

$$\frac{u^{\bar{*}}_{ij}+u^{\bar{\bar{*}}}_{ij}}{2}=u^n_{ij}+\frac{ak}{2}\delta_y^2u^n_{ij}+\frac{3a^2k^2}{16}\delta_y^2(\delta_y^2u^n_{ij})+O(k^3+kh^2). \qquad (18)$$

On the other hand, to define the operator $L_x(k)$, we should consider the equation

$$u_t=au_{xx}. \qquad (19)$$

It comes from equation (19), that

$$u_{2t}=au_{xx,t}=a^2u_{4x}. \qquad (20)$$

Applying the Taylor series expansion about (x_i,y_j,t^*) (where $t^*\in(t^n,t^{n+1})$ is the time used at the beginning of the next step in a time-split MacCormack scheme) with mesh size h using central difference representation, we obtain

$$u^*_{xx,ij}=\delta_x^2u^*_{ij}+O(h^2);\ u^*_{4x,ij}=\delta_x^2(\delta_x^2u^*_{ij})+O(h^2);$$
$$u^{\overline{**}}_{xx,ij}=\delta_x^2u^{\overline{**}}_{ij}+O(h^2);$$
$$u^{\overline{**}}_{4x,ij}=\delta_x^2(\delta_x^2u^{\overline{**}}_{ij})+O(h^2), \qquad (21)$$

where $\delta_x^2u^l_{ij}$ is defined by equation (8). Moreover, expanding the Taylor series at the predictor and corrector steps about (x_i,y_j,t^*) with time step k using forward difference, it becomes easy to observe that

$$u^{\overline{**}}_{ij}=u^*_{ij}+ku_t)^*_{ij}+\frac{k^2}{2}u_{2t})^*_{ij}+O(k^3);$$
$$u^{\overline{\overline{**}}}_{ij}=u^*_{ij}+ku_t)^{\overline{**}}_{ij}+\frac{k^2}{2}u_{2t})^{\overline{**}}_{ij}+O(k^3). \qquad (22)$$

A combination of equations (22), (21), (19) and (20) provides

$$u^{\overline{**}}_{ij}=u^*_{ij}+ak\delta_x^2u^*_{ij}+\frac{a^2k^2}{2}\delta_x^2(\delta_x^2u^*_{ij})+O(k^3+kh^2);$$
$$u^{\overline{\overline{**}}}_{ij}=u^*_{ij}+ak\delta_x^2u^{\overline{**}}_{ij}+\frac{a^2k^2}{2}\delta_x^2(\delta_x^2u^{\overline{**}}_{ij})+O(k^3+kh^2) (23)$$

To obtain simple expressions of $\delta_x^2u^{\overline{**}}_{ij}$ and $\delta_x^2(\delta_x^2u^{\overline{**}}_{ij})$, we should use the first equation in (23). Tracking the infinitesimal term in this equation, direct computations give

$$\delta_x^2u^{\overline{**}}_{ij}\approx\delta_x^2u^*_{ij}+ak\delta_x^2\left(\delta_x^2u^*_{ij}\right)+\frac{a^2k^2}{2}\delta_x^2(\delta_x^4u^*_{ij})$$
$$\text{and}\ \ \delta_x^2(\delta_x^2u^{\overline{**}}_{ij})\approx\delta_x^2(\delta_x^2u^*_{ij}).$$

The truncation of this error term does not compromise the result. This fact together with relation (23) yield

$$u^{\overline{\overline{**}}}_{ij}=u^*_{ij}+ak\delta_x^2u^*_{ij}+\frac{3a^2k^2}{2}\delta_x^2(\delta_x^2u^*_{ij})+O(k^3+kh^2). \qquad (24)$$

Taking the average of $u^{\overline{**}}_{ij}$ and $u^{\overline{\overline{**}}}_{ij}$, it becomes easy to see that

$$\frac{u^{\overline{**}}_{ij}+u^{\overline{\overline{**}}}_{ij}}{2}=u^*_{ij}+ak\delta_x^2u^*_{ij}+a^2k^2\delta_x^2(\delta_x^2u^*_{ij})+O(k^3+kh^2). \qquad (25)$$

Similarly, starting with the one-dimensional equation: $u_t-au_{yy}=0$, expanding the Taylor series about (x_i,y_j,t^{**}) (where t^{**} represents the time used at the last step in a time-split MacCormack approach) at the predictor and corrector steps with time step $k/2$ and mesh size h, using forward difference representations to get

$$\frac{u^{\overline{n+1}}_{ij}+u^{\overline{\overline{n+1}}}_{ij}}{2}=u^{**}_{ij}+\frac{ak}{2}\delta_y^2u^n_{ij}+\frac{3a^2k^2}{16}\delta_y^2(\delta_y^2u^{**}_{ij})+O(k^3+kh^2). \qquad (26)$$

To construct a three-level explicit time-split MacCormack method for solving the heat conduction equation (1)-(3), we must follow the ideas presented in the literature to construct the explicit MacCormack scheme [16, 20, 27, 30]. Specifically, we should neglect the terms of second order together with the infinitesimal term $O(k^3+kh^2)$ in equations (18), (25) and (26). In addition, the terms u^*_{ij}, u^{**}_{ij} and u^{n+1}_{ij} must be defined as the average of predicted and corrected values. That is,

$$u^*_{ij}=\frac{u^{\bar{*}}_{ij}+u^{\bar{\bar{*}}}_{ij}}{2};\ \ u^{**}_{ij}=\frac{u^{\overline{**}}_{ij}+u^{\overline{\overline{**}}}_{ij}}{2}\ \ \text{and}\ \ u^{n+1}_{ij}=\frac{u^{\overline{n+1}}_{ij}+u^{\overline{\overline{n+1}}}_{ij}}{2}. \qquad (27)$$

Thus, equations

$$u^*_{ij}=L_y(k/2)u^n_{ij};\ \ u^{**}_{ij}=L_x(k)u^*_{ij}\ \ \text{and}\ \ u^{n+1}_{ij}=L_y(k/2)u^{**}_{ij}, \qquad (28)$$

are by definition equivalent to

$$u^*_{ij}=u^n_{ij}+\frac{ak}{2}\delta_y^2u^n_{ij};\ \ u^{**}_{ij}=u^*_{ij}+ak\delta_x^2u^*_{ij}$$
$$\text{and}\ \ u^{n+1}_{ij}=u^{**}_{ij}+\frac{ak}{2}\delta_y^2u^{**}_{ij}. \qquad (29)$$

The property of the operator $L_y(k/2)L_x(k)L_y(k/2)$ related to the symmetric, together with equations (18), (25) and (26) suggest that the considered method is a three-level explicit predictor-corrector numerical scheme, second order accurate in time and fourth order convergent in space. This result is asserted by a large set of numerical examples (see section 5 for more details). Utilizing the definition of the linear operators "δ_x^2" and "δ_y^2" given by relation (8), equation (29) can be rewritten as follows: For $n = 0, 1, ..., N-1$;

$$u_{ij}^{*} = u_{ij}^{n} + \frac{ak}{2h^2}(u_{i,j+1}^{n} - 2u_{ij}^{n} + u_{i,j-1}^{n}),$$
$$i = 0,1,...,M; \;\; j = 1,2,...,M-1; \tag{30}$$

$$u_{ij}^{**} = u_{ij}^{*} + \frac{ak}{h^2}(u_{i+1,j}^{*} - 2u_{ij}^{*} + u_{i-1,j}^{*}),$$
$$i = 1,2,...,M-1; \;\; j = 0,1,...,M; \tag{31}$$

$$u_{ij}^{n+1} = u_{ij}^{**} + \frac{ak}{2h^2}(u_{i,j+1}^{**} - 2u_{ij}^{**} + u_{i,j-1}^{**}),$$
$$i = 0,1,...,M; \;\; j = 1,2,...,M-1, \tag{32}$$

with the initial and boundary conditions. For $i, j = 0, 1, ..., M$,

$$u_{ij}^{0} = u_0(x_i, y_j); u_{i0}^{n} = \varphi_{i0}^{n};\; u_{iM}^{n} = \varphi_{iM}^{n};\; u_{0j}^{n} = \varphi_{0j}^{n};$$
$$u_{Mj}^{n} = \varphi_{Mj}^{n};\; u_{0j}^{*} = \varphi_{0j}^{n+1}; u_{Mj}^{*} = \varphi_{Mj}^{n+1};$$
$$u_{j0}^{*} = \varphi_{j0}^{n+1}; u_{jM}^{*} = \varphi_{jM}^{n+1}; u_{0j}^{**} = \varphi_{0j}^{n+1};\; u_{Mj}^{**} = \varphi_{Mj}^{n+1};$$
$$u_{j0}^{**} = \varphi_{j0}^{n+1}; u_{jM}^{**} = \varphi_{jM}^{n+1}; u_{i0}^{N} = \varphi_{i0}^{N};$$
$$u_{iM}^{N} = \varphi_{iM}^{N}; u_{0j}^{N} = \varphi_{0j}^{N};\; u_{Mj}^{N} = \varphi_{Mj}^{N}, \tag{33}$$

which represent a full description of a three-level explicit time-split MacCormack method for solving the 2D heat conduction equations (1)-(3).

In the following, we prove the stability, the error estimates and the convergence rate of our method under the time step restriction

$$\frac{2ak}{h^2} \leq 1, \tag{34}$$

where a is the thermal diffusivity given in equation (1). Estimate (34) is well known in literature as Courant-Friedrich-Lewy condition. We assume that the analytical solution $\overline{u} \in L^\infty(0,T;L^2(\Omega)) \cap H^1(0,T;H^3(\Omega)) \cap H^2(0,T;H^1(\Omega)) \cap H^2(0,T;L^2(\Omega)) \cap L^2(0,T;H^4(\Omega))$. Thus, there exists a nonzero constant $\widetilde{C} > 0$, independent of time step k and mesh size h, that satisfies

$$|||\overline{u}|||_{L^\infty(0,T;L^2(\Omega))} + |||\overline{u}|||_{H^1(0,T;H^3(\Omega))}$$
$$+|||\overline{u}|||_{H^2(0,T;H^1(\Omega))} + |||\overline{u}|||_{H^2(0,T;L^2(\Omega))}$$
$$+|||\overline{u}|||_{L^2(0,T;H^4(\Omega))} \leq \widetilde{C}. \tag{35}$$

To prove the main results of this paper (namely Theorem 1-2), we need some intermediate results (Lemma 1-3).

