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Estimating the Parameters of Reliability Function based on Rayleigh Distribution

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Abstract: In this paper, reliability function based on the Rayleigh distribution with two parameters is obtained. The location and scale parameters are estimated using different methods. The maximum likelihood, the first and second modified moment and Bayesian estimation methods are established. A numerical study depends on simulated observation is introduced. Rayleigh distribution was conducted to compare different estimation methods based on their mean square error.

Keywords: Bayesian estimation, Maximum likelihood estimation, Moment estimation, Rayleigh distribution, Reliability function, Simulation.

1 Introduction

As a special case of the three parameters Weibull distribution, Lord Rayleigh introduced the Rayleigh distribution [1]. Life time of the random phenomenon can be modeled by the model theory reliability using the Rayleigh distribution, which has several applications in medicine and industry. It is used as survival data analysis and in large scale test in life and reliability [2]. Thus, various studies addressed this type of distribution. For instance, it is used in physics for studying various types of radiation, such as sound and light measurement. An important characteristic of the Rayleigh distribution is related to its hazard function or failure rate, which is an increasing linear function of time. It means that when failure time is distributed according to the Rayleigh distribution an intense aging of item occurs in a system. It is important to address reliability to develop the efficient future and improve the performance of the systems. Mousa and Al-Sagheer [3] calculated the maximum likelihood estimators and Bayes estimators for the parameters and the reliability function. Balakrishnan et al. [4] introduced the parameter estimation for the reliability function of the Weibull and Rayleigh distributions. Ragab and Madi [5] discussed the Bayesian predictive methods for the total time on test using doubly censored data with a Rayleigh distribution and the scale parameters as well as applied the methods to a real data set. Kim and Han [6] applied Bayesian inference method based on the conjugate prior of the scale parameter of the Rayleigh distribution under general progressive censoring. Khan et al. [7] predicated the inference from two parameters Rayleigh life model for doubly censored sample.

Sultan and Balakrishnan [8] explored the higher order moments of record values from Rayleigh and Weibull distributions. Kundu and Ragab [9] obtained the generalized Rayleigh distribution. Ragab and Ahsanullah [10] estimated the location and scale parameters of the generalized exponential distribution based on order statistics. For additional details and recent results for this distribution, see [11,12].

In this paper, we derived the reliability function for the two parameters Rayleigh distribution and estimated its parameters using different methods. A program of simulation was presented depending on observation with different parameter's values and sample sizes from Rayleigh distribution. Comparing the different methods depends on the mean square errors (MES). It was shown that the second modified maximum likelihood method is the best method to estimate reliability function, [13,14, 15].

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The probability density function (pdf) for the Rayleigh distribution with two parameters α, β takes the form:

$$f(t,\alpha,\beta) = \frac{(t-\alpha)}{\beta^2} \exp\left\{-\frac{(t-\alpha)^2}{2\beta^2}\right\}, \quad \alpha < t < \infty, \tag{1}$$

where $\alpha(>0)$ is the location parameter, and $\beta(>0)$ is the scale parameter.

The Rayleigh distribution can be applied to the systems and equipment that have a hazard rate changes with times and when failure time starts with certain time $\alpha \neq 0$. α represents the guarantee for the item or component. The cumulative distribution function is

$$F(t,\alpha,\beta) = 1 - \exp\left\{-\frac{(t-\alpha)^2}{2\beta^2}\right\}, \tag{2}$$

then the reliability function is

$$R(t) = \exp\left\{-\frac{(t-\alpha)^2}{2\beta^2}\right\}. \tag{3}$$

And the hazard function is:

$$h(t) = \frac{f(t)}{R(t)} = \frac{t-\alpha}{\beta^2}. \tag{4}$$

From Equation (4) the hazard rate is a function of time t.

2 Methods of Estimation

In this section, we derived the different methods for estimating the parameters of reliability function for Rayleigh distribution.

