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Coefficients Bounds of Multivalent Function Connected with q-Analogue of Salagean Operator

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Abstract: Using q-analogue of Salagean operator, we investigate subclass of multivalent functions in the open unit disk Δ. We obtain Fekete-Szegő inequalities for a certain class of analytic functions f satisfying $1+\frac{1}{\zeta}\left[\frac{1}{[p]_q}\frac{z(\mathscr{D}_q(\mathscr{D}^n_{p,q}f(z))}{\mathscr{D}^n_{p,q}f(z)}-1\right]\prec\Upsilon(z)$. Applications of our results to certain functions defined by convolution products with a normalized analytic function are given. Moreover, Fekete-Szegő inequalities for certain subclasses of functions defined through Poisson distribution are obtained.

Keywords: Analytic function, Fekete-Szegő problem, p-valent function, q-analogue of Salagean operator.

1 Introduction

In [18] Srivastava presented a brief overview of the classical q-analysis versus the so-called (p,q)-analysis with an obviously redundant additional parameter p. We also briefly consider several other families of such extensively investigated linear convolution operators as (for example) the Dziok–Srivastava, Srivastava–Wright and Srivastava–Attiya linear convolution operators, together with their extended and generalized versions. The theory of (p,q)-analysis has an important role in various areas of mathematics and physics. Stages of the q-calculus and the fractional qcalculus in geometric function theory of complex analysis encourage significant further developments on these and other relevant topics (see Srivastava and Karlsson [20, pp. 350–351]& Srivastava [16,17,19]). Our main objective in this survey-cum-expository article is based chiefly upon the fact that the recent and future usages of the classical q-calculus and the fractional q-calculus in geometric function theory of complex analysis encourage conducting significant further researches on many of these and other relevant subjects. Jackson [6,7] was the first one to present some applications of q-calculus and introduce the q-analogue of derivative and integral operator (see also [1]).

Let $\mathscr{A}(p)$ denote the class of analytic and multivalent functions in the open unit disk $\Delta=\{z\in\mathbb{C}:|z|<1\}$ of the form

$$f(z)=z^p+\sum_{k=1}^{\infty}a_{k+p}z^{k+p}\ (p\in\mathbb{N}=\{1,2,3,...\}). \quad (1)$$

In particular, we write

$$\mathscr{A}(1)=\mathscr{A}.$$

For functions $f\in\mathscr{A}(p)$ given by (1) and $g\in\mathscr{A}(p)$ given by

$$g(z)=z^p+\sum_{k=1}^{\infty}b_{k+p}z^{k+p}, \quad (2)$$

the Hadamard product (or convolution) of f and g is defined by

$$(f*g)(z)=z^p+\sum_{k=1}^{\infty}a_{k+p}b_{k+p}z^{k+p}=(g*f)(z). \quad (3)$$

A function $f\in A(p)$ is said to be p-valently starlike of order α denoted by $S_p^*(\alpha)$ if and only if f satisfies the following inequality:

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha\ (0\leq\alpha<p;\ p\in\mathbb{N};\ z\in\Delta).$$

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Definition 1. *If f and $\mathscr{F}$ are analytic functions in Δ, f is subordinate to $\mathscr{F}$, written $f \prec \mathscr{F}$, if there exists a* Schwarz function w, *which is analytic in Δ, with $w(0)=0$ and $|w(z)|<1$ for all $z \in \Delta$, such that $f(z)=\mathscr{F}(w(z))$, $z \in \Delta$. Furthermore, if the function $\mathscr{F}$ is univalent in Δ, we have the following equivalence (see [3] and [11]):*

$$f(z) \prec \mathscr{F}(z) \Leftrightarrow f(0)=\mathscr{F}(0) \text{ and } f(\Delta) \subset \mathscr{F}(\Delta).$$

Srivastava [18] made use of various operators of q-calculus and fractional q-calculus. We recall the definitions and notations as follows:

$$(\lambda;q)_k = \begin{cases} 1 & k=0, \\ (1-\lambda)(1-\lambda q)\ldots(1-\lambda q^{k-1}) & k \in \mathbb{N}. \end{cases}$$

Using the q-gamma function $\Gamma_q(z)$, we get

$$\left(q^{\lambda};q\right)_k = \frac{(1-q)^k \, \Gamma_q(\lambda+k)}{\Gamma_q(\lambda)}, \quad (k \in \mathbb{N}_0),$$

where (see [5])

$$\Gamma_q(z) = (1-q)^{1-z} \frac{(q;q)_\infty}{(q^z;q)_\infty}, \quad (|q|<1).$$

Furthermore, we note that

$$(\lambda;q)_\infty = \prod_{k=0}^{\infty}\left(1-\lambda q^k\right), \quad (|q|<1),$$

and, the q-gamma function $\Gamma_q(z)$ is known

$$\Gamma_q(z+1) = [z]_q \, \Gamma_q(z),$$

where $[k]_q$ denotes the basic q-number defined as follows

$$[k]_q := \begin{cases} \frac{1-q^k}{1-q}, & k \in \mathbb{C}, \\ 1+\sum\limits_{j=1}^{k-1} q^j, & k \in \mathbb{N}. \end{cases} \tag{4}$$

Using the definition formula (4) we have the next two products:
(i) For any non negative integer k, the *q-shifted factorial* is given by

$$[k]_q! := \begin{cases} 1, & \text{if } k=0, \\ \prod\limits_{n=1}^{k} [n]_q, & \text{if } k \in \mathbb{N}. \end{cases}$$

(ii) For any positive number r, the *q-generalized Pochhammer symbol* is defined by

$$[r]_{q,k} := \begin{cases} 1, & \text{if } k=0, \\ \prod\limits_{n=r}^{r+k-1} [n]_q, & \text{if } k \in \mathbb{N}. \end{cases}$$

It is known in terms of the classical (Euler's) gamma function $\Gamma(z)$ that

$$\Gamma_q(z) \to \Gamma(z) \qquad \text{as } q \to 1^-.$$

Also, we observe that

$$\lim_{q\to 1^-}\left\{\frac{(q^{\lambda};q)_k}{(1-q)^k}\right\} = (\lambda)_k,$$

where $(\lambda)_k$ is the familiar Pochhammer symbol defined by

$$(\lambda)_k = \begin{cases} 1, & \text{if } k=0, \\ \lambda(\lambda+1)\ldots(\lambda+k-1), & \text{if } k \in \mathbb{N}. \end{cases}.$$

The q-derivative of a function $f(z)$ is $\mathscr{D}_q f(z)$ defined as follows

Definition 2. *The* q-derivative operator *for f is defined by (see [7])*

$$\mathscr{D}_q f(z) := \begin{cases} \frac{f(qz)-f(z)}{z(q-1)} & z \neq 0 \\ f'(0) & z=0, \end{cases} \tag{5}$$

provided that $f'(0)$ exists.