Lemma 1. *Let $u_{ij}^n = u(x_i, y_j, t^n)$ be the numerical solution provided by the scheme (30)-(33), $\overline{u}_{ij}^n = \overline{u}(x_i, y_j, t^n)$ be the exact one and let $e_{ij}^n = u_{ij}^n - \overline{u}_{ij}^n$ be the error. We recall that $\overline{u}_{ij}^{*} = \frac{\overline{u}_{ij}^{\overline{*}} + \overline{u}_{ij}^{\overline{\overline{*}}}}{2}$, $\overline{u}_{ij}^{**} = \frac{\overline{u}_{ij}^{\overline{**}} + \overline{u}_{ij}^{\overline{\overline{**}}}}{2}$, satisfy relations (18) and (25), respectively. u_{ij}^{*} and u_{ij}^{**} are given by equations (30) and (31), respectively. The following equalities hold:*

$$< \delta_x^2 e_{ij}^n, e_{ij}^n >_x = h^2 \sum_{j,i=1}^{M-1} \frac{1}{h^2} \left(e_{i+1,j}^n - 2e_{ij}^n + e_{i-1,j}^n\right) e_{ij}^n$$
$$= -\|\delta_x e^n\|_{L^2(\Omega)}^2, \tag{36}$$

and

$$< \delta_y^2 e_{ij}^n, e_{ij}^n >_y = \; h^2 \sum_{j,i=1}^{M-1} \frac{1}{h^2} \left(e_{i,j+1}^n - 2e_{ij}^n + e_{i,j-1}^n\right) e_{ij}^n$$
$$= -\|\delta_y e^n\|_{L^2(\Omega)}^2, \tag{37}$$

where the operators δ_x and δ_y are defined in relation (8).

Proof. (of Lemma 1). We should prove equation (36) only. The proof of relation (37) is similar.

Using the definition of the operator δ_x^2 and the scalar product $< \cdot, \cdot >_x$ given by (8) and (10), respectively, it becomes easy to show that

$$< \delta_x^2 e_{ij}^n, e_{ij}^n >_x = h^2 \sum_{i,j=1}^{M-1} \frac{1}{h^2} \left(e_{i+1,j}^n - 2e_{ij}^n + e_{i-1,j}^n\right) e_{ij}^n$$
$$= \sum_{i,j=1}^{M-1} \left[\left(e_{i+1,j}^n - e_{ij}^n\right) e_{ij}^n - \left(e_{i,j}^n - e_{i-1,j}^n\right) e_{ij}^n\right] =$$
$$h \sum_{i,j=1}^{M-1} \left[\left(\frac{e_{i+1,j}^n - e_{ij}^n}{h}\right) e_{ij}^n - \left(\frac{e_{i,j}^n - e_{i-1,j}^n}{h}\right) e_{ij}^n\right]$$
$$= h \sum_{i,j=1}^{M-1} \left[\left(\delta_x e_{i+\frac{1}{2},j}^n\right) e_{ij}^n - \left(\delta_x e_{i-\frac{1}{2},j}^n\right) e_{ij}^n\right] =$$
$$h \sum_{j=1}^{M-1} \left((\delta_x e_{\frac{3}{2},j}^n) e_{1j}^n - (\delta_x e_{\frac{1}{2},j}^n) e_{1j}^n\right) +$$
$$\left((\delta_x e_{\frac{5}{2},j}^n) e_{2j}^n - (\delta_x e_{\frac{3}{2},j}^n) e_{2j}^n\right)$$
$$+ \left((\delta_x e_{\frac{7}{2},j}^n) e_{3j}^n - (\delta_x e_{\frac{5}{2},j}^n) e_{5j}^n\right) + \cdots +$$
$$\left((\delta_x e_{M-\frac{3}{2},j}^n) e_{M-2,j}^n - (\delta_x e_{M-\frac{5}{2},j}^n) e_{M-2,j}^n\right) + \left((\delta_x e_{M-\frac{1}{2},j}^n) e_{M-1,j}^n - (\delta_x e_{M-\frac{3}{2},j}^n) e_{M-1,j}^n\right)\Bigg\} =$$
$$h \sum_{j=1}^{M-1} \Bigg\{ -\left(e_{2j}^n - e_{1j}^n\right) \delta_x e_{\frac{3}{2},j}^n - \left(e_{3j}^n - e_{2j}^n\right) \delta_x e_{\frac{5}{2},j}^n - \left(e_{4j}^n - e_{3j}^n\right) \delta_x e_{\frac{7}{2},j}^n - \cdots -$$
$$\left(e_{M-1,j}^n - e_{M-2,j}^n\right) \delta_x e_{M-\frac{3}{2},j}^n + (\delta_x e_{M-\frac{1}{2},j}^n) e_{M-1,j}^n - (\delta_x e_{\frac{1}{2},j}^n) e_{1,j}^n \Bigg\}. \tag{38}$$

From the boundary condition (33), we have that $e^n_{Mj} = e^n_{0j} = 0$. Hence $(\delta_x e^n_{M-\frac{1}{2},j})e^n_{Mj} = 0$ and $(\delta_x e^n_{\frac{1}{2},j})e^n_{0j} = 0$. This fact together with equation (38) result in

$$h^2\sum_{i,j=1}^{M-1}\frac{1}{h^2}\left(e^n_{i+1,j}-2e^n_{ij}+e^n_{i-1,j}\right)e^n_{ij}$$
$$=h^2\sum_{j,i=1}^{M-1}\left\{-\left(\delta_x e^n_{\frac{3}{2},j}\right)^2-\left(\delta_x e^n_{\frac{5}{2},j}\right)^2-\cdots-\left(\delta_x e^n_{M-\frac{3}{2},j}\right)^2\right.$$
$$\left.-\left(\frac{e^n_{Mj}-e^n_{M-1,j}}{h}\right)\delta_x e^n_{M-\frac{1}{2},j}-\left(\frac{e^n_{1j}-e^n_{0,j}}{h}\right)\delta_x e^n_{\frac{1}{2},j}\right\}$$
$$=-ah^2\sum_{i=0}^{M-1}\sum_{j=1}^{M-1}\left(\delta_x e^n_{i+\frac{1}{2},j}\right)^2=-\|\delta_x e^n\|^2_{L^2(\Omega)}.$$

Lemma 2. *Consider $v\in H^4(\Omega)$ as a function satisfying $v|_{[x_i,x_{i+1}]}\in\mathscr{C}^6[x_i,x_{i+1}]$, for $i=0,1,2,...,M-1$. Then, it holds*

$$\frac{1}{h^2}(v_{i+1}-2v_i+v_{i-1})-v_{2x,i}$$
$$=\frac{h^2}{12}v_{4x,i}-\frac{h^4}{720}\left[v_{6x}(\theta_i^{(3)})+v_{6x}(\theta_i^{(4)})\right],$$
$$\text{for}\quad i=1,2,...,M-1,$$

where $\theta_i^{(4)}\in(x_{i-1},x_i)$, $\theta_i^{(3)}\in(x_i,x_{i+1})$ and v_{mx} denotes the derivative of order m of v. Furthermore, for $i=2,3,...,M-2$,

$$\frac{1}{h^4}(v_{i+2}-4v_{i+1}+6v_i-4v_{i-1}+v_{i-2})-v_{4x,i}$$
$$=h^2\left\{\frac{1}{720}\left[v_{6x}(\theta_i^{(1)})+v_{6x}(\theta_i^{(2)})\right]+\frac{241}{3220}\left[v_{6x}(\theta_i^{(3)})+v_{6x}(\theta_i^{(4)})\right]\right\},$$

where $\theta_i^{(2)}\in(x_{i-2},x_{i-1})$, $\theta_i^{(4)}\in(x_{i-1},x_i)$, $\theta_i^{(3)}\in(x_i,x_{i+1})$ and $\theta_i^{(1)}\in(x_{i+1},x_{i+2})$.

Proof.(of Lemma 2) Expanding the Taylor series about x_i with grid spacing h using both forward and backward differences to obtain

$$v_{i+2}=v_{i+1}+hv_{x,i+1}+\frac{h^2}{2}v_{2x,i+1}+\frac{h^3}{6}v_{3x,i+1}+\frac{h^4}{24}v_{4x,i+1}$$
$$+\frac{h^5}{120}v_{5x,i+1}+\frac{h^6}{720}v_{6x}(\theta_i^{(1)}),\tag{39}$$

where $\theta_i^{(1)}\in(x_{i+1},x_{i+2})$;

$$v_{i-2}=v_{i-1}-hv_{x,i-1}+\frac{h^2}{2}v_{2x,i-1}-\frac{h^3}{6}v_{3x,i-1}+\frac{h^4}{24}v_{4x,i-1}$$
$$-\frac{h^5}{120}v_{5x,i-1}+\frac{h^6}{720}v_{6x}(\theta_i^{(2)}),\tag{40}$$

where $\theta_i^{(2)}\in(x_{i-2},x_{i-1})$;

$$v_i=v_{i+1}-hv_{x,i+1}+\frac{h^2}{2}v_{2x,i+1}-\frac{h^3}{6}v_{3x,i+1}+\frac{h^4}{24}v_{4x,i+1}$$
$$-\frac{h^5}{120}v_{5x,i+1}+\frac{h^6}{720}v_{6x}(\theta_i^{(3)}),\tag{41}$$

where $\theta_i^{(3)}\in(x_i,x_{i+1})$;

$$v_i=v_{i-1}+hv_{x,i-1}+\frac{h^2}{2}v_{2x,i-1}+\frac{h^3}{6}v_{3x,i-1}+\frac{h^4}{24}v_{4x,i-1}$$
$$+\frac{h^5}{120}v_{5x,i-1}+\frac{h^6}{720}v_{6x}(\theta_i^{(4)}),\tag{42}$$

where $\theta_i^{(4)}\in(x_{i-1},x_i)$.

Similarly, applying the Taylor expansion for both derivative and higher order derivatives of v to obtain

$$v_{x,i+1}=v_{x,i}+hv_{2x,i}+\frac{h^2}{2}v_{3x,i}+\frac{h^3}{6}v_{4x,i}+\frac{h^4}{24}v_{5x,i}$$
$$+\frac{h^5}{120}v_{6x}(\theta_i^{(5)}),\tag{43}$$

where $\theta_i^{(5)}\in(x_i,x_{i+1})$;

$$v_{x,i-1}=v_{x,i}-hv_{2x,i}+\frac{h^2}{2}v_{3x,i}-\frac{h^3}{6}v_{4x,i}+\frac{h^4}{24}v_{5x,i}$$
$$-\frac{h^5}{120}v_{6x}(\theta_i^{(6)}),\tag{44}$$

where $\theta_i^{(6)}\in(x_{i-1},x_i)$;

$$v_{2x,i+1}=v_{2x,i}+hv_{3x,i}+\frac{h^2}{2}v_{4x,i}+\frac{h^3}{6}v_{5x,i}+\frac{h^4}{24}v_{6x}(\theta_i^{(7)}),\tag{45}$$

where $\theta_i^{(7)}\in(x_i,x_{i+1})$;

$$v_{2x,i-1}=v_{2x,i}-hv_{3x,i}+\frac{h^2}{2}v_{4x,i}-\frac{h^3}{6}v_{5x,i}+\frac{h^4}{24}v_{6x}(\theta_i^{(8)}),\tag{46}$$

where $\theta_i^{(8)}\in(x_{i-1},x_i)$;