2.1 Maximum likelihood estimation

The maximum likelihood estimation (MLE) is the most important method. This technique aims to maximize the likelihood function $L(t;\alpha,\beta)$ and then get the values $\hat{\alpha}$, $\hat{\beta}$ which maximize $L(t_i;\alpha,\beta)$. The log-likelihood function based on the random sample $(t_1,\cdots,t_n)$ of two parameters Rayleigh distribution is given by

$$\begin{aligned} L(t_1,\cdots,t_n;\alpha,\beta) &= \prod_{i=1}^{n} f(t;\alpha,\beta) \\ &= \left[\prod_{i=1}^{n} \frac{(t_i-\alpha)}{\beta^2}\right] e^{-\sum_{i=1}^{n} \frac{(t_i-\alpha)^2}{2\beta^2}}. \end{aligned} \tag{5}$$

By taking the logarithm for both sides of equation (5), we get

$$\ln(L) = -2n\ln(\beta) + \sum_{i=1}^{n} \ln(t_i-\alpha) - \sum_{i=1}^{n} \frac{(t_i-\alpha)^2}{2\beta^2}. \tag{6}$$

Then by taking the partial derivative, we get the derivative of equation (6) with respect to α and β, respectively, and then equating the results to zero, we get

$$-\sum_{i=1}^{n}\left(\frac{1}{t_i-\hat{\alpha}}\right) + \frac{1}{\hat{\beta}^2}\sum_{i=1}^{n}(t_i-\hat{\alpha}) = 0, \tag{7}$$

$$-\frac{2n}{\hat{\beta}} + \frac{1}{\hat{\beta}^2}\sum_{i=1}^{n}(t_i-\hat{\alpha})^2 = 0, \tag{8}$$

and because α is the minimum value of t_i,

$$\hat{\alpha}_{\text{MLE}} = \min(t_1,t_2,\cdots,t_n) = t_{(1)}. \tag{9}$$

Thus, from equation (9), we have the value of parameter α which makes the likelihood function maximize, where $t_{(1)}$ is the first order statistic form the random sample. Substituting from equation (9) in (8) and solving it with respect to α and β, we get

$$\hat{\beta}_{\text{MLE}} = \sqrt{\frac{1}{2n}\sum_{i=1}^{n}(t_i-\hat{\alpha}_{\text{MLE}})^2}. \tag{10}$$

Depending on property invariant by the maximum likelihood estimation, the estimator maximum likelihood to reliability function will be as

$$\hat{R}_{\text{MLE}}(t) = \exp\left\{\frac{1}{2\hat{\beta}^2_{\text{MLE}}}\sum_{i=1}^{n}(t_i-\hat{\alpha}_{\text{MLE}})^2\right\}. \tag{11}$$

2.2 The first modified maximum likelihood method

In this method we notice that the cumulative distribution for the random variable $T_{(1)}$ is

$$F_{T_{(1)}}(t) = P[T_{(1)} \leq t] = 1 - P[T_{(1)} > t] = 1 - [R(t)]^n. \tag{12}$$

Also, the probability density function for first order statistics, $T_{(1)}$, is given by

$$f_{T_{(1)}}(t) = n\left(\frac{t-\alpha}{\beta^2}\right)\exp\left\{-n\frac{(t-\alpha)^2}{2\beta^2}\right\}. \tag{13}$$

From equation (13), the expected value for $T_{(1)}$ is

$$E[T_{(1)}] = \int_{\alpha}^{\infty} t f_{T_{(1)}} dt = \alpha + \beta\sqrt{\frac{\pi}{2n}}, \tag{14}$$

where $t_{(1)}$ the statistic represents the first order of values t.

If $(\hat{\alpha},\hat{\beta})$ are unbiased estimators for (α,β), we get

$$\hat{\alpha} = t_{(1)} - \beta\sqrt{\frac{\pi}{2n}}. \tag{15}$$

Supposing that $\hat{\beta} = s$, the standard deviation for the random sample values of t we get

$$\hat{\alpha}_{\text{MLE1}} = t_{(1)} - s\sqrt{\frac{\pi}{2n}}. \tag{16}$$

Using equation (7) and solving it with respect to $\hat{\beta}$, we get

$$\hat{\beta}_{\text{MLE1}} = \sqrt{\frac{\sum_{i=1}^{n}(t_i - \alpha^{\text{MLE1}})}{\sum_{i=1}^{n}\left(\frac{1}{t_i - \alpha^{\text{MLE1}}}\right)}}. \tag{17}$$

The reliability function will be as follows

$$\hat{R}_{\text{MLE1}}(t) = \exp\left\{\frac{\sum_{i=1}^{n}(t_i - \hat{\alpha}_{\text{MLE1}})^2}{2\hat{\beta}_{\text{MLE1}}^2}\right\}. \tag{18}$$