We note that

$$\lim_{q\to 1^-} \mathscr{D}_q f(z) := \lim_{q\to 1^-} \frac{f(qz)-f(z)}{z(q-1)} = f'(z).$$

From (1) and (5), we have

$$\mathscr{D}_q f(z) := [p]_q z^{p-1} + \sum_{k=1}^{\infty} [k+p]_q a_{k+p} z^{k+p-1}, \ z \neq 0, \tag{6}$$

where $[p]_q$ is defined by (4). Now, using the q-derivative operator $\mathscr{D}_q$, we introduce the operator $\mathscr{D}^n_{p,q} : \mathscr{A}(p) \to \mathscr{A}(p)$ as follows:

$$\mathscr{D}^0_{p,q} f(z) = f(z),$$

$$\mathscr{D}^1_{p,q} f(z) = \frac{z}{[p]_q}\mathscr{D}_q f(z) = z^p + \sum_{k=1}^{\infty} \frac{[k+p]_q}{[p]_q} a_{k+p} z^{k+p}$$

$$\mathscr{D}^2_{p,q} f(z) = \frac{z}{[p]_q}\mathscr{D}_q\left(\mathscr{D}^1_{p,q} f(z)\right) = z^p + \sum_{k=1}^{\infty}\left(\frac{[k+p]_q}{[p]_q}\right)^2 a_{k+p} z^{k+p}$$

$$\vdots$$

$$\mathscr{D}^n_{p,q} f(z) = \frac{z}{[p]_q}\mathscr{D}_q\left(\mathscr{D}^{n-1}_{p,q} f(z)\right)$$

$$= z^p + \sum_{k=1}^{\infty}\left(\frac{[k+p]_q}{[p]_q}\right)^n a_{k+p} z^{k+p} \tag{7}$$

$(p \in \mathbb{N},\ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ 0<q<1)$.

Specializing the parameters p and q, we obtain the following operators:

$$(i)\ \lim_{q\to 1^-} \mathscr{D}^n_{p,q} f(z) = \mathscr{D}^n_p f(z) = z^p + \sum_{k=1}^{\infty} \left(\frac{k+p}{p}\right)^n a_{k+p} z^{k+p},$$

such that $\mathscr{D}^n_p$ is called Salagean in p-valent (see Kamali and Orhan [8], Orhan and Kiziltunc [12]);

$$(ii)\ \mathscr{D}^n_{1,q} f(z) = \mathscr{D}^n_q f(z) = z + \sum_{k=2}^{\infty} \left([k]_q\right)^n a_k z^k; \tag{8}$$

$$(iii)\ \lim_{q\to 1^-} \mathscr{D}^n_{1,q} f(z) = \mathscr{D}^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k,$$

where $\mathscr{D}^n$ is called Salagean opeator (see Salagean [15]). Using the operator $\mathscr{D}^n_{p,q}$ given by (7), we introduce the subclass $\mathscr{S}^*_{n,q}(p,\alpha)$ of p-valently (n,q) starlike functions of order α in Δ as follows

$$f(z) \in \mathscr{S}^*_{n,q}(p,\alpha) \Leftrightarrow \Re\left\{\frac{1}{[p]_q} \frac{z\,\mathscr{D}_q\left(\mathscr{D}^n_{p,q} f(z)\right)}{\mathscr{D}^n_{p,q} f(z)}\right\} > \alpha,$$

$$(p \in \mathbb{N},\ n \in \mathbb{N}_0,\ 0<q<1,\ 0 \le \alpha < 1,\ z \in \Delta) \tag{9}$$

The following classes are included in the class $\mathscr{S}^*_{n,q}(p,\alpha)$:

(i) $\mathscr{S}^*_{0,q}(p,\alpha) = \mathscr{S}^*_q(p,\alpha)$ defined and studied by Srivastava et al. [21];

(ii) $\lim_{q\to 1^-} \mathscr{S}^*_{0,q}(p,\alpha) = \mathscr{S}^*_p(\alpha)$ defined and studied by Patil and Thakare [13].

Now, we define the following subclass of functions $\mathscr{G}^{n,q}_p(\zeta,\Upsilon)$ as follows:

Definition 3. *Let* $\Upsilon(z) = 1 + \mathscr{B}_1 z + \mathscr{B}_2 z^2 + \mathscr{B}_3 z^3 + \dots,\ z \in \Delta,\ \mathscr{B}_1 > 0,$ *be a starlike (univalent) function with respect to 1, which maps the unit disk Δ onto a region included in the right half plane which is symmetric with respect to the real axis. For $\zeta \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$, and the function $f \in \mathscr{A}$ is said to be in the class $\mathscr{G}^{n,q}_p(\zeta,\Upsilon)$ if*

$$1 + \frac{1}{\zeta}\left[\frac{1}{[p]_q} \frac{z(\mathscr{D}_q(\mathscr{D}^n_{p,q} f(z))}{\mathscr{D}^n_{p,q} f(z)} - 1\right]$$

is analytic in Δ and satisfies

$$1 + \frac{1}{\zeta}\left[\frac{1}{[p]_q} \frac{z(\mathscr{D}_q(\mathscr{D}^n_{p,q} f(z))}{\mathscr{D}^n_{p,q} f(z)} - 1\right] \prec \Upsilon(z),$$

$$(p \in \mathbb{N},\ 0<q<1,\ n \in \mathbb{N}_0,\ \zeta \in \mathbb{C}^*).$$

We note that:

$$(i)\quad \mathscr{G}^{n,q}_1(\zeta,\Upsilon) = \mathscr{U}^{n,q}(\zeta,\Upsilon)$$

$$= \left\{ f(z) \in \mathscr{A} :\ 1 + \frac{1}{\zeta}\left[\frac{z(\mathscr{D}_q(\mathscr{D}^n_q f(z))}{\mathscr{D}^n_q f(z)} - 1\right] \prec \Upsilon(z) \right\},$$

where $(0<q<1,\ n \in \mathbb{N}_0,\ \zeta \in \mathbb{C}^*)$ and the operator $\mathscr{D}^n_q$ is given by (8);

$$(ii)\ \mathscr{G}^{0,q}_p(\zeta,\Upsilon) = \mathscr{G}^q_p(\zeta,\Upsilon)$$

$$= \left\{ f(z) \in \mathscr{A}(p) :\ 1 + \frac{1}{\zeta}\left[\frac{1}{[p]_q} \frac{z\,\mathscr{D}_q f(z)}{f(z)} - 1\right] \prec \Upsilon(z) \right\},$$

where $(p \in \mathbb{N},\ 0<q<1,\ \zeta \in \mathbb{C}^*)$;