$$v_{3x,i+1}=v_{3x,i}+hv_{4x,i}+\frac{h^2}{2}v_{5x,i}+\frac{h^3}{6}v_{6x}(\theta_i^{(9)}),$$
$$v_{3x,i-1}=v_{3x,i}-hv_{4x,i}+\frac{h^2}{2}v_{5x,i}-\frac{h^3}{6}v_{6x}(\theta_i^{(10)}),\tag{47}$$

where $\theta_i^{(9)}\in(x_i,x_{i+1})$, $\theta_i^{(10)}\in(x_{i-1},x_i)$;

$$v_{4x,i+1}=v_{4x,i}+hv_{5x,i}+\frac{h^2}{2}v_{6x}(\theta_i^{(11)}),$$
$$v_{4x,i-1}=v_{4x,i}-hv_{5x,i}+\frac{h^2}{2}v_{6x}(\theta_i^{(12)}),\tag{48}$$

where $\theta_i^{(11)}\in(x_i,x_{i+1})$, $\theta_i^{(12)}\in(x_{i-1},x_i)$;

$$v_{5x,i+1}=v_{5x,i}+hv_{6x}(\theta_i^{(13)}),\quad v_{5x,i-1}=v_{5x,i}-hv_{6x}(\theta_i^{(12)}),\tag{49}$$

where $\theta_i^{(13)}\in(x_i,x_{i+1})$, $\theta_i^{(14)}\in(x_{i-1},x_i)$.
Now, adding equations (41)-(42) side by side gives

$$2v_i=v_{i+1}+v_{i-1}-h(v_{x,i+1}-v_{x,i-1})+\frac{h^2}{2}(v_{2x,i+1}+v_{2x,i-1})$$
$$-\frac{h^3}{6}(v_{3x,i+1}-v_{3x,i-1})+$$

$$\frac{h^4}{24}(v_{4x,i+1}+v_{4x,i-1})-\frac{h^5}{120}(v_{5x,i+1}-v_{5x,i-1})$$
$$+\frac{h^6}{720}(v_{6x}(\theta_i^{(3)})+v_{6x}(\theta_i^{(4)})). \quad (50)$$

Subtracting (44) from (43), adding (45) and (46), using equations (47), (48) and (55), simple calculations provide

$$v_{x,i+1}-v_{x,i-1}=2hv_{2x,i}+\frac{h^3}{3}v_{4x,i}+\frac{h^5}{720}\left(v_{6x}(\theta_i^{(5)})+v_{6x}(\theta_i^{(6)})\right); \quad (51)$$

$$v_{2x,i+1}+v_{2x,i-1}=2v_{2x,i}+h^2v_{4x,i}+\frac{h^4}{24}\left(v_{6x}(\theta_i^{(7)})+v_{6x}(\theta_i^{(8)})\right); \quad (52)$$

$$v_{3x,i+1}-v_{3x,i-1}=2hv_{4x,i}+\frac{h^3}{6}\left(v_{6x}(\theta_i^{(9)})+v_{6x}(\theta_i^{(10)})\right); \quad (53)$$

$$v_{4x,i+1}+v_{4x,i-1}=2v_{4x,i}+\frac{h^2}{2}\left(v_{6x}(\theta_i^{(11)})+v_{6x}(\theta_i^{(12)})\right);$$
$$v_{5x,i+1}-v_{5x,i-1}=h\left(v_{6x}(\theta_i^{(13)})+v_{6x}(\theta_i^{(14)})\right). \quad (54)$$

Combining equations (43)-(54), straightforward computations result in

$$2v_i=v_{i+1}+v_{i-1}-h^2v_{2x,i}-\frac{h^4}{12}v_{4x,i}$$
$$+h^6\left\{\frac{1}{720}(v_{6x}(\theta_i^{(3)})+v_{6x}(\theta_i^{(4)}))-\frac{1}{120}(v_{6x}(\theta_i^{(5)})+v_{6x}(\theta_i^{(6)}))\right.$$
$$+\frac{1}{48}(v_{6x}(\theta_i^{(7)})+v_{6x}(\theta_i^{(8)}))-\frac{1}{36}(v_{6x}(\theta_i^{(9)})+v_{6x}(\theta_i^{(10)}))$$
$$+\frac{1}{48}(v_{6x}(\theta_i^{(11)})+v_{6x}(\theta_i^{(12)}))-$$
$$\left.\frac{1}{120}(v_{6x}(\theta_i^{(13)})+v_{6x}(\theta_i^{(14)}))\right\}. \quad (55)$$

Since $\theta_i^{(3)},\theta_i^{(5)},\theta_i^{(7)},\theta_i^{(9)},\theta_i^{(11)},\theta_i^{(13)}\in(x_i,x_{i+1})$ and $\theta_i^{(4)},\theta_i^{(6)},\theta_i^{(8)},\theta_i^{(10)},\theta_i^{(12)},\theta_i^{(14)}\in(x_{i-1},x_i)$, without loss of generality, we can assume that $\theta_i^{(3)}=\theta_i^{(5)}=\theta_i^{(7)}=\theta_i^{(9)}=\theta_i^{(11)}=\theta_i^{(13)}$ and $\theta_i^{(4)}=\theta_i^{(6)}=\theta_i^{(8)}=\theta_i^{(10)}=\theta_i^{(12)}=\theta_i^{(14)}$. Using this, relation (55) becomes

$$\frac{1}{h^2}(v_{i+1}-2v_i+v_{i-1})-v_{2x,i}=\frac{h^2}{12}v_{4x,i}$$
$$-\frac{h^4}{720}\left[v_{6x}(\theta_i^{(3)})+v_{6x}(\theta_i^{(4)})\right].$$

This completes the proof of the first item of Lemma 2. Now, we can prove the second item of Lemma 2.

Plugging equations (39) and (41), (40) and (42), (41) and (42), respectively, it becomes easy to see that

$$v_{i+2}-2v_{i+1}+v_i=h^2v_{2x,i+1}+\frac{h^4}{12}v_{4x,i+1}$$
$$+\frac{h^6}{720}\left(v_{6x}(\theta_i^{(1)})+(\theta_i^{(3)})\right); \quad (56)$$

$$v_i-2v_{i-1}+v_{i-2}=h^2v_{2x,i-1}+\frac{h^4}{12}v_{4x,i-1}$$
$$+\frac{h^6}{720}v_{6x}(\theta_i^{(2)})+\frac{h^6}{720}v_{6x}(\theta_i^{(3)});$$

and

$$4v_i=2(v_{i+1}+v_{i-1})-2h(v_{x,i+1}-v_{x,i-1})$$
$$+h^2(v_{2x,i+1}+v_{2x,i-1})$$
$$-\frac{h^3}{3}(v_{3x,i+1}-v_{3x,i-1})+\frac{h^4}{12}(v_{4x,i+1}+v_{4x,i-1})$$
$$-\frac{h^5}{60}(v_{5x,i+1}-v_{5x,i-1})$$
$$+\frac{h^6}{720}\left(v_{6x}(\theta_i^{(3)})+v_{6x}(\theta_i^{(4)})\right). \quad (57)$$

A combination of equations (56)-(57) yields

$$v_{i+2}-4v_{i+1}+6v_i-4v_{i-1}+v_{i-2}$$
$$=-2h(v_{x,i+1}-v_{x,i-1})+2h^2(v_{2x,i+1}+v_{2x,i-1})$$
$$-\frac{h^3}{3}(v_{3x,i+1}-v_{3x,i-1})+\frac{h^4}{6}(v_{4x,i+1}+v_{4x,i-1})$$
$$-\frac{h^5}{60}(v_{5x,i+1}-v_{5x,i-1})$$
$$+\frac{h^6}{720}\left[v_{6x}(\theta_i^{(1)})+v_{6x}(\theta_i^{(2)})+3\left(v_{6x}(\theta_i^{(3)})+v_{6x}(\theta_i^{(4)})\right)\right] \quad (58)$$

Substituting (51)-(54) into (58),simple computations result in

$$v_{i+2}-4v_{i+1}+6v_i-4v_{i-1}+v_{i-2}$$
$$=-4h^2v_{2x,i}-\frac{2h^4}{3}v_{4x,i}+4h^2v_{2x,i}+2h^4v_{4x,i}-\frac{2h^4}{3}v_{4x,i}$$
$$+h^6\left\{\frac{1}{720}\left[v_{6x}(\theta_i^{(1)})\ +v_{6x}(\theta_i^{(2)})+3\left(v_{6x}(\theta_i^{(3)})+v_{6x}(\theta_i^{(4)})\right)\right]\right.$$
$$-\frac{1}{60}\left(v_{6x}(\theta_i^{(5)})+v_{6x}(\theta_i^{(6)})\right)+\frac{1}{12}\left(v_{6x}(\theta_i^{(7)})+v_{6x}(\theta_i^{(8)})\right)$$
$$-\frac{1}{18}\left(v_{6x}(\theta_i^{(9)})+v_{6x}(\theta_i^{(10)})\right)+\frac{1}{12}\left(v_{6x}(\theta_i^{(11)})+v_{6x}(\theta_i^{(12)})\right)$$
$$\left.-\frac{1}{60}\left(v_{6x}(\theta_i^{(13)})+v_{6x}(\theta_i^{(14)})\right)\right\},$$

which is equivalent to

$$v_{i+2}-4v_{i+1}+6v_i-4v_{i-1}+v_{i-2}=h^4v_{4x,i}$$
$$+h^6\left\{\frac{1}{720}\left[v_{6x}(\theta_i^{(1)})+v_{6x}(\theta_i^{(2)})+3\left(v_{6x}(\theta_i^{(3)})+v_{6x}(\theta_i^{(4)})\right)\right]-\right.$$
$$\frac{1}{60}\left(v_{6x}(\theta_i^{(5)})+v_{6x}(\theta_i^{(6)})\right)+\frac{1}{12}\left(v_{6x}(\theta_i^{(7)})+v_{6x}(\theta_i^{(8)})\right)$$
$$-\frac{1}{18}\left(v_{6x}(\theta_i^{(9)})+v_{6x}(\theta_i^{(10)})\right)+\frac{1}{12}\left(v_{6x}(\theta_i^{(11)})+v_{6x}(\theta_i^{(12)})\right)$$
$$\left.-\frac{1}{60}\left(v_{6x}(\theta_i^{(13)})+v_{6x}(\theta_i^{(14)})\right)\right\}. \quad (59)$$

Assuming that $\theta_i^{(3)} = \theta_i^{(5)} = \theta_i^{(7)} = \theta_i^{(9)} = \theta_i^{(11)} = \theta_i^{(13)}$ and $\theta_i^{(4)} = \theta_i^{(6)} = \theta_i^{(8)} = \theta_i^{(10)} = \theta_i^{(12)} = \theta_i^{(14)}$, equation (59) becomes

$$v_{i+2} - 4v_{i+1} + 6v_i - 4v_{i-1} + v_{i-2}$$
$$= h^4 v_{4x,i} + h^6 \left\{ \frac{1}{720} \left[v_{6x}(\theta_i^{(1)}) + v_{6x}(\theta_i^{(2)}) + \right] + \frac{241}{3220} \left[v_{6x}(\theta_i^{(3)}) + v_{6x}(\theta_i^{(4)}) \right] \right\},$$

which is equivalent to

$$\frac{1}{h^4}(v_{i+2} - 4v_{i+1} + 6v_i - 4v_{i-1} + v_{i-2}) - v_{4x,i}$$
$$= h^2 \left\{ \frac{1}{720} \left[v_{6x}(\theta_i^{(1)}) + v_{6x}(\theta_i^{(2)}) + \right] + \frac{241}{3220} \left[v_{6x}(\theta_i^{(3)}) + v_{6x}(\theta_i^{(4)}) \right] \right\}.$$

This ends the proof of Lemma 2.