2.3 The second modified maximum likelihood method

Using equation (14) and if $\hat{\beta}$ equals the standard deviation for (t) values we get

$$\hat{\alpha}_{\text{MLE2}} = t_{(1)} - s\sqrt{\frac{\pi}{2n}}, \tag{19}$$

and

$$\hat{\beta}_{\text{MLE2}} = \sqrt{\frac{\sum_{i=1}^{n}(t_i - \hat{\alpha}_{\text{MLE2}})^2}{2n}}. \tag{20}$$

Hence, reliability function is

$$\hat{R}_{\text{MLE2}}(t) = \exp\left\{-\frac{(t_i - \hat{\alpha}_{\text{MLE2}})^2}{2\hat{\beta}_{\text{MLE2}}^2}\right\}. \tag{21}$$

2.4 Method of moments

The method of moments is an old and simple technique on the idea that the sample moments are estimation of population moments. We derived the estimators in this method from equation (14). If we suppose that $\hat{\beta} = s$ is the first estimator for β and s is the standard deviation, we get

$$\mu_1 = E[T] = \beta\sqrt{\frac{\pi}{2}} + \alpha \tag{22}$$

$$\mu_2 = E[T^2] = 2\beta^2 + \sqrt{2\pi}\alpha\beta + \alpha^2. \tag{23}$$

Consequently, the variance is given by

$$V(T) = \beta^2\left(2 - \frac{\pi}{2}\right). \tag{24}$$

By equating the sample first and second moments by the population of first and second moment, we get

$$\bar{T} = \bar{\alpha} + \hat{\beta}\sqrt{\frac{\pi}{2}} \tag{25}$$

$$S^2 = \hat{\beta}^2\left(2 - \frac{\pi}{2}\right). \tag{26}$$

From equation (26), the parameter estimator for β takes the form

$$\hat{\beta}_{\text{ME}} = \frac{s}{\sqrt{2 - \frac{\pi}{2}}}. \tag{27}$$

And from equations (25), (27), we get α estimator, as follows:

$$\hat{\alpha}_{\text{ME}} = \bar{T} - \hat{\beta}_{\text{ME}}\sqrt{\frac{\pi}{2}}. \tag{28}$$

Hence, the estimated reliability function by the moment method is

$$\hat{R}_{\text{ME}}(t) = \exp\left\{-\frac{(t_i - \hat{\alpha}_{\text{ME}})^2}{2\hat{\beta}_{\text{ME}}^2}\right\}. \tag{29}$$

2.5 Bayesian estimation

In this section, the two unknown parameters for the Rayleigh distribution are estimated using Bayesian estimation when both parameters are unknown.

Let the prior distribution for the parameters α and β be

$$g(\beta) \propto \frac{1}{\beta}, \tag{30}$$

$$g(\alpha) = \min t_i = t_{(1)}. \tag{31}$$

The joint prior distribution for α and β is

$$g(\alpha,\beta) \propto \frac{1}{\beta}. \tag{32}$$

And the posterior distribution is

$$p(\alpha,\beta|t) \propto L(t;\alpha,\beta)g(\alpha,\beta). \tag{33}$$

Substituting from equations (5) and (32) into equation (33), we have

$$\begin{aligned} p(\alpha,\beta|t) &\propto \frac{1}{\beta}\prod_{i=1}^{n}\frac{(t_i-\alpha)}{\beta^2}\exp\left\{-\sum_{i=1}^{n}\frac{(t_i-\alpha)^2}{2\beta^2}\right\} \\ &\propto \frac{1}{\beta^{2n+1}}\prod_{i=1}^{n}(t_i-\alpha)\exp\left\{-\sum_{i=1}^{n}\frac{(t_i-\alpha)^2}{2\beta^2}\right\} \\ &= K\frac{1}{\beta^{2n+1}}\exp\left\{-\sum_{i=1}^{n}\frac{(t_i-\alpha)^2}{2\beta^2}\right\}, \end{aligned} \tag{34}$$

where

$$K^{-1} = \int_0^\infty \frac{1}{\beta^{2n+1}} \exp\left\{-\sum_{i=1}^{n}\frac{(t_i-\alpha)^2}{2\beta^2}\right\} d\beta$$
$$= \frac{\Gamma(n)}{2^{1-n}}\left(\sum_{i=1}^{n}(t_i-\alpha)\right)^{-n},$$

and

$$K = \frac{2^{1-n}}{\Gamma(n)}\left(\sum_{i=1}^{n}(t_i-\alpha)\right)^{n}. \quad (35)$$