$$(iii)\ \mathscr{G}^{n,q}_p((1 - \tfrac{\lambda}{[p]_q} e^{-i\alpha}\cos\alpha),\Phi) = \mathscr{W}^{n,q}_p(\alpha,\lambda,\Phi)$$

$$= \left\{ f(z) \in \mathscr{A}(p) : \frac{e^{i\alpha}\left(\frac{1}{[p]_q}\frac{z(\mathscr{D}_q(\mathscr{D}^n_{p,q} f(z))}{\mathscr{D}^n_{p,q} f(z)}\right) - \lambda\cos\alpha - i\,[p]_q \sin\alpha}{([p]_q - \lambda)\cos\alpha} \prec \Upsilon(z) \right\},$$

where $\left(p \in \mathbb{N},\ n \in \mathbb{N}_0,\ 0<q<1,\ |\alpha| < \frac{\pi}{2},\ 0 \le \lambda < [p]_q\right)$;

$$(iv)\ \mathscr{G}^{n,q}_1((1 - \frac{\lambda}{[p]_q} e^{-i\alpha}\cos\alpha),\Upsilon) = \mathscr{U}^{n,q}(\alpha,\lambda,\Upsilon)$$

$$= \left\{ f(z) \in \mathscr{A} : \frac{e^{i\alpha}\left(\frac{z(\mathscr{D}_q(\mathscr{D}^n_q f(z))}{\mathscr{D}^n_q f(z)}\right) - \lambda\cos\alpha - i\ \sin\alpha}{(1-\lambda)\cos\alpha} \prec \Upsilon(z) \right\},$$

where $\left(n \in \mathbb{N}_0,\ 0<q<1,\ |\alpha| < \frac{\pi}{2},\ 0 \le \lambda < 1\right)$.

2 Fekete-Szegő problem for functions in the class $\mathscr{G}^{n,q}_p(\gamma,\Phi)$

Let Ω be the class of functions $w(z)$ of the form

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \dots\dots, z \in \Delta. \tag{10}$$

The following lemmas will be needed to prove our results.

Lemma 1.([9]) *Let the function $w(z) \in \Omega$ be given by (10), then*

$$\left|w_2 - \tau w_1^2\right| \le \max\{1; |\tau|\},\ (\tau \in \mathbb{C}).$$

The result is sharp for the functions given by

$$w(z) = z \text{ or } w(z) = z^2, \qquad (z \in \Delta).$$

Lemma 2.([2] and [10]) *Let $w(z) \in \Omega$, then*

$$\left|w_2 - k w_1^2\right| \le \begin{cases} -k, & k \le -1 \\ 1, & -1 \le k \le 1, \\ k, & k \ge 1. \end{cases} \tag{11}$$

When $k < -1$ or $k > 1$, the equality holds if and only if $w(z) = z$ or one of its rotations. If $-1 < k < 1$, then the equality holds true in (11) if and only if $w(z) = z^2$ or one of its rotations. If $k = -1$, the equality holds true in (11) if and only if

$$w(z)=\frac{z(z+\eta)}{1+\eta z}\quad (0\leq\eta\leq 1),$$

or one of its rotations. If $k=1$, the equality holds if and only if

$$w(z)=-\frac{z(z+\eta)}{1+\eta z}\quad (0\leq\eta\leq 1).$$

Moreover, the above-mentioned upper bound in (11) is sharp and can be improved when $-1<k<1$ as follows:

$$\left|w_2-kw_1^2\right|+(k+1)\left|w_1\right|^2\leq 1 \qquad (-1<k<0), \tag{12}$$

and

$$\left|w_2-kw_1^2\right|+(1-k)\left|w_1\right|^2\leq 1 \quad (0<k<1). \tag{13}$$

Lemma 3.[14] *Let the function $w(z)\in\Omega$ given by (10). Then, for any real numbers ρ_1 and ρ_2, the following sharp estimates hold true:*

$$\left|w_3+\rho_1 w_1 w_2+\rho_2 w_1^3\right|\leq \mathscr{H}(\rho_1,\rho_2),$$

where
$\mathscr{H}(\rho_1,\rho_2)=$

$$\begin{cases} 1 & ((\rho_1,\rho_2)\in D_1\cup D_2) \\ |\rho_2| & ((\rho_1,\rho_2)\in \cup_{k=3}^{7} D_k) \\ \frac{2}{3}(|\rho_1|+1)(\frac{|\rho_1|+1}{3(|\rho_1|+1+\rho_2})^{\frac{1}{2}} & ((\rho_1,\rho_2)\in D_8\cup D_9) \\ \frac{\rho_2}{3}\left(\frac{\rho_1^2-4}{\rho_1^2-4\rho_2}\right)\left(\frac{\rho_1^2-4}{3(\rho_2-1)}\right)^{\frac{1}{2}} & (\rho_1\rho_2)\in D_{10}\cup D_{11}\in\{\pm 2,1\} \\ \frac{2}{3}\left(|\rho_1|-1\right)\left(\frac{|\rho_1|+1}{3(|\rho_1|+1+\rho_2)}\right)^{\frac{1}{2}} & ((\rho_1,\rho_2)\in D_{12}). \end{cases}$$

The extremal functions, up to rotations, are of the form given by

$$\begin{aligned}
w(z) &= z^3,\ w(z)=z,\\
w(z) &= w_0(z)=\frac{z[(1-\lambda)\varepsilon_2+\lambda\varepsilon_1]z-\varepsilon_1\varepsilon_2}{1-[(1-\lambda)\varepsilon_1+\lambda\varepsilon_2]z},\\
w(z) &= w_1(z)=\frac{z(t_1-z)}{1-t_1 z},\\
w(z) &= w_2(z)=\frac{z(t_2+z)}{1+t_2 z},\ |\varepsilon_1|=|\varepsilon_2|=1,\\
\varepsilon_1 &= t_0-e^{-i(\frac{\theta_0}{2})}(a\mp b),\quad \varepsilon_2=-e^{-i(\frac{\theta_0}{2})}(ia\pm b),\\
a &= t_0\cos\left(\frac{\theta_0}{2}\right),\ b=\sqrt{1-t_0^2\sin^2\left(\frac{\theta_0}{2}\right)},\ \lambda=\frac{b\pm a}{2b},\\
t_0 &= \left(\frac{2\rho_2(\rho_1^2+2)-3\rho_1^2}{3(\rho_2-1)(\rho_1^2-4\rho_2)}\right)^{\frac{1}{2}},\\
t_1 &= \left(\frac{|\rho_1|+1}{3\left(|\rho_1|+1+\rho_2\right)}\right)^{\frac{1}{2}},\\
t_2 &= \left(\frac{|\rho_1|-1}{3\left(|\rho_1|-1-\rho_2\right)}\right)^{\frac{1}{2}}
\end{aligned}$$

and

$$\cos\left(\frac{\theta_0}{2}\right)=\frac{\rho_1}{2}\left(\frac{\rho_2(\rho_1^2+8)-2(\rho_1^2+2)}{2\rho_2(\rho_1^2+2)-3\rho_1^2}\right).$$