Lemma 3. *The term ρ_{ij}^{α} given by equation (16) can be bounded as*

$$|\rho_{ij}^{\alpha}| = \frac{a^2}{8} |\delta_y^2(\delta_y^2 \overline{u}_{ij}^{\alpha})| \leq \widehat{C}_1 \left[1 + \widehat{C}_2 h^2 \right], \quad (60)$$

where $\alpha = n, *$. $\widehat{C}_l$, $l = 1, 2$, are positive constant independent of the time step k and the mesh size h.

Proof. (of Lemma 3). It comes from the definition of the operator "δ_y^2" that

$$\rho_{ij}^{\alpha} = \frac{a^2}{8} \delta_y^2(\delta_y^2 \overline{u}_{ij}^{\alpha})$$
$$= \frac{a^2}{8h^4} \left(\overline{u}_{i,j+2}^{\alpha} - 4\overline{u}_{i,j+1}^{\alpha} + 6\overline{u}_{ij}^{\alpha} - 4\overline{u}_{i,j-1}^{\alpha} + \overline{u}_{i,j-2}^{\alpha} \right).$$

This fact together with Lemma 2 give

$$\rho_{ij}^{\alpha} =$$
$$\frac{a^2}{8} \left[\overline{u}_{4y,ij}^{\alpha} + h^2 \left(\frac{1}{720} \left[\overline{u}_{6y}^{\alpha}(x_i, \theta_j^{(1)}) + \overline{u}_{6y}^{\alpha}(x_i, \theta_j^{(2)}) \right] + \frac{241}{3220} \left[\overline{u}_{6y}^{\alpha}(x_i, \theta_j^{(3)}) + \overline{u}_{6y}^{\alpha}(x_i, \theta_j^{(4)}) \right] \right) \right],$$

where $\theta_j^{(2)} \in (y_{i-2}, y_{i-1})$, $\theta_j^{(4)} \in (y_{i-1}, y_i)$, $\theta_j^{(3)} \in (y_i, y_{i+1})$ and $\theta_j^{(1)} \in (y_{i+1}, y_{i+2})$. On the other hand, $\overline{u}(x, \cdot, t)|_{[y_j, y_{j+1}]} \in \mathscr{C}^6([y_j, y_{j+1}])$, for every $x \in (0,1)$, $t \in (0,T)$ and $j = 0, 1, 2, ..., M-1$ and $\||\overline{u}|\|_{L^\infty(0,T;L^2(\Omega))} \leq \widetilde{C}$ (according to estimate (35)). Taking the absolute value, there exist positive constants $\widehat{C}_l$, $l = 1, 2$, independent of the time step k and the mesh grid h so that

$$|\rho_{ij}^{\alpha}| \leq \widehat{C}_1 \left(1 + \widehat{C}_2 h^2 \right).$$

This completes the proof of Lemma 3.

Using Lemmas 1, 2 and 3, we can prove the main results of this paper (Theorems 1-2).

3 Stability analysis of a three-level time-split MacCormack scheme

In this section we analyze stability of the three-level time-split MacCormack scheme (30)-(33) applied to problem (1)-(3).

Theorem 1. *Suppose u is the solution provided by the scheme (30)-(33). Under the time step restriction (34), it holds*

$$\max_{0 \leq n \leq N} \|u^n\|_{L^2(\Omega)} \leq \widetilde{C},$$

where $\widetilde{C}$ is given by estimate (35).

Proof. Combining equations (18), (25) and (30), simple calculations yield

$$e_{ij}^{*} = e_{ij}^n + \frac{ak}{2} \delta_y^2 e_{ij}^n + \frac{3ak^2}{16} \delta_y^2(\delta_y^2 \overline{u}_{ij}^n) + O(k^3 + kh^2). \quad (61)$$

Utilizing the definition of the operator "δ_y^2", equation (61) is equivalent to

$$e_{ij}^{*} = e_{ij}^n + \frac{ak}{2h^2}(e_{i,j+1}^n - 2e_{ij}^n + e_{i,j-1}^n)$$
$$+ \frac{3ak^2}{16h^4}(\overline{u}_{i,j+2}^n - 4\overline{u}_{i,j+1}^n + 6\overline{u}_{ij}^n - 4\overline{u}_{i,j-1}^n + \overline{u}_{i,j-2}^n)$$
$$+ O(k^3 + kh^2). \quad (62)$$

We recall that the present paper aims to provide a general picture of the stability analysis of the scheme (30)-(33). Since the formulas can become quite heavy, for the sake of readability, we must neglect the higher order terms in both time step k and mesh grid h. However, the truncation of these terms does not compromise the result on the stability analysis. Using this fact, equation (62) provides

$$e_{ij}^{*} = e_{ij}^n + \frac{ak}{2h^2}(e_{i,j+1}^n - 2e_{ij}^n + e_{i,j-1}^n).$$

Taking the square, it holds

$$(e_{ij}^{*})^2 = (e_{ij}^n)^2 + \frac{ak}{h^2}(e_{i,j+1}^n - 2e_{ij}^n + e_{i,j-1}^n)e_{ij}^n$$
$$+ \frac{a^2k^2}{4h^4}(e_{i,j+1}^n - 2e_{ij}^n + e_{i,j-1}^n)^2. \quad (63)$$

Now, using inequality $(a \pm b)^2 \leq 2(a^2 + b^2)$, for any $a, b \in \mathbb{R}$, by simple computations, it becomes easy to observe that

$$(e_{i,j+1}^n - 2e_{ij}^n + e_{i,j-1}^n)^2$$
$$leq2 \left[(e_{i,j+1}^n - e_{ij}^n)^2 + (e_{i,j-1}^n - e_{ij}^n)^2 \right]. \quad (64)$$

A combination of equation (37) and estimates (63)-(64) results in

$$(e_{ij}^{*})^2 \leq (e_{ij}^n)^2 + \frac{ak}{h^2}(e_{i,j+1}^n - 2e_{ij}^n + e_{i,j-1}^n)e_{ij}^n$$
$$+ \frac{a^2k^2}{h^2} \left[\left(\delta_y e_{i,j+\frac{1}{2}}^n \right)^2 + \left(\delta_y e_{i,j-\frac{1}{2}}^n \right)^2 \right]. \quad (65)$$

Summing estimate (65) up from $i, j = 1, 2, ..., M-1$, this gives

$$\sum_{i,j=1}^{M-1} (e_{ij}^{*})^2 \leq \sum_{i,j=1}^{M-1} (e_{ij}^n)^2 + \frac{ak}{h^2} \sum_{i,j=1}^{M-1} (e_{i,j+1}^n - 2e_{ij}^n + e_{i,j-1}^n)e_{ij}^n$$
$$+ \frac{a^2k^2}{h^2} \sum_{i,j=1}^{M-1} \left[\left(\delta_y e_{i,j+\frac{1}{2}}^n \right)^2 + \left(\delta_y e_{i,j-\frac{1}{2}}^n \right)^2 \right],$$

which implies

$$\sum_{i,j=1}^{M-1}(e^*_{ij})^2 \le \sum_{i,j=1}^{M-1}(e^n_{ij})^2 + \frac{ak}{h^2}\sum_{i,j=1}^{M-1}(e^n_{i,j+1}-2e^n_{ij}+e^n_{i,j-1})e^n_{ij} + \frac{2a^2k^2}{h^2}\sum_{i=1}^{M-1}\sum_{j=0}^{M-1}\left(\delta_y e^n_{i,j+\frac{1}{2}}\right)^2. \quad (66)$$

Multiplying both sides of inequality (66) by h^2 and using equation (37) we obtain

$$\|e^*\|^2_{L^2(\Omega)} \le \|e^n\|^2_{L^2(\Omega)} - ak\|\delta_y e^n\|^2_{L^2(\Omega)} + \frac{2a^2k^2}{h^2}\|\delta_y e^n\|^2_{L^2(\Omega)}.$$

From the time step restriction (34), i.e. $1-\frac{2ak}{h^2}\ge 0$, it follows

$$\|e^*\|^2_{L^2(\Omega)} \le \|e^n\|^2_{L^2(\Omega)}. \quad (67)$$

Similarly, combining equations (25), (27) and (32) (respectively, equations (26), (27) and (32)), it becomes easy to show that

$$\|e^{**}\|^2_{L^2(\Omega)} \le \|e^*\|^2_{L^2(\Omega)}, \text{ and } \|e^{n+1}\|^2_{L^2(\Omega)} \le \|e^{**}\|^2_{L^2(\Omega)}. \quad (68)$$

Now, plugging estimates (67) and (68), straightforward computations yield

$$\|e^{n+1}\|^2_{L^2(\Omega)} \le \|e^n\|^2_{L^2(\Omega)}.$$

Summing this up from $n = 0,1,2,..,p-1$, for any nonnegative integer p satisfying $1 \le p \le N$, to get

$$\|e^p\|^2_{L^2(\Omega)} \le \|e^0\|^2_{L^2(\Omega)}. \quad (69)$$

It stems from the initial condition given in (33) that $e^0_{ij} = 0$, for $0 \le i,j \le M$. This fact together with estimate (69) result in

$$\|e^p\|_{L^2(\Omega)} = 0. \quad (70)$$

We have that $\|u^p\|_{L^2(\Omega)} - \|\overline{u}^p\|_{L^2(\Omega)} \le \|u^p-\overline{u}^p\|_{L^2(\Omega)} = \|e^p\|_{L^2(\Omega)}$. A combination of this inequality together with equation (70) give

$$\|u^p\|_{L^2(\Omega)} \le \|\overline{u}^p\|_{L^2(\Omega)}.$$

Since $\overline{u}$ is the exact solution, using estimate (35), the proof of Theorem 1 is complete.

4 Convergence of the Method

This section considers the error estimates of a three-level explicit time-split method (30)-(33) for solving equations (1)-(3), under the time step restriction (34). We assume that the exact solution $\overline{u}$ satisfies estimate (35). Let

$$\mathscr{U}_h = \{u^n_{ij},\ n=0,1,2,...,N;\ i,j=0,1,2,...,M\}, \quad (71)$$

be the space of grid functions defined on $\Omega_h \times \Omega_k$, where $\Omega_k = \{t^n,\ 0\le n\le N\}$ and $\Omega_h = \{(x_i,y_j),\ 0\le i,j\le M\}\cap\Omega$.