Substituting (35) into (34), the joint posterior distribution

$$p(\alpha,\beta|t) = \frac{2}{\beta\Gamma(n)}\left(\sum_{i=1}^{n}\frac{(t_i-\alpha)}{2\beta^2}\right)^{n} \exp\left\{-\sum_{i=1}^{n}\frac{(t_i-\alpha)^2}{2\beta^2}\right\}. \quad (36)$$

Using squared error loss function:

$$L(\hat{\beta},\beta) = C(\beta^*-\beta)^2, \quad (37)$$

where C is a constant.

$$R(\hat{\beta}) = \int_0^\infty C(\beta^*-\beta)^2 p(\beta,\alpha|t)d\beta$$
$$= C\beta^{*2} + \frac{C}{n-1}\left(\frac{\sum_{i=1}^{n}(t_i-\alpha)^2}{2}\right) - \frac{2\beta^* C\Gamma\left(\frac{2n-1}{2}\right)}{\Gamma(n)}\sqrt{\frac{\sum_{i=1}^{n}(t_i-\alpha)^2}{2}}.$$

Using

$$\frac{\partial R(\beta^*)}{\partial\beta} = 0.$$

Then

$$\hat{\alpha} = \min t_i = t_{(1)},$$
$$\hat{\beta} = \frac{\Gamma\left(\frac{2n-1}{2}\right)}{\Gamma(n)}\sqrt{\frac{\sum_{i=1}^{n}(t_i-\alpha)^2}{2}}.$$

3 Simulation

In this section, we carried out simulation to compare mean square error of all parameters, the program is written in Fortran language, the results are based on 1000 simulation runs.

	I	II	III	IV	V	VI
α	0.5	1	1	1.5	2	2
β	1	0.5	1	2	1.5	2

Tables 1-6 show the mean square errors (MSEs) for the different cases. Figures 1-6 represent the plots for the MSEs for each case all methods.

Table 1 and Figure 1 indicate the following:

1. For $n = 10$, the mean square error, MMLE2 $<$ ME $<$ MMLE1 $<$ Bayes $<$ MLE.

Table 1: The MSE for the estimation methods, for Case I.

n	MLE	MMLE1	MMLE2	ME	BAYES
10	0.01835	0.01133	0.01023	0.01036	0.01350
20	0.00862	0.00446	0.00427	0.00502	0.00579
30	0.00559	0.00289	0.00152	0.00341	0.00400
50	0.00318	0.00256	0.00212	0.01083	0.00254
100	0.00157	0.00137	0.00098	0.00114	0.00121

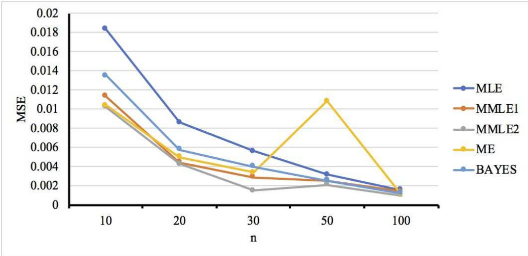

Fig. 1: The MSE for the estimation methods, for Case I.

2. For $n = 20$, the mean square error, MMLE2 $<$ MMLE1 $<$ ME $<$ Bayes $<$ MLE.
3. For $n = 30$, the mean square error, MMLE2 $<$ MMLE1 $<$ ME $<$ Bayes $<$ MLE.
4. For $n = 50$, the mean square error, MMLE2 $<$ Bayes $<$ MMLE1 $<$ MLE $<$ ME.
5. For $n = 100$, the mean square error, MMLE2 $<$ ME $<$ Bayes $<$ MMLE1 $<$ MLE.
6. When the value of n is increased the mean square error is decreased.