The sets D_k $(k=1,2,\ldots\ldots,12)$ are defined as follows

$$D_1:=\left\{(\rho_1,\rho_2):|\rho_1|\leq\frac{1}{2}\ and\ |\rho_2|\leq\frac{1}{2}\right\},$$

$D_2 := \{(\rho_1,\rho_2) : \frac{1}{2}\leq|\rho_1|\leq 2\ and\ \frac{4}{27}\leq (|\rho_1|+1)^3-(|\rho_1|+1)\leq\rho_2\leq 1\}$,

$$D_3:=\left\{(\rho_1,\rho_2):|\rho_1|\leq\frac{1}{2}\ and\ \rho_2\leq -1\right\},$$

$$D_4:=\left\{(\rho_1,\rho_2):|\rho_1|\geq\frac{1}{2}\ and\ \rho_2\leq -\frac{2}{3}(|\rho_1|+1)\right\},$$

$$D_5:=\{(\rho_1,\rho_2):|\rho_1|\leq 2\ and\ \rho_2\geq 1\},$$

$$D_6:=\left\{(\rho_1,\rho_2):2\leq|\rho_1|\leq 4\ and\ \rho_2\geq\frac{1}{12}(\rho_1^2+8)\right\}$$

$$D_7:=\left\{(\rho_1,\rho_2):|\rho_1|\geq 4\ and\ \rho_2\leq -\frac{2}{3}(|\rho_1|-1)\right\},$$

$D_8:=\{(\rho_1,\rho_2):\frac{1}{2}\leq|\rho_1|\leq 2\ and\ -\frac{2}{3}(|\rho_1|+1)\leq\rho_2\leq \frac{4}{27}(|\rho_1|+1)^3-(|\rho_1|+1)\}$,

$$D_9:=\left\{(\rho_1,\rho_2):|\rho_1|\geq 2\ and\ -\frac{2}{3}(|\rho_1|+1)\leq\rho_2\leq\frac{2|\rho_1|(|\rho_1|+1)}{\rho_1^2+2|\rho_1|+4}\right\},$$

$D_{10}:=$
$$\left\{(\rho_1,\rho_2):2\leq|\rho_1|\leq 4\ and\ \frac{2|\rho_1|(|\rho_1|+1)}{\rho_1^2+2|\rho_1|+4}\leq\rho_2\leq\frac{1}{12}(\rho_1^2+8)\right\},$$

$$D_{11}:=\left\{(\rho_1,\rho_2):|\rho_1|\geq 4\ and\ \frac{2|\rho_1|(|\rho_1|+1)}{\rho_1^2+2|\rho_1|+4}\leq\rho_2\leq\frac{2|\rho_1|(|\rho_1|-1)}{\rho_1^2-2|\rho_1|+4}\right\}$$

and

$$D_{12}:=\left\{(\rho_1,\rho_2):|\rho_1|\geq 4\ and\ \frac{2|\rho_1|(|\rho_1|-1)}{\rho_1^2-2|\rho_1|+4}\leq\rho_2\leq\frac{2}{3}(|\rho_1|-1)\right\}.$$

Otherwise, we shall assume in the remainder of this paper that $p\in\mathbb{N}$, $0<q<1$, $n\in\mathbb{N}_0$ and $z\in\Delta$.

Theorem 1. *Let the function f given by (1) belong to the class $\mathscr{G}_p^{n,q}(\zeta,\Upsilon)$, with $\Upsilon(z)=1+\mathscr{B}_1 z+\mathscr{B}_2 z^2+\ldots$ satisfying the conditions of the Definition 3, and ω is a complex number, then*

$$\begin{aligned}\left|a_{p+2}-\omega a_{p+1}^2\right| &\leq \frac{|\zeta|\mathscr{B}_1[p]_q}{[p+2]_q-[p]_q}\left(\frac{[p]_q}{[p+2]_q}\right)^n\\ &\cdot\max\left\{1,\left|\frac{\mathscr{B}_2}{\mathscr{B}_1}+\frac{\zeta\mathscr{B}_1[p]_q}{[p+1]_q-[p]_q}(1-\omega\frac{([p+2]_q-[p]_q)}{[p+1]_q-[p]_q}\left(\frac{[p]_q}{[p+1]_q}\right)^n\right|\right\}.\end{aligned} \tag{14}$$

and

$$\left|a_{p+3}\right|\leq\frac{|\zeta|\mathscr{B}_1[p]_q}{[p+3]_q-[p]_q}\left(\frac{[p]_q}{[p+3]_q}\right)^n\ \mathscr{H}(\rho_1,\rho_2), \tag{15}$$

where

$$\rho_1=\frac{2\mathscr{B}_2}{\mathscr{B}_1}-\frac{\zeta\mathscr{B}_1[p]_q(2[p]_q-[p+1]_q-[p+2]_q)}{([p+1]_q-[p]_q)([p+2]_q-[p]_q)}, \tag{16}$$

and

$$\begin{aligned}\rho_2=\frac{\mathscr{B}_3}{\mathscr{B}_1}-\frac{\zeta\mathscr{B}_1[p]_q(2[p]_q-[p+1]_q-[p+2]_q)}{([p+1]_q-[p]_q)([p+2]_q-[p]_q)}\cdot\left(\frac{\mathscr{B}_2}{\mathscr{B}_1}+\frac{\zeta\mathscr{B}_1[p]_q}{[p+1]_q-[p]_q}\right)\\ -\left(\frac{\zeta\mathscr{B}_1[p]_q}{[p+1]_q-[p]_q}\right)^2.\end{aligned} \tag{17}$$

This result is sharp.