We introduce the following discrete norms

$$|||u|||_{L^\infty(0,T;L^2(\Omega))} = \max_{0\le n\le N}\|u^n\|_{L^2(\Omega)};$$

$$|||u|||_{L^2(0,T;L^2(\Omega))} = \left(k\sum_{n=0}^{N}\|u^n\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}},$$

and

$$|||u|||_{L^1(0,T;L^2(\Omega))} = k\sum_{n=0}^{N}\|u^n\|_{L^2(\Omega)};\ \text{ for } u\in\mathscr{U}_h. \quad (72)$$

Theorem 2. *Let u be the solution provided by a three-level time-split MacCormack approach (30)-(33). Under the time step restriction (34), the error term $e = u-\overline{u}$ satisfies*

$$|||e|||_{L^\infty(0,T;L^2(\Omega))} \le O(k+h^4).$$

Proof. We recall that the error term provided by the scheme (30)-(33) is denoted by $e^n_{ij} = u^n_{ij} - \overline{u}^n_{ij}$, where $\overline{u}$ satisfies equations (18), (25) and (26) and u are given by relations (30)-(33). Thus, it comes from equation (62) that

$$e^*_{ij} = e^n_{ij} + \frac{ak}{2h^2}(e^n_{i,j+1}-2e^n_{ij}+e^n_{i,j-1}) + \frac{1}{2}k^2(2\rho^n_{ij} + \rho^{\overline{*}}_{ij}) + O(k^3+kh^2),$$

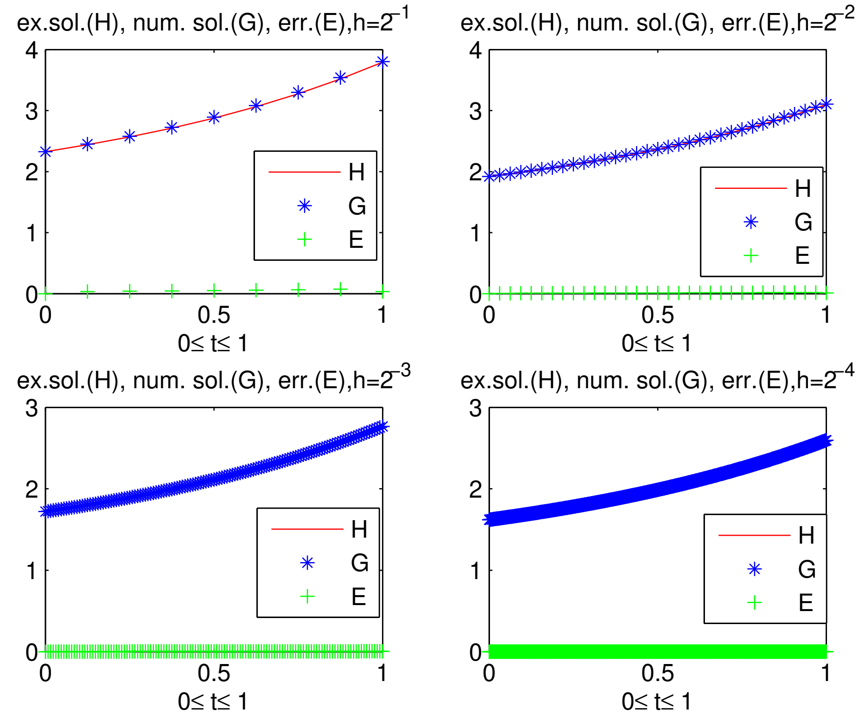
which is equivalent to

$$e^*_{ij} = e^n_{ij} + \tfrac{ak}{2h^2}(e^n_{i,j+1}-2e^n_{ij}+e^n_{i,j-1}) + \tfrac{1}{2}k^2(2\rho^n_{ij}+\rho^{\overline{*}}_{ij}) + C_r(k^3+kh^2),$$

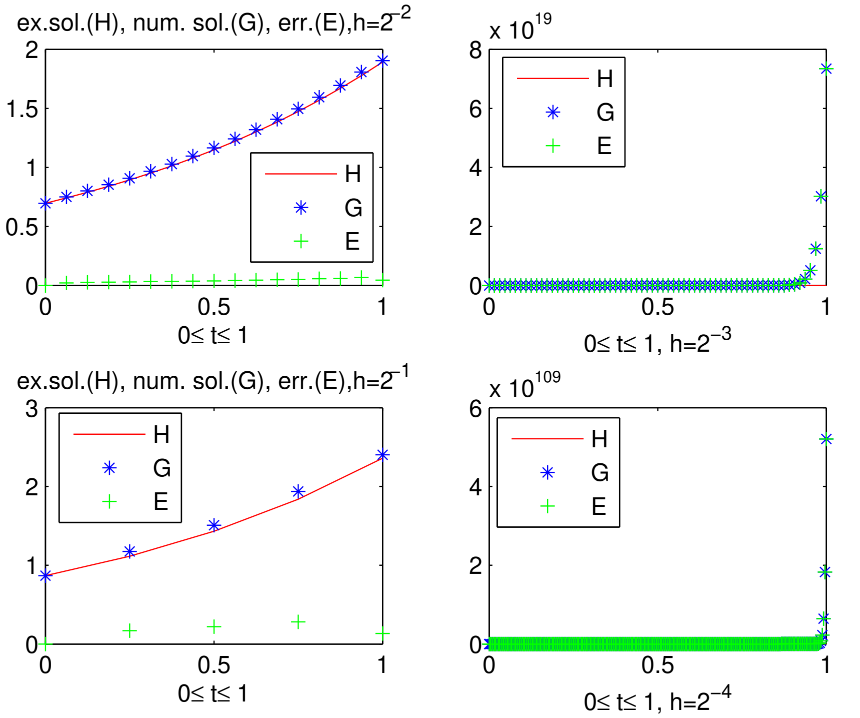
where C_r is a parameter that depends neither on the time step k nor the grid spacing h and ρ^α_{ij} is defined by (16). Taking the square, it is becomes easy to show that

$$\begin{aligned}(e^*_{ij})^2 = {} & (e^n_{ij})^2 + \frac{ak}{h^2}\left(e^n_{i,j+1}-2e^n_{ij}+e^n_{i,j-1}\right)e^n_{ij} \\ & + k^2(2\rho^n_{ij}+\rho^{\overline{*}}_{ij})e^n_{ij} + 2C_r(k^3+kh^2)e^n_{ij} \\ & + \frac{a^2k^2}{4h^4}(e^n_{i,j+1}-2e^n_{ij}+e^n_{i,j-1})^2 \\ & + \frac{ak^3}{2h^2}(e^n_{i,j+1}-2e^n_{ij}+e^n_{i,j-1})(2\rho^n_{ij}+\rho^{\overline{*}}_{ij}) \\ & + \frac{aC_r}{h^2}(e^n_{i,j+1}-2e^n_{ij}+e^n_{i,j-1})(k^4+k^2h^2) \\ & + \frac{k^4}{4}(2\rho^n_{ij}+\rho^{\overline{*}}_{ij})^2 \\ & + C_r^2(k^3+kh^2)^2 + C_r(k^5+k^3h^2)(2\rho^n_{ij}+\rho^{\overline{*}}_{ij}). \quad (73)\end{aligned}$$

Applying the inequalities: $2ab \le a^2+b^2$, $(a\pm b)^2 \le 2(a^2+b^2)$ and $(a\pm b\pm c)^2 \le 3(a^2+b^2+c^2)$, Analysis of stability and convergence of a three-level explicit time-split MacCormack method with $a = 1$.



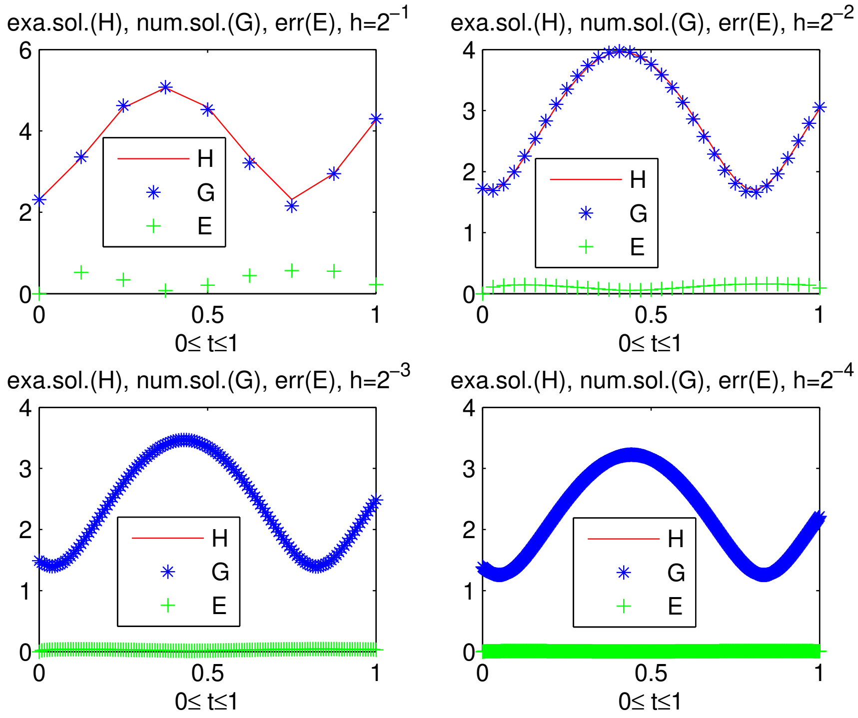
Case: $k = \frac{1}{2}h^2$



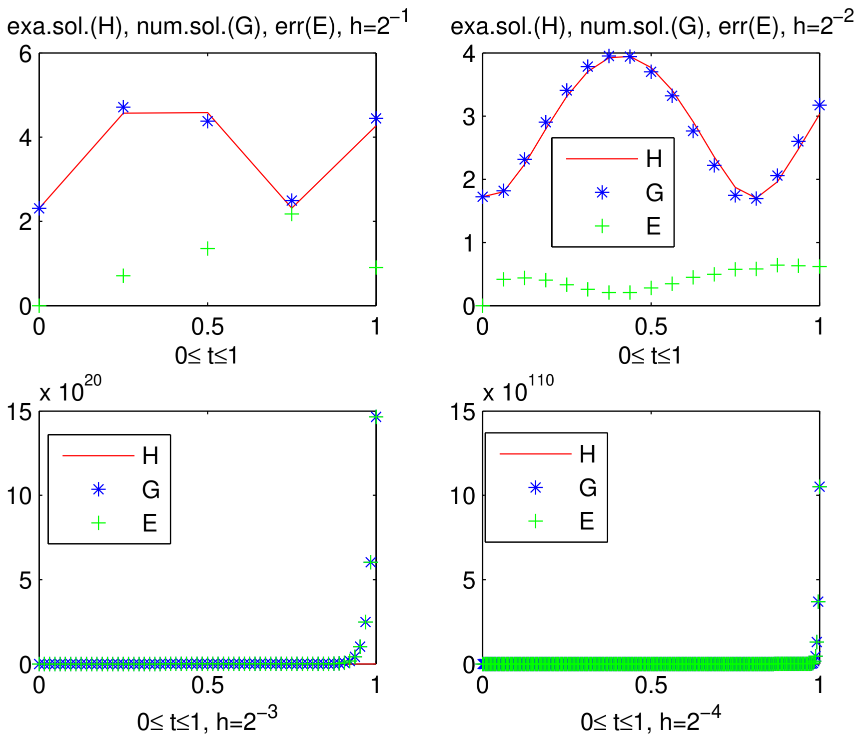
Case: $k = h^2$

Fig. 1 $u(x,y,t) = 1 + \exp\left(t - \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y\right)$

Analysis of stability and convergence of a three-level explicit time-split MacCormack method with $a=1$.