Table 2: The MSE for the estimation methods, for Case II.

n	MLE	MMLE1	MMLE2	ME	BAYES
10	0.017056	0.01016	0.01013	0.01027	0.01303
20	0.00773	0.00468	0.00435	0.00450	0.00443
30	0.00558	0.00362	0.00334	0.00446	0.00669
50	0.00320	0.00246	0.00102	0.00210	0.00235
100	0.00145	0.00147	0.00098	0.00100	0.00127

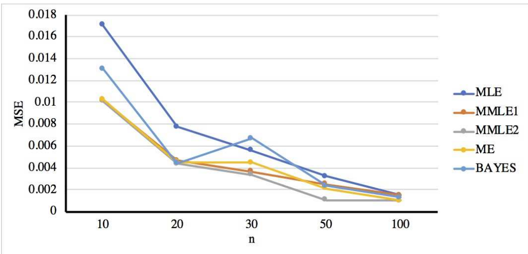

Fig. 2: The MSE for the estimation methods, for Case II.

Table 2 and Figure 2 show the following:

1. For $n = 10$, the mean square error, MMLE2 < MMLE1 < ME < Bayes < MLE.
2. For $n = 20$, the mean square error, MMLE2 < Bayes < ME < MMLE1 < MLE.
3. For $n = 30$, the mean square error, MMLE2 < MMLE1 < ME < MLE < Bayes.
4. For $n = 50$, the mean square error, MMLE2 < ME < Bayes < MMLE1 < MLE.
5. For $n = 100$, the mean square error, MMLE2 < ME < Bayes < MLE < MMLE1.
6. When the value of n increases, the mean square error is decreases.

Table 3: The MSE for the estimation methods, for Case III.

n	MLE	MMLE1	MMLE2	ME	BAYES
10	0.01768	0.01192	0.01167	0.01170	0.01940
20	0.00903	0.00577	0.00552	0.00560	0.00663
30	0.00558	0.00377	0.00213	0.00216	0.00306
50	0.00304	0.00249	0.00203	0.00210	0.00220
100	0.00153	0.00140	0.00077	0.00105	0.00150

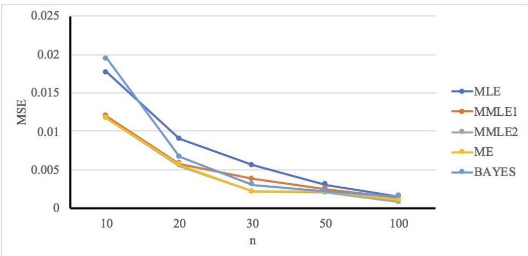

Fig. 3: The MSE for the estimation methods, for Case III.

Table 3 and Figure 3 reveal the following:

1. For $n = 10$, the mean square error, MMLE2 < ME < MMLE1 < MLE < Bayes.
2. For $n = 20$, the mean square error, MMLE2 < ME < MMLE1 < Bayes < MLE.
3. For $n = 30$, the mean square error, MMLE2 < ME < Bayes < MMLE1 < MLE.
4. For $n = 50$, the mean square error, MMLE2 < ME < Bayes < MMLE1 < MLE.
5. For $n = 100$, the mean square error, MMLE2 < ME < MMLE1 < Bayes < MLE.
6. When the value of n increases, the mean square error decreases.

Table 4 and Figure 4 demonstrate the following:

1. For $n = 10$, the mean square error, MMLE2 < MMLE1 < ME < Bayes < MLE.

Table 4: The MSE for the estimation methods, for Case IV.

n	MLE	MMLE1	MMLE2	ME	BAYES
10	0.01864	0.01141	0.01107	0.01162	0.01308
20	0.00843	0.00578	0.00428	0.00521	0.00582
30	0.00449	0.00357	0.00313	0.00321	0.00404
50	0.00320	0.00205	0.00172	0.00199	0.00221
100	0.00150	0.00131	0.00027	0.00102	0.00151

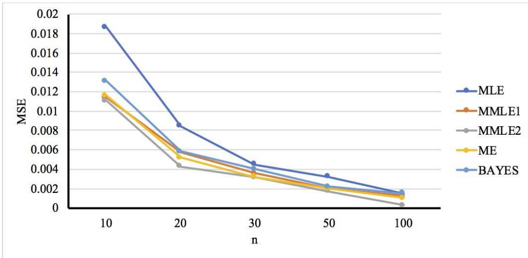

Fig. 4: The MSE for the estimation methods, for Case IV.