Proof. Let $f(z) \in \mathscr{G}_p^{n;q}(\zeta,\Upsilon)$, then there exists a Schwarz function $w(z) \in \Omega$ such that

$$1+\frac{1}{\zeta}\left[\frac{1}{[p]_q}\frac{z(\mathscr{D}_q(\mathscr{D}^n_{p,q}f(z))}{\mathscr{D}^n_{p,q}f(z)}-1\right]=\Upsilon(w(z)). \tag{18}$$

It follows that

$$\frac{1}{[p]_q}\frac{z(\mathscr{D}_q(\mathscr{D}^n_{p,q}f(z))}{\mathscr{D}^n_{p,q}f(z)}-1=\zeta(\Upsilon(w(z))-1). \tag{19}$$

Since

$$\begin{aligned}
&\frac{1}{[p]_q}\frac{z(\mathscr{D}_q(\mathscr{D}^n_{p,q}f(z))}{\mathscr{D}^n_{p,q}f(z)}\\
&=1+\left(\frac{[p+1]_q}{[p]_q}\right)^n\left(\frac{[p+1]_q}{[p]_q}-1\right)a_{p+1}z\\
&+\left[\left(\frac{[p+2]_q}{[p]_q}\right)^n\left(\frac{[p+2]_q}{[p]_q}-1\right)a_{p+2}\right]z^2\\
&-\left[\left(\frac{[p+1]_q}{[p]_q}\right)^{2n}\left(\frac{[p+1]_q}{[p]_q}-1\right)a_{p+1}^2\right]z^2\\
&+\left[\left(\frac{[p+3]_q}{[p]_q}\right)^n\left(\frac{[p+3]_q}{[p]_q}-1\right)a_{p+3}\right]z^3\\
&+\left[\left(\frac{[p+1]_q}{[p]_q}\right)^{3n}\left(\frac{[p+1]_q}{[p]_q}-1\right)a_{p+1}^3\right]z^3\\
&+\left[\left(\frac{[p+1]_q}{[p]_q}\right)^n\left(\frac{[p+2]_q}{[p]_q}\right)^n\right]\\
&\times\left[\left(2-\frac{[p+1]_q}{[p]_q}-\frac{[p+2]_q}{[p]_q}\right)a_{p+1}a_{p+2}\right]z^3+\ldots,
\end{aligned} \tag{20}$$

and

$$\begin{aligned}
&\zeta(\Upsilon(w(z))-1)\\
&=\zeta\mathscr{B}_1w_1z+\zeta\left[\mathscr{B}_1w_2+\mathscr{B}_2w_1^2\right]z^2\\
&+\zeta[\mathscr{B}_1w_3+2w_1w_2\mathscr{B}_2+\mathscr{B}_3w_1^3]z^3+\ldots
\end{aligned} \tag{21}$$

Substituting (20) and (21) in (19) then equating the coefficients of like powers of z, we obtain

$$a_{p+1}=\left(\frac{\zeta\mathscr{B}_1[p]_qw_1}{[p+1]_q-[p]_q}\right)\left(\frac{[p]_q}{[p+1]_q}\right)^n, \tag{22}$$

$$\begin{aligned}
a_{p+2}=&\left(\frac{\zeta\mathscr{B}_1[p]_q}{[p+2]_q-[p]_q}\right)\left(\frac{[p]_q}{[p+2]_q}\right)^n\\
&\cdot\left[w_2+w_1^2\left(\frac{\mathscr{B}_2}{\mathscr{B}_1}+\frac{\zeta\mathscr{B}_1[p]_q}{[p+1]_q-[p]_q}\right)\right],
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
a_{p+3}=&\left(\frac{\zeta\mathscr{B}_1[p]_q}{[p+3]_q-[p]_q}\right)\left(\frac{[p]_q}{[p+3]_q}\right)^n\left\{w_3+\left[\frac{2\mathscr{B}_2}{\mathscr{B}_1}\right.\right.\\
&\left.-\frac{\zeta\mathscr{B}_1[p]_q(2[p]_q-[p+1]_q-[p+2]_q)}{([p+1]_q-[p]_q)([p+2]_q-[p]_q)}\right]w_1w_2\\
&+\left[\frac{\mathscr{B}_3}{\mathscr{B}_1}-\left(\frac{\zeta\mathscr{B}_1[p]_q}{[p+1]_q-[p]_q}\right)^2\right.\\
&-\frac{\zeta B_1[p]_q(2[p]_q-[p+1]_q-[p+2]_q)}{([p+1]_q-[p]_q)([p+2]_q-[p]_q)}\\
&\left.\left.\cdot\left(\frac{\mathscr{B}_2}{\mathscr{B}_1}+\frac{\zeta\mathscr{B}_1[p]_q}{[p+1]_q-[p]_q}\right)\right]w_1^3\right\}.
\end{aligned} \tag{24}$$

From (22) and (23), we get

$$a_{p+2}-\omega a_{p+1}^2=\frac{\zeta\mathscr{B}_1[p]_q}{([p+2]_q-[p]_q)}\left(\frac{[p]_q}{[p+2]_q}\right)^n\cdot\left\{w_2-\nu w_1^2\right\}, \tag{25}$$

where

$$\nu=\frac{\zeta\mathscr{B}_1[p]_q}{([p+1]_q-[p]_q)}\left(\omega\frac{[p+2]_q-[p]_q}{[p+1]_q-[p]_q}\left(\frac{[p]_q[p+2]_q}{([p+1]_q)^2}\right)^n-1\right)-\frac{\mathscr{B}_2}{\mathscr{B}_1}. \tag{26}$$

By application of Lemma 1 and Lemma 3 we get (14) and (15), respectively. These results are sharp for the functions

$$1+\frac{1}{\zeta}\left[\frac{1}{[p]_q}\frac{z(\mathscr{D}_q(\mathscr{D}^n_{p,q}f(z))}{\mathscr{D}^n_{p,q}f(z)}-1\right]=\Upsilon(z^2).$$

and

$$1+\frac{1}{\zeta}\left[\frac{1}{[p]_q}\frac{z(\mathscr{D}_q(\mathscr{D}^n_{p,q}f(z))}{\mathscr{D}^n_{p,q}f(z)}-1\right]=\Upsilon(z).$$

The proof is complete.

The next results can be obtained using Lemma 2.

Theorem 2. *Let the function f given by (1) belongs to the class $\mathscr{G}_p^{n,q}(\zeta,\Upsilon)$, with $\Upsilon(z)=1+\mathscr{B}_1z+\mathscr{B}_2z^2+\ldots$ satisfying the conditions of the Definition 3 and $\omega,\mathscr{B}_2\in\mathbb{R}$, and $\zeta>0$, then*

$$\left|a_{p+2}-\omega a_{p+1}^2\right|\leq\begin{cases}\frac{\zeta[p]_q}{[p+2]_q-[p]_q}\left(\frac{[p]_q}{[p+2]_q}\right)^n\\ \cdot\left(\mathscr{B}_2+\frac{\zeta\mathscr{B}_1^2[p]_q}{[p+1]_q-[p]_q}(1-\omega\mathscr{K})\right) & \text{if } \omega\leq\sigma_1\\ \frac{\zeta\mathscr{B}_1[p]_q}{[p+2]_q-[p]_q}\left(\frac{[p]_q}{[p+2]_q}\right)^n & \text{if } \sigma_1\leq\omega\leq\sigma_2,\\ \frac{\zeta[p]_q}{[p+2]_q-[p]_q}\left(\frac{[p]_q}{[p+2]_q}\right)^n\\ \cdot\left(\frac{\zeta\mathscr{B}_1^2[p]_q}{[p+1]_q-[p]_q}(\omega\mathscr{K}-1)-\mathscr{B}_2\right) & \text{if } \omega\geq\sigma_2,\end{cases} \tag{27}$$