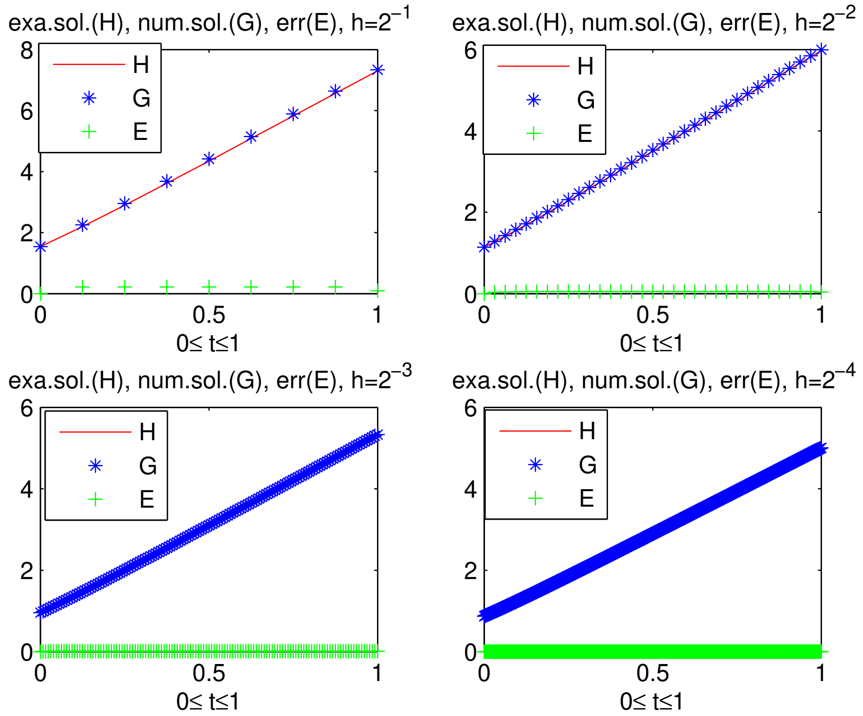
Case: $k=\frac{1}{2}h^2$



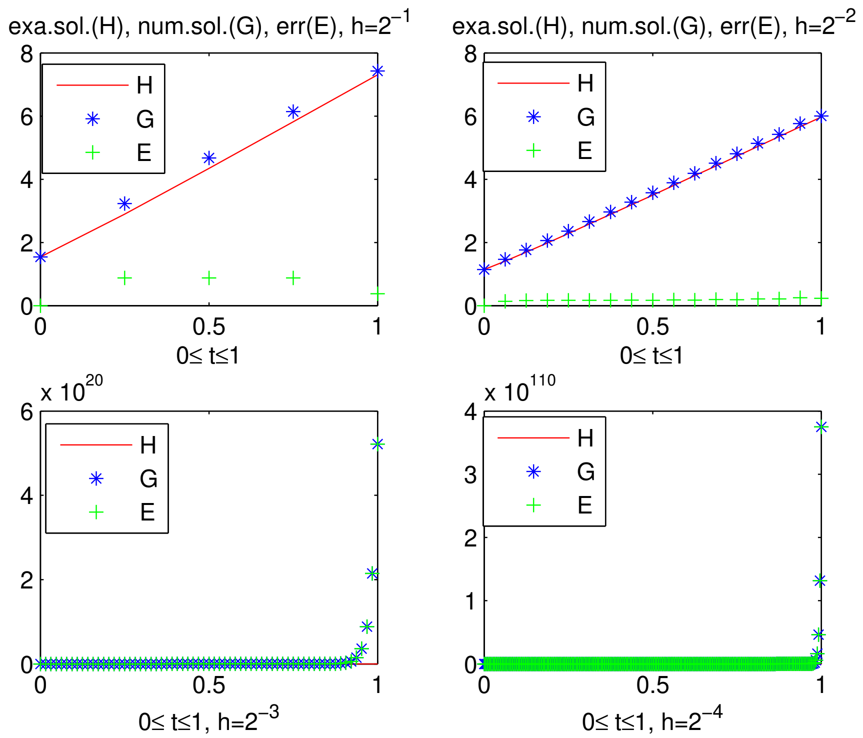
Case: $k=h^2$

Fig. 2 $\overline{u}(x,y,t)=\exp(x+y)\cos(4t+x+y)$

Analysis of stability and convergence of a three-level explicit time-split MacCormack method with $a = 1$.



Case: $k = \frac{1}{2}h^2$



Case: $k = h^2$

Fig. 3 $\overline{u}(x,y,t) = 4t + x^2 + y^2$

Table 1 Case: $k = \frac{1}{2}h^2$.

h	$\|\|E(u)\|\|_{L^2}$	r_u^2	$\|\|E(u)\|\|_{L^\infty}$	r_u^∞	$\|\|E(u)\|\|_{L^1}$	r_u^1
2^{-1}	$0.516\times10^{-}$	—-	0.735×10^{-1}	—-	0.497×10^{-1}	—-
2^{-2}	0.129×10^{-1}	4.0000	0.192×10^{-1}	3.8281	0.123×10^{-1}	4.0407
2^{-3}	0.34×10^{-2}	3.7941	0.52×10^{-2}	3.6923	0.33×10^{-2}	3.7273
2^{-4}	0.9×10^{-3}	3.7778	0.14×10^{-2}	3.7143	0.8×10^{-3}	4.1250
2^{-5}	0.2×10^{-3}	4.5000	0.3×10^{-3}	4.6667	0.2×10^{-3}	4.0000

Table 2 Case: $k = h^2$.

h	$\|\|E(u)\|\|_{L^2}$	r_u^2	$\|\|E(u)\|\|_{L^\infty}$	r_u^∞	$\|\|E(u)\|\|_{L^1}$	r_u^1
2^{-1}	0.2079	—-	0.2805	—-	0.5611	—-
2^{-2}	0.43×10^{-1}	4.8349	0.663×10^{-1}	4.2308	0.412×10^{-1}	13.6189
2^{-3}	0.1007×10^{20}	0.427×10^{-20}	0.7345×10^{20}	0.9×10^{-21}	0.195×10^{19}	0.213×10^{-19}
2^{-4}	0.348×10^{109}	0.2894×10^{-89}	0.5207×10^{110}	0.1411×10^{-89}	0.31×10^{108}	0.629×10^{-89}

Table 3 Case: $k = \frac{1}{2}h^2$.

h	$\|\|E(u)\|\|_{L^2}$	r_u^2	$\|\|E(u)\|\|_{L^\infty}$	r_u^∞	$\|\|E(u)\|\|_{L^1}$	r_u^1
2^{-1}	0.4054	—-	0.5676	—-	0.3672	—-
2^{-2}	0.1214	3.3394	0.1589	3.5721	0.1166	3.1492
2^{-3}	0.336×10^{-1}	3.6131	0.432×10^{-1}	3.6782	0.334×10^{-1}	3.4910
2^{-4}	0.88×10^{-2}	3.8182	0.112×10^{-1}	3.8571	0.83×10^{-2}	4.0241
2^{-5}	0.22×10^{-2}	4.0000	0.29×10^{-2}	3.8621	0.204×10^{-2}	4.0686

for every $a,b,c \in \mathbb{R}$, together with the time step restriction (34) (i.e. $2ak \leq h^2$), equation (73) provides

$$(e^*_{ij})^2 \leq (e^n_{ij})^2 + \frac{ak}{h^2}\left(e^n_{i,j+1} - 2e^n_{ij} + e^n_{i,j-1}\right)e^n_{ij} + k^2|(2\rho^n_{ij} + \rho^{\bar{*}}_{ij})e^n_{ij}| + 2C_r(k^3 + kh^2)|e^n_{ij}| + \frac{a^2k^2}{4h^4}(e^n_{i,j+1} - 2e^n_{ij} + e^n_{i,j-1})^2 + \frac{k^2}{4}|e^n_{i,j+1} - 2e^n_{ij} + e^n_{i,j-1}||2\rho^n_{ij} + \rho^{\bar{*}}_{ij}| + \frac{C_r}{2}|e^n_{i,j+1} - 2e^n_{ij} + e^n_{i,j-1}|(k^3 + kh^2) + \frac{k^4}{2}(2\rho^n_{ij} + \rho^{\bar{*}}_{ij})^2 + 2C_r^2(k^3 + kh^2)^2.$$

which implies

$$(e^*_{ij})^2 \leq (e^n_{ij})^2 + \frac{ak}{h^2}\left(e^n_{i,j+1} - 2e^n_{ij} + e^n_{i,j-1}\right)e^n_{ij} + k^2|(2\rho^n_{ij} + \rho^{\bar{*}}_{ij})e^n_{ij}| + 2C_r(k^3 + kh^2)|e^n_{ij}| + \frac{a^2k^2}{2h^4}\left[(e^n_{i,j+1} - e^n_{ij})^2 + (e^n_{ij} - e^n_{i,j-1})^2\right] + \frac{k^2}{4}|e^n_{i,j+1} - 2e^n_{ij} + e^n_{i,j-1}||2\rho^n_{ij} + \rho^{\bar{*}}_{ij}| + \frac{C_r}{2}|e^n_{i,j+1} - 2e^n_{ij} + e^n_{i,j-1}|(k^3 + kh^2)$$

$$+\frac{k^4}{2}(2\rho_{ij}^n+\rho_{ij}^{\bar{*}})^2+2C_r^2(k^3+kh^2)^2$$
$$\leq (e_{ij}^n)^2+\frac{ak}{h^2}\left(e_{i,j+1}^n-2e_{ij}^n+e_{i,j-1}^n\right)e_{ij}^n$$
$$+\frac{1}{2}\left[k^3(2\rho_{ij}^n+\rho_{ij}^{\bar{*}})^2+8C_r^2(k^5+kh^4)\right]$$
$$+k(e_{ij}^n)^2+\frac{a^2k^2}{2h^2}\left[(\delta_y e_{i,j+\frac{1}{2}}^n)^2+(\delta_y e_{i,j-\frac{1}{2}}^n)^2\right]$$
$$+\frac{k^3}{4}(2\rho_{ij}^n+\rho_{ij}^{\bar{*}})^2+\frac{3k}{8}\left[(e_{i,j+1}^n)^2+4(e_{ij}^n)^2+(e_{i,j-1}^n)^2\right]$$
$$+C_r^2(k^{\frac{3}{2}}+k^{\frac{1}{2}}h^2)^2+\frac{3k}{16}\left((e_{i,j+1}^n)^2+4(e_{ij}^n)^2+(e_{i,j-1}^n)^2\right)$$
$$+\frac{k^4}{2}(2\rho_{ij}^n+\rho_{ij}^{\bar{*}})^2+2C_r^2(k^3+kh^2)^2. \quad (74)$$