2. For $n = 20$, the mean square error, MMLE2 < ME < MMLE1 < Bayes < MLE.
3. For $n = 30$, the mean square error, MMLE2 < ME < MMLE1 < Bayes < MLE.
4. For $n = 50$, the mean square error, MMLE2 < ME < MMLE1 < Bayes < MLE.
5. For $n = 100$, the mean square error, MMLE2 < ME < MMLE1 < MLE < Bayes.
6. When the value of n increases, the mean square error decreases.

Table 5: The MSE for the estimation methods, for Case V.

n	MLE	MMLE1	MMLE2	ME	BAYES
10	0.01824	0.01191	0.01163	0.01526	0.01448
20	0.00807	0.00503	0.00326	0.00569	0.00581
30	0.00602	0.00405	0.00304	0.00364	0.00403
50	0.00320	0.00239	0.00200	0.00215	0.00226
100	0.00157	0.00185	0.00011	0.00401	0.00131

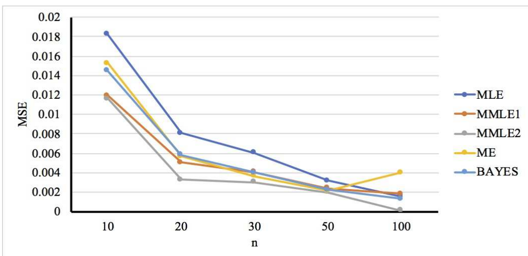

Fig. 5: The MSE for the estimation methods, for Case V.

Table 5 and Figure 5 show the following:

1. For $n = 10$, the mean square error, MMLE2 < MMLE1 < Bayes < ME < MLE.
2. For $n = 20$, the mean square error, MMLE2 < MMLE1 < ME < Bayes < MLE.
3. For $n = 30$, the mean square error, MMLE2 < ME < Bayes < MMLE1 < MLE.
4. For $n = 50$, the mean square error, MMLE2 < ME < Bayes < MMLE1 < MLE.
5. For $n = 100$, the mean square error, MMLE2 < Bayes < MLE < MMLE1 < ME.
6. When the value of n increases, the mean square error decreases.

Table 6: The MSE for the estimation methods, for Case VI.

n	MLE	MMLE1	MMLE2	ME	BAYES
10	0.01796	0.01119	0.01110	0.01176	0.01316
20	0.00901	0.00590	0.00543	0.00551	0.00608
30	0.00562	0.00388	0.00340	0.00353	0.00392
50	0.00312	0.00230	0.00200	0.00203	0.00227
100	0.00152	0.00150	0.00087	0.00104	0.00109

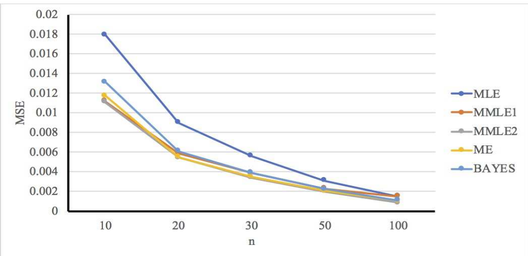

Fig. 6: The MSE for the estimation methods, for Case VI.

Table 6 and Figure 6 illustrate the following:

1. For $n = 10$, the mean square error, MMLE2 < MMLE1 < ME < Bayes < MLE.
2. For $n = 20$, the mean square error, MMLE2 < ME < MMLE1 < Bayes < MLE.
3. For $n = 30$, the mean square error, MMLE2 < ME < MMLE1 < Bayes < MLE.
4. For $n = 50$, the mean square error, MMLE2 < ME < Bayes < MMLE1 < MLE.
5. For $n = 100$, the mean square error, MMLE2 < ME < Bayes < MMLE1 < ME.
6. When the value of n increases, the mean square error decreases.

4 Conclusion

In this article, reliability function for the two parameters Rayleigh distribution was obtained. The location and scale parameters in reliability function were estimated by maximum likelihood, Bayesian estimation, first and second modified moment methods. Also, a numerical study that depends on simulated observation was analyzed with different parameter values and several sample's sizes from reliability function. Rayleigh distribution was conducted to compare different methods. Thus, the second modified moment method is the best method to estimate reliability function.

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The authors declare that there is no conflict of interest regarding the publication of this article.

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