where

$$\sigma_1=\frac{([p+1]_q-[p]_q)\left[(\mathscr{B}_2-\mathscr{B}_1)([p+1]_q-[p]_q)+\zeta\mathscr{B}_1^2[p]_q\right]}{\zeta\mathscr{B}_1^2[p]_q([p+2]_q-[p]_q)}\left(\frac{([p+1]_q)^2}{[p]_q[p+2]_q}\right)^n, \tag{28}$$

$$\sigma_2=\frac{\left([p+1]_q-[p]_q\right)\left[(\mathscr{B}_2+\mathscr{B}_1)\left([p+1]_q-[p]_q\right)+\zeta\mathscr{B}_1^2[p]_q\right]}{\zeta\mathscr{B}_1^2[p]_q\left([p+2]_q-[p]_q\right)}\left(\frac{\left([p+1]_q\right)^2}{[p]_q[p+2]_q}\right)^n, \tag{29}$$

and

$$\mathscr{K}=\frac{[p+2]_q-[p]_q}{\left([p+1]_q-[p]_q\right)}\left(\frac{[p]_q[p+2]_q}{\left([p+1]_q\right)^2}\right)^n.$$

The result is sharp.

Proof. With the same technique that El-Deeb and Bulboaca [4, Theorem 2.2]adopted, we prove our result. Also, (25) and (26) are satisfied.

(i) According to the first part of Lemma 2, we have

$$\left|w_2-kw_1^2\right|\le -k,\ \text{if}\ k\le -1.$$

Using (26), simple computation shows that the inequality $k\le -1$ is equivalent to $\mu\le\sigma_1$, and from (25) combined with the inequality $\left|w_2-kw_1^2\right|\le -k$ the first part of our theorem is proved.

(ii) The second part of Lemma 2 shows that

$$\left|w_2-kw_1^2\right|\le 1,\ \text{if}-1\le k\le 1.$$

From (26), it is easy to see that the inequality $-1\le k\le 1$ is equivalent to $\sigma_1\le\mu\le\sigma_2$. From the relation (25), the inequality $\left|w_2-kw_1^2\right|\le 1$ proves the second part of our result.

(iii) Finally, form the third part of Lemma 2 we have

$$\left|w_2-kw_1^2\right|\le -k,\ \text{if}\ k\ge 1.$$

The relation (26) shows that $k\ge 1$ is equivalent to $\mu\ge\sigma_2$, while (25) combined with the inequality $\left|w_2-kw_1^2\right|\le -k$ proves the last part of our result.

Theorem 3. *Let the function f given by (1) belong to the class $\mathscr{G}_p^{n,q}(\zeta,\Upsilon)$, with $\Upsilon(z)=1+\mathscr{B}_1z+\mathscr{B}_2z^2+\ldots$ satisfying the conditions of the Definition 3 and $\omega,\mathscr{B}_2\in\mathbb{R}$, and $\zeta>0$, then the following inequalities hold:*

(i) for $\sigma_1<\omega\le\sigma_3$, we have

$$\left|a_{p+2}-\omega a_{p+1}^2\right|+\frac{\left([p+1]_q-[p]_q\right)^2}{\zeta\mathscr{B}_1^2[p]_q([p+2]_q-[p]_q)}\left(\frac{[p+1]_q}{[p]_q}\right)^{2n}\cdot\left[(\mathscr{B}_1-\mathscr{B}_2)-\left(1-\omega\frac{([p+2]_q-[p]_q)}{[p+1]_q-[p]_q}\left(\frac{[p]_q[p+2]_q}{\left([p+1]_q\right)^2}\right)^n\right)\right]\left|a_{p+1}\right|^2\le\frac{\zeta\mathscr{B}_1[p]_q}{[p+2]_q-[p]_q}\left(\frac{[p]_q}{[p+2]_q}\right)^n, \tag{30}$$

(ii) for $\sigma_3\le\omega\le\sigma_2$, we have

$$\left|a_{p+2}-\omega a_{p+1}^2\right|+\frac{\left([p+1]_q-[p]_q\right)^2}{\zeta\mathscr{B}_1^2[p]_q([p+2]_q-[p]_q)}\left(\frac{[p+1]_q}{[p]_q}\right)^{2n}\cdot\left[(\mathscr{B}_1-\mathscr{B}_2)-\left(\omega\frac{([p+2]_q-[p]_q)}{[p+1]_q-[p]_q}\left(\frac{[p]_q[p+2]_q}{\left([p+1]_q\right)^2}\right)^n-1\right)\right]\left|a_{p+1}\right|^2,\le\frac{\zeta\mathscr{B}_1[p]_q}{[p+2]_q-[p]_q}\left(\frac{[p]_q}{[p+2]_q}\right)^n, \tag{31}$$

where σ_1 and σ_2 are defined by (28) and (29), respectively,

$$\sigma_3=\frac{\left([p+1]_q-[p]_q\right)\left[\mathscr{B}_2\left([p+1]_q-[p]_q\right)+\zeta\mathscr{B}_1^2[p]_q\right]}{\zeta\mathscr{B}_1^2[p]_q\left([p+2]_q-[p]_q\right)}\left(\frac{\left([p+1]_q\right)^2}{[p]_q[p+2]_q}\right)^n.$$

Proof. Form the proof of Theorem 1 we have the relations (22) and (23). Also, from (22), we conclude that

$$w_1=\left(\frac{\left([p+1]_q-[p]_q\right)}{\zeta\mathscr{B}_1[p]_q}\right)\left(\frac{[p+1]_q}{[p]_q}\right)^n a_{p+1}. \tag{32}$$

(i) Using the inequality (12), we prove the first part of the result. Thus, according to (22) , (23) and the above-mentioned relation, it is easy to see that (12) could be written in the equivalent form (30), while the assumption $-1<k\le 0$ is equivalent to $\sigma_1<\omega\le\sigma_3$;
(ii) Considering the second part of the result we will use the inequality (13). In view of (22) , (23) and (32), it implies that (13) can be written in the form (31), and the assumption $0<k<1$ is equivalent to $\sigma_3<\omega\le\sigma_2$.