Utilizing the time step restriction (34), i.e. $2ak\leq h^2$, estimate (74) results in

$$(e_{ij}^*)^2\leq (e_{ij}^n)^2+\frac{ak}{h^2}\left(e_{i,j+1}^n-2e_{ij}^n+e_{i,j-1}^n\right)e_{ij}^n$$
$$+\frac{1}{2}\left[k^3(2\rho_{ij}^n+\rho_{ij}^{\bar{*}})^2+8C_r^2(k^5+kh^4)\right]+k(e_{ij}^n)^2$$
$$+\frac{ak}{4}\left[(\delta_y e_{i,j+\frac{1}{2}}^n)^2+(\delta_y e_{i,j-\frac{1}{2}}^n)^2\right]+\frac{k^3}{4}(2\rho_{ij}^n$$
$$+\rho_{ij}^{\bar{*}})^2+\frac{3k}{8}\left[(e_{i,j+1}^n)^2+4(e_{ij}^n)^2+(e_{i,j-1}^n)^2\right]$$
$$+C_r^2(k^{\frac{3}{2}}+k^{\frac{1}{2}}h^2)^2+\frac{3k}{16}\left((e_{i,j+1}^n)^2+4(e_{ij}^n)^2+(e_{i,j-1}^n)^2\right)$$
$$+\frac{k^4}{2}(2\rho_{ij}^n+\rho_{ij}^{\bar{*}})^2+2C_r^2(k^3+kh^2)^2.$$

Summing this up from $i,j=1,2,...M-1$, provides

$$\sum_{i,j=1}^{M-1}(e_{ij}^*)^2\leq\sum_{i,j=1}^{M-1}(e_{ij}^n)^2$$
$$+ak\sum_{i,j=1}^{M-1}\frac{1}{h^2}\left(e_{i,j+1}^n-2e_{ij}^n+e_{i,j-1}^n\right)e_{ij}^n$$
$$+\frac{ak}{4}\sum_{i,j=1}^{M-1}\left[(\delta_y e_{i,j+\frac{1}{2}}^n)^2+(\delta_y e_{i,j-\frac{1}{2}}^n)^2\right]+k\sum_{i,j=1}^{M-1}(e_{ij}^n)^2$$
$$+\frac{9k}{16}\sum_{i,j=1}^{M-1}\left[(e_{i,j+1}^n)^2+4(e_{ij}^n)^2+(e_{i,j-1}^n)^2\right]$$
$$+\frac{k^3}{2}(3+2k)\sum_{i,j=1}^{M-1}\left[4(\rho_{ij}^n)^2+(\rho_{ij}^{\bar{*}})^2\right]$$
$$+2C_r^2k\sum_{i,j=1}^{M-1}\left[k^2+2k^4+3h^4+2k(k^4+h^4)\right].$$

Combining the boundary condition (33), i.e. $e_{Mj}^n=e_{0j}^n=0$, for all $j=0,1,...,M$, Lemmas 1 and 3, and multiplying both sides of inequality (75) by h^2, straightforward computations yield

$$h^2\sum_{i,j=1}^{M-1}(e_{ij}^*)^2\leq h^2\sum_{i,j=1}^{M-1}(e_{ij}^n)^2-ak\|\delta_y e^n\|_{L^2(\Omega)}^2$$
$$+\frac{ak}{2}\|\delta_y e^n\|_{L^2(\Omega)}^2+\frac{25kh^2}{16}\sum_{i,j=1}^{M-1}(e_{ij}^n)^2+$$
$$\frac{k^3h^2}{2}(3+2k)(M-1)^2$$
$$\left[4\widehat{C}_1^2(1+\widehat{C}_2h^2)^2+\widehat{C}_1^2(1+\widehat{C}_2h^2)^2\right]$$
$$+2C_r^2kh^2(M-1)^2\left[k^2+2k^4+3h^4+2k(k^4+h^4)\right].$$

Since $h=\frac{1}{M}$, $k\leq 1+k^2$ and $h^2\leq 1+h^4$, this becomes

$$h^2\sum_{j,i=1}^{M-1}(e_{ij}^*)^2\leq h^2\sum_{j,i=1}^{M-1}(e_{ij}^n)^2-\frac{ak}{2}\|\delta_y e^n\|_{L^2(\Omega)}^2$$
$$+\frac{25kh^2}{16}\sum_{j,i=1}^{M-1}(e_{ij}^n)^2+\frac{5\widehat{C}_1^2k^3}{2}(5+2k^2)(1+\widehat{C}_2+\widehat{C}_2h^4)^2$$
$$+2C_r^2k\left[k^2+2k^4+3h^4+2k(k^4+h^4)\right],$$

which implies

$$\|e^*\|_{L^2(\Omega)}^2\leq\|e^n\|_{L^2(\Omega)}^2$$
$$+\widehat{C}_4\left\{k\|e^n\|_{L^2(\Omega)}^2+k^3\left[1+k^2+k^3+h^4+h^8+k^2h^4+k^2h^8\right]+k(1+k)h^4\right\}, \quad (75)$$

where we absorb all constants into a constant $\widehat{C}_4$.

Similarly, one shows that

$$\|e^{**}\|_{L^2(\Omega)}^2\leq\|e^*\|_{L^2(\Omega)}^2$$
$$+\widehat{C}_5\left\{k\|e^*\|_{L^2(\Omega)}^2+k^3\left[1+k^2+k^3+h^4+h^8+k^2h^4+k^2h^8\right]+k(1+k)h^4\right\}, \quad (76)$$

where all the constants have been absorbed into a constant $\widehat{C}_5$, and

$$\|e^{n+1}\|_{L^2(\Omega)}^2\leq\|e^{**}\|_{L^2(\Omega)}^2$$
$$+\widehat{C}_6\left\{k\|e^{**}\|_{L^2(\Omega)}^2+k^3\left[1+k^2+k^3+h^4+h^8+k^2h^4+k^2h^8\right]+k(1+k)h^4\right\}, \quad (77)$$

where all the constants have been absorbed into a constant $\widehat{C}_6$.
Now, setting

$$\varphi_1(k,h)=k^3\left[1+k^2+k^3+h^4+h^8+k^2h^4+k^2h^8\right]+k(1+k)h^4, \quad (78)$$

plugging estimates (75)-(77), straightforward computations give

$$\|e^{n+1}\|_{L^2(\Omega)}^2\leq\|e^n\|_{L^2(\Omega)}^2$$

$$+k\left[\widehat{C}_4+\widehat{C}_5+\widehat{C}_6+k\left(\widehat{C}_4\widehat{C}_5+\widehat{C}_6(\widehat{C}_4+\widehat{C}_5)+k\widehat{C}_4\widehat{C}_5\widehat{C}_6\right)\right]\|e^n\|_{L^2(\Omega)}^2$$

$$+\left[\widehat{C}_4+\widehat{C}_5+\widehat{C}_6+k\left(\widehat{C}_4\widehat{C}_5+\widehat{C}_4\widehat{C}_6+\widehat{C}_5\widehat{C}_6+k\widehat{C}_4\widehat{C}_5\widehat{C}_6\right)\right]\varphi_1(k,h).$$

Absorbing all the constants into a constant $\widehat{C}_7$ yields

$$\|e^{n+1}\|^2_{L^2(\Omega)} \leq \|e^n\|^2_{L^2(\Omega)}$$
$$+\widehat{C}_7 \left\{ k(1+k(1+k))\|e^n\|^2_{L^2(\Omega)} + [1+k(1+k)]\varphi_1(k,h) \right\}.$$

Summing this up from $n = 0,1,2,..,p-1$, for any nonnegative integer p such that $1 \leq p \leq N$, to get

$$\|e^p\|^2_{L^2(\Omega)} \leq \|e^0\|^2_{L^2(\Omega)}$$
$$+\widehat{C}_7 \{k[1+k(1+k)]$$
$$\left.\sum_{n=0}^{p-1}\|e^n\|^2_{L^2(\Omega)} + p[1+k(1+k)]\varphi_1(k,h)\right\}. \quad (79)$$

It comes from the initial condition given in (33), that $e^0_{ij} = 0$, for $0 \leq i,j \leq M$. Applying the Gronwall Lemma, estimate (79) provides

$$\|e^p\|^2_{L^2(\Omega)} \leq \widehat{C}_7 \exp\left\{\widehat{C}_7 pk[1+k(1+k)]\right\} p[1+k(1+k)]\varphi_1(k,h). \quad (80)$$

However, $k = \frac{T}{N}$, so $\widehat{C}_7 kp = \widehat{C}_7 T\frac{p}{N} \leq \widehat{C}_7 T$ (since $p \leq N$). This fact, together with estimate (80) yield

$$\|e^p\|^2_{L^2(\Omega)} \leq \widehat{C}_7 T \exp\left\{\widehat{C}_7 T[1+k(1+k)]\right\}[1+k(1+k)]\varphi_2(k,h)^2,$$

where $\varphi_2(k,h)^2 = k^{-1}\varphi_1(k,h)$, $\varphi_1(k,h)$ is defined by equation (78). Taking the square root, it becomes easy to observe that

$$\|e^p\|_{L^2(\Omega)} \leq \sqrt{\widehat{C}_7 T[1+k(1+k)]}\exp\left\{\frac{\widehat{C}_7 T}{2}[1+k(1+k)]\right\}\varphi_2(k,h). \quad (81)$$

It comes from equality $\varphi_2(k,h)^2 = k^{-1}\varphi_1(k,h)$, and equation (78) that

$$\varphi_2(k,h)^2 = k^2\left[1+k^2+k^3+h^4+h^8+k^2h^4+k^2h^8\right]$$
$$+(1+k)h^4 \leq (k+h^4)^2(\widetilde{C}_8+\varphi_3(k,h)),$$

where $\widetilde{C}_8$ is a positive constant independent of k and h, and $\varphi_3(k,h)$ tends to zero when $k,h \to 0$. Taking the maximum over p of estimate (81), for $0 \leq p \leq N$, the proof of Theorem 2 is completed thanks to equation (72).

5 Numerical Experiments and Convergence Rate

We construct an exact solution to the initial value problem (1)-(3). Some numerical experiments in two-dimensional case are performed using Matlab. We observe satisfactory results, so our algorithm provides good performances for multidimensional problems. More precisely, we consider the constructed solution which is associated with the thermal diffusivity $a = 1$, together with two examples introduced in [26]. The numerical evidences assert both stability and predicted convergence rate from the theory (see Theorem 1 and section 2, Page 6, last paragraph). This convergence rate is obtained by listing in Tables 1-6 the errors between the computed solution and the exact one with different values of mesh size h and time step k, satisfying $k = \frac{1}{2}h^2$.

Now, assuming that the exact solution to problem (1)-(3) is of the form $u(x,y,t) = [1+\exp(ct+dx+by)]^n$, where n is an integer. By straightforward computations, it is easy to see that

$$u_t(x,y,t) = nc\exp(ct+dx+by)[1+\exp(ct+dx+by)]^{n-1}, \quad (82)$$
$$u_x(x,y,t) = nd\exp(ct+dx+by)[1+\exp(ct+dx+by)]^{n-1},$$

and

$$u_{xx}(x,y,t) = nd^2\exp(ct+dx+by)$$
$$[1+n\exp(ct+dx+by)][1+\exp(ct+dx+by)]^{n-2}. \quad (83)$$

Analogously

$$u_{yy}(x,y,t) = nb^2\exp(ct+dx+by)$$
$$[1+n\exp(ct+dx+by)][1+\exp(ct+dx+by)]^{n-2}. \quad (84)$$

Combining equations (82)-(84), it is easy to see that

$$u_t - (u_{xx}+u_{yy})$$
$$= n\exp(ct+dx+by)(1+\exp(ct+dx+by))^{n-1}$$
$$\{c-(d^2+b^2)[1+n\exp(ct+dx+by)]$$
$$\left.(1+\exp(ct+dx+by))^{-1}\right\}$$

Setting $c = d^2+b^2$, this becomes

$$u_t - (u_{xx}+u_{yy}) = n(d^2+b^2)\exp(ct+dx+by)$$
$$\times(1+\exp(ct+dx+by))^{n-1}\{1-[1+n\exp(ct+dx+by)]$$
$$\left.(1+\exp(ct+dx+by))^{-1}\right\}$$
$$= n(1-n)(d^2+b^2)\exp[2(ct+dx+by)]$$
$$\times(1+\exp(ct+dx+by))^{n-2}. \quad (85)$$

For $n = 1$, equation (85) provides

$$u_t - (u_{xx}+u_{yy}) = 0.$$

For instance, taking $d^2+b^2 = 1$, this is equivalent to $b^2 = 1-d^2$. Since b^2 must be greater than zero, this implies $d^2 \leq 1$. For $d = \pm\frac{\sqrt{2}}{2}$, this results in $b = \pm\frac{\sqrt{2}}{2}$ and $c = 1$. Thus, the exact solution is given by $u(x,y,t) = 1+\exp\left(t-\frac{\sqrt{2}}{2}x-\frac{\sqrt{2}}{2}y\right)$, for $t \in [0,1]$ and $(x,y) \in [0,1]^2$. The initial and boundary conditions are defined by this solution.

We take the mesh size $h \in \{\frac{1}{2},\frac{1}{2^2},\frac{1}{2^3},\frac{1}{2^4},\frac{1}{2^5}\}$ and time step $k \in \{\frac{1}{2^2},\frac{1}{2^3},\frac{1}{2^4},\frac{1}{2^5},\frac{1}{2^6},\frac{1}{2^7},\frac{1}{2^8},\frac{1}{2^9},\frac{1}{2^{10}}\frac{1}{2^{11}}\}$. In addition, we set $k = \frac{1}{2}h^2$ (because of the time step restriction (34)) and we compute the error estimates: $|||E(u)|||_{L^2(0,T;L^2)}$, $|||E(u)|||_{L^\infty(0,T;L^2)}$ and $|||E(u)|||_{L^1(0,T;L^2)}$ associated with the time-split method to see that the algorithm is stable,

Table 4 Case: $k = h^2$.

h	$\|\|\|E(u)\|\|\|_{L^2}$	r_u^2	$\|\|\|E(u)\|\|\|_{L^\infty}$	r_u^∞	$\|\|\|E(u)\|\|\|_{L^1}$	r_u^1
2^{-1}	1.4054	—-	2.1771	—-	1.2870	—-
2^{-2}	0.4541	3.0949	0.6393	3.4054	0.4103	2.9909
2^{-3}	0.2009×10^{19}	0.226×10^{-20}	0.1465×10^{22}	0.44×10^{-21}	0.389×10^{20}	0.1106×10^{-19}
2^{-4}	0.701×10^{110}	0.2866×10^{-88}	0.1051×10^{112}	0.1394×10^{-88}	0.63×10^{109}	0.6175×10^{-88}

Table 5 Case: $k = \frac{1}{2}h^2$.

h	$\|\|\|E(u)\|\|\|_{L^2}$	r_u^2	$\|\|\|E(u)\|\|\|_{L^\infty}$	r_u^∞	$\|\|\|E(u)\|\|\|_{L^1}$	r_u^1
2^{-1}	0.2073	—-	0.2188	—-	0.2031	—-
2^{-2}	0.522×10^{-1}	3.9713	0.539×10^{-1}	4.0594	0.52×10^{-1}	3.9058
2^{-3}	0.14×10^{-1}	3.7286	0.143×10^{-1}	3.7892	0.139×10^{-1}	3.7510
2^{-4}	0.36×10^{-2}	3.8889	0.37×10^{-2}	3.8649	0.36×10^{-2}	3.8611
2^{-5}	0.9×10^{-3}	4.0000	0.9×10^{-3}	4.1111	0.9×10^{-3}	4.0000

Table 6 Case: $k = h^2$.

h	$\|\|\|E(u)\|\|\|_{L^2}$	r_u^2	$\|\|\|E(u)\|\|\|_{L^\infty}$	r_u^∞	$\|\|\|E(u)\|\|\|_{L^1}$	r_u^1
2^{-1}	0.7806	—-	0.8750	—-	0.75	—-
2^{-2}	0.1861	4.1945	0.2474	3.5368	0.184	4.0761
2^{-3}	0.715×10^{20}	0.26×10^{-20}	0.5216×10^{21}	0.47×10^{-21}	0.138×10^{20}	0.1333×10^{-19}
2^{-4}	0.25×10^{110}	0.286×10^{-89}	0.375×10^{111}	0.1391×10^{-89}	0.23×10^{109}	0.6000×10^{-88}

second order accurate in time and fourth order convergent in space. Furthermore, we plot the approximate solution, the exact one together with the errors versus n. Analysis shows that the three-level explicit time-split method is more effective than the method of fundamental solutions [26]. In fact, although the authors proved that their method provide good results, they did not give the convergence rate of their algorithm. Finally, when h varies in the given range, we observe from Tables 1-6 that the approximation errors $O(k^\beta)+O(h^\theta)$ are dominated by the k-terms $O(k^\theta)$ (or h-terms $O(h^\beta)$). Consequently, the ratio r_u^p, where $p = 1,2,\infty$, of the approximation errors on two adjacent mesh levels Ω_{2h} and Ω_h is approximately $(2h)^\theta/h^\theta = 2^\theta$, where p refers to the $L^p(0,T;L^2(\Omega)$-error norm. Thus, we should use r_u^p to estimate the corresponding convergence rate with respect to h. Define the norms for the approximate solution u, the exact one $\overline{u}$, and the errors $E(u)$, as follows

$$|||u|||_{L^2(0,T;L^2)} = \left[k\sum_{n=0}^{N}\|u^n\|^2_{L^2_f}\right]^{\frac{1}{2}};$$

$$|||\overline{u}|||_{L^2(0,T;L^2)} = \left[k\sum_{n=0}^{N}\|\overline{u}^n\|^2_{L^2_f}\right]^{\frac{1}{2}};$$

$$|||E(u)|||_{L^2(0,T;L^2)} = \left[k\sum_{n=0}^{N}\|u^n-\overline{u}^n\|^2_{L^2_f}\right]^{\frac{1}{2}};$$

$$|||E(u)|||_{L^1(0,T;L^2)} = k\sum_{n=0}^{N}\|u^n-\overline{u}^n\|_{L^2_f};$$

and

$$|||E(u)|||_{L^\infty(0,T;L^2)} = \max_{0\le n\le N}\|u^n-\overline{u}^n\|_{L^2_f}.$$

• **Test** 1. Suppose Ω is the unit square $(0,1)\times(0,1)$ and $T = 1$. We assume that the thermal diffusivity $a = 1$, so the exact solution $\overline{u}$ is given by

$$\overline{u}(x,y,t) = 1+\exp\left(t-\frac{\sqrt{2}}{2}x-\frac{\sqrt{2}}{2}y\right).$$

The initial and boundary conditions are given by this solution.

Tables 1-2. Analyzing convergence rate $O(h^\theta+\Delta t^\beta)$ for time-split MacCormack by r_u^p, with varying time step $k = \Delta t$ and mesh grid $h = \Delta x$.

• **Test** 2. In this example, we choose the domain Ω to be the unit square $(0,1)^2$ and $T = 1$. The thermal diffusivity a is assumed equals 1. The exact solution is taken in [26]

$$\overline{u}(x,y,t) = \exp(x+y)\cos(4t+x+y),$$

the initial and boundary conditions (2) and (3) have been obtained from this solution.

Tables 3-4. Convergence rates $O(h^{\theta}+\Delta t^{\beta})$ for time-split MacCormack by r_u^p, with varying spacing h and time step k.

• **Test** 3. In this example, we choose the domain Ω as in Test 1 and we consider the exact solution given in [26]

$$\overline{u}(x,y,t) = 4t + x^2 + y^2.$$

The initial and boundary conditions are also given by the exact solution $\overline{u}$.

Tables 5-6. Convergence rates $O(h^{\theta}+\Delta t^{\beta})$ for time-split MacCormack by r_u^p, with varying spacing h and time step k.

Section 4 shows that the algorithm is first order convergent in time and fourth order accurate in space. If the result provided in Section 2, page 6, last paragraph is asserted, the considered method is inconsistent. Surprisingly, **Figures** 1-3 and **Tables 1-6** show that the three-level explicit time-split approach is stable, second order accurate in time and fourth order convergent in space under the time step restriction (34). This confirms the theoretical result provided in Section 2, pages 6-7. Thus, the time-split MacCormack scheme for solving the initial-boundary value problem (1)-(3) is stable, consistent, second order convergent in time and fourth order accurate in space.

6 Conclusion and Further Research

In this paper, we provided a detailed study of stability, error estimates and convergence rate of a three-level explicit time-split MacCormack method for solving the 2D heat conduction equation (1)-(2). The analysis illustrated that our method was stable, consistent, second order accurate in time and fourth order convergent in space under the time step restriction (34). This convergence rate was asserted by a wide set of numerical evidence (see **Figures** 1-3 and **Tables 1-6**). Numerical examples also showed that the new algorithm was (1) More effective than the method of fundamental solutions introduced in [26], (2) Fast and robust tools for the integration of general systems of parabolic PDEs. For high Reynolds number flows where the viscous region becomes very thin, MacCormack developed a hybrid version of his scheme (i.e. MacCormack rapid solver method [20]). This hybrid scheme is an explicit-implicit method which proved to be from 10 to 100 more faster than a time-split MacCormack algorithm (see [33], P. 632). We will apply the rapid solver method to the two-dimensional heat equations in our future investigations.

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Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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