3 Applications to functions defined by Poisson distribution

Definition 4. *For the function g given by (2) and the function $f\in\mathscr{A}(p)$ is said to be in the class $\mathscr{G}_p^{n,q}(\zeta,g,\Upsilon)$ if $f*g\in\mathscr{G}_p^{n,q}(\zeta,\Upsilon)$ that is*

$$1+\frac{1}{\zeta}\left[\frac{1}{[p]_q}\frac{z\left(\mathscr{D}_q\left(\mathscr{D}_{p,q}^n(f*g)(z)\right)\right)}{\mathscr{D}_{p,q}^n(f*g)(z)}-1\right]\prec\Upsilon(z),$$

$$\left(p\in\mathbb{N},\ n\in\mathbb{N}_0,\ 0<q<1,\ \zeta\in\mathbb{C}^*\right).$$

A variable y has Poisson distribution if it takes the values 0, 1, 2, 3,...with probabilities e^{-m}, $\frac{me^{-m}}{1!}$, $\frac{m^2e^{-m}}{2!}$, $\frac{m^3e^{-m}}{3!}$, , respectively, where m is called the parameter. Thus

$$\mathscr{P}(y=k)=\frac{m^ke^{-m}}{k!},\quad k=0,1,2,3,...,$$

we introduce a power series whose coefficients are probabilities of the Poisson distribution:

$$\mathscr{K}^m(z)=z+\sum_{k=2}^{\infty}\frac{m^{k-1}e^{-m}}{(k-1)!}z^k,$$

and

$$\mathscr{I}_p^m(z)=z^{p-1}\mathscr{K}^m(z)=z^p+\sum_{k=1}^{\infty}\frac{m^k}{k!}e^{-m}z^{k+p}.$$

Now, we introduce $\mathscr{P}_p^m:\mathscr{A}(p)\to\mathscr{A}(p)$ defined by

$$\mathscr{P}_p^m f(z):=\mathscr{I}_p^m(z)*f(z)=z^p+\sum_{k=1}^{\infty}\frac{m^k}{k!}e^{-m}a_{k+p}z^{k+p},\ z\in\mathbb{U}.$$

Applying Theorem 1, Theorem 2 and Theorem 3 for the function $f*g$ given by (3) we get following results respectively:

Theorem 4. *Let the function f given by (1) belong to the class $\mathscr{G}_p^{n,q}(\zeta,g,\Upsilon)$, with $\Upsilon(z)=1+\mathscr{B}_1z+\mathscr{B}_2z^2+\ldots$ satisfying the conditions of the Definition 3, and ω is a complex number, then*

$$\left|a_{p+2}-\omega a_{p+1}^2\right|\leq\frac{|\zeta|\mathscr{B}_1[p]_q}{b_{p+2}([p+2]_q-[p]_q)}\left(\frac{[p]_q}{[p+2]_q}\right)^n$$

$$\cdot\max\left\{1,\left|\frac{\mathscr{B}_2}{\mathscr{B}_1}+\frac{\zeta\mathscr{B}_1[p]_q}{[p+1]_q-[p]_q}(1-\omega\frac{b_{p+2}([p+2]_q-[p]_q)}{b_{p+1}^2([p+1]_q-[p]_q)}\left(\frac{[p]_q}{[p+1]_q}\right)^n\right|\right\}.$$

and

$$\left|a_{p+3}\right|\leq\frac{|\zeta|\mathscr{B}_1[p]_q}{b_{p+3}([p+3]_q-[p]_q)}\left(\frac{[p]_q}{[p+3]_q}\right)^n\mathscr{H}(\rho_1,\rho_2),$$

where ρ_1 and ρ_2 are given by (16) and (17). This result is sharp.

Theorem 5. *Let the function f given by (1) belong to the class $\mathscr{G}_p^{n,q}(\zeta,g,\Upsilon)$, with $\Upsilon(z)=1+\mathscr{B}_1z+\mathscr{B}_2z^2+\ldots$ satisfying the conditions of the Definition 3 and $\omega,\mathscr{B}_2\in\mathbb{R}$, and $\zeta>0$, then*

$$\left|a_{p+2}-\omega a_{p+1}^2\right|$$

$$\leq\begin{cases}\frac{\zeta[p]_q}{b_{p+2}([p+2]_q-[p]_q)}\left(\frac{[p]_q}{[p+2]_q}\right)^n\cdot\left[\mathscr{B}_2+\frac{\zeta\mathscr{B}_1^2[p]_q}{[p+1]_q-[p]_q}(1-\omega\mathscr{K}_1)\right] & \text{if } \omega\leq\sigma_1^*,\\ \frac{\zeta\mathscr{B}_1[p]_q}{b_{p+2}([p+2]_q-[p]_q)}\left(\frac{[p]_q}{[p+2]_q}\right)^n & \text{if } \sigma_1^*\leq\omega\leq\sigma_2^*,\\ \frac{\zeta[p]_q}{b_{p+2}([p+2]_q-[p]_q)}\left(\frac{[p]_q}{[p+2]_q}\right)^n\cdot\left[\frac{\zeta\mathscr{B}_1^2[p]_q}{[p+1]_q-[p]_q}(\omega\mathscr{K}_1-1)-\mathscr{B}_2\right] & \text{if } \omega\geq\sigma_2^*,\end{cases}$$

where

$$\sigma_1^*=\frac{b_{p+1}^2([p+1]_q-[p]_q)[(\mathscr{B}_2-\mathscr{B}_1)([p+1]_q-[p]_q)+\zeta\mathscr{B}_1^2[p]_q]}{\zeta\mathscr{B}_1^2[p]_q([p+2]_q-[p]_q)b_{p+2}}\left(\frac{([p+1]_q)^2}{[p]_q[p+2]_q}\right)^n, \tag{33}$$

$$\sigma_2^*=\frac{b_{p+1}^2([p+1]_q-[p]_q)[(\mathscr{B}_2+\mathscr{B}_1)([p+1]_q-[p]_q)+\zeta\mathscr{B}_1^2[p]_q]}{\zeta\mathscr{B}_1^2[p]_q([p+2]_q-[p]_q)b_{p+2}}\left(\frac{([p+1]_q)^2}{[p]_q[p+2]_q}\right)^n, \tag{34}$$

and

$$\mathscr{K}_1=\frac{b_{p+2}([p+2]_q-[p]_q)}{b_{p+1}^2([p+1]_q-[p]_q)}\left(\frac{[p]_q[p+2]_q}{([p+1]_q)^2}\right)^n.$$

The result is sharp.

Theorem 6. *Let the function f given by (1) belong to the class $\mathscr{G}_p^{n,q}(\zeta,g,\Upsilon)$, with $\Upsilon(z)=1+\mathscr{B}_1z+\mathscr{B}_2z^2+\ldots$ satisfying the conditions of the Definition 3 and $\omega,\mathscr{B}_2\in\mathbb{R}$, and $\zeta>0$, then the following inequalities hold:*

(i) for $\sigma_1^*<\omega\leq\sigma_3^*$, we have

$$\left|a_{p+2}-\omega a_{p+1}^2\right|+\frac{([p+1]_q-[p]_q)^2}{\zeta\mathscr{B}_1^2[p]_q([p+2]_q-[p]_q)}\left(\frac{[p+1]_q}{[p]_q}\right)^{2n}$$
$$\cdot\left[\mathscr{B}_1-\mathscr{B}_2-\left(1-\omega\frac{b_{p+2}([p+2]_q-[p]_q)}{b_{p+1}^2([p+1]_q-[p]_q)}\left(\frac{[p]_q[p+2]_q}{([p+1]_q)^2}\right)^n\right)\right]\left|a_{p+1}\right|^2$$
$$\leq\frac{\zeta\mathscr{B}_1[p]_q}{b_{p+2}([p+2]_q-[p]_q)}\left(\frac{[p]_q}{[p+2]_q}\right),$$

(ii) for $\sigma_3^*\leq\mu\leq\sigma_2^*$, we have

$$\left|a_{p+2}-\omega a_{p+1}^2\right|+\frac{([p+1]_q-[p]_q)^2}{\zeta\mathscr{B}_1^2[p]_q([p+2]_q-[p]_q)}\left(\frac{[p+1]_q}{[p]_q}\right)^{2n}$$
$$\cdot\left[\mathscr{B}_1-\mathscr{B}_2-\left(\omega\frac{b_{p+2}([p+2]_q-[p]_q)}{b_{p+1}^2([p+1]_q-[p]_q)}\left(\frac{[p]_q[p+2]_q}{([p+1]_q)^2}\right)^n-1\right)\right]\left|a_{p+1}\right|^2$$
$$\leq\frac{\zeta\mathscr{B}_1[p]_q}{b_{p+2}([p+2]_q-[p]_q)}\left(\frac{[p]_q}{[p+2]_q}\right)^n,$$

where σ_1^* and σ_2^* are defined by (33) and (34) respectively,

$$\sigma_3^*=\frac{b_{p+1}^2([p+1]_q-[p]_q)[\mathscr{B}_2([p+1]_q-[p]_q)+\zeta\mathscr{B}_1^2[p]_q]}{\zeta\mathscr{B}_1^2[p]_q([p+2]_q-[p]_q)b_{p+2}}\left(\frac{([p+1]_q)^2}{[p]_q[p+2]_q}\right).$$

For $g:=\mathscr{I}_p^m$ we have

$$b_{p+1}=me^{-m},\quad b_{p+2}=\frac{m^2}{2}e^{-m}\quad\text{and}\quad b_{p+3}=\frac{m^3}{6}e^{-m},$$

and for this special case from Theorem 4, Theorem 5 and Theorem 6, we deduce the following result:

Theorem 7. *Let the function f given by (1) belong to the class $\mathscr{G}_p^{n,q}(\zeta,\mathscr{P}_p^m,\Upsilon)$, with $\Upsilon(z)=1+\mathscr{B}_1z+\mathscr{B}_2z^2+\ldots$ satisfying the conditions of the Definition 3, and ω is a complex number, then*

$$\left|a_{p+2}-\omega a_{p+1}^2\right|\leq\frac{2|\zeta|\mathscr{B}_1[p]_q}{m^2e^{-m}([p+2]_q-[p]_q)}\left(\frac{[p]_q}{[p+2]_q}\right)^n$$

$$\cdot\max\left\{1,\left|\frac{\mathscr{B}_2}{\mathscr{B}_1}+\frac{\zeta\mathscr{B}_1[p]_q}{[p+1]_q-[p]_q}(1-\omega\frac{([p+2]_q-[p]_q)}{2e^{-m}([p+1]_q-[p]_q)}\left(\frac{[p]_q}{[p+1]_q}\right)^n\right|\right\}.$$

and

$$\left|a_{p+3}\right|\leq\frac{6|\zeta|\mathscr{B}_1[p]_q}{m^3e^{-m}([p+3]_q-[p]_q)}\left(\frac{[p]_q}{[p+3]_q}\right)^n\mathscr{H}(\rho_1,\rho_2),$$

where ρ_1 and ρ_2 are given by (16) and (17). This result is sharp.

Theorem 8. *Let the function f given by (1) belong to the class $\mathscr{G}_p^{n,q}(\zeta,\mathscr{P}_p^m,\Upsilon)$, with $\Upsilon(z)=1+\mathscr{B}_1z+\mathscr{B}_2z^2+\ldots$ satisfying the conditions of the Definition 3 and $\omega,\mathscr{B}_2\in\mathbb{R}$, and $\zeta>0$, then*

$$\left|a_{p+2}-\omega a_{p+1}^2\right|\leq$$

$$\begin{cases} \frac{2\zeta[p]_q}{m^2e^{-m}\left([p+2]_q-[p]_q\right)}\left(\frac{[p]_q}{[p+2]_q}\right)^n \\ \cdot\left[\mathscr{B}_2+\frac{\zeta\mathscr{B}_1^2[p]_q}{[p+1]_q-[p]_q}(1-\omega\mathscr{K}_2)\right] & if\ \omega\le\eta_1^*, \\ \frac{2\zeta\mathscr{B}_1[p]_q}{m^2e^{-m}\left([p+2]_q-[p]_q\right)}\left(\frac{[p]_q}{[p+2]_q}\right)^n & if\ \eta_1^*\le\omega\le\eta_2^*, \\ \frac{2\zeta[p]_q}{m^2e^{-m}\left([p+2]_q-[p]_q\right)}\left(\frac{[p]_q}{[p+2]_q}\right)^n \\ \cdot\left[\frac{\zeta\mathscr{B}_1^2[p]_q}{[p+1]_q-[p]_q}(\omega\mathscr{K}_2-1)-\mathscr{B}_2\right] & if\ \omega\le\eta_1^*, \end{cases}$$

where

$$\eta_1^*=\frac{2e^{-m}\left([p+1]_q-[p]_q\right)\left[(\mathscr{B}_2-\mathscr{B}_1)\left([p+1]_q-[p]_q\right)+\zeta\mathscr{B}_1^2[p]_q\right]}{\zeta[p]_q\mathscr{B}_1^2\left([p+2]_q-[p]_q\right)}\left(\frac{\left([p+1]_q\right)^2}{[p]_q[p+2]_q}\right)^n, \tag{35}$$

$$\eta_2^*=\frac{2e^{-m}\left([p+1]_q-[p]_q\right)\left[(\mathscr{B}_2+\mathscr{B}_1)\left([p+1]_q-[p]_q\right)+\zeta\mathscr{B}_1^2[p]_q\right]}{\zeta[p]_q\mathscr{B}_1^2\left([p+2]_q-[p]_q\right)}\left(\frac{\left([p+1]_q\right)^2}{[p]_q[p+2]_q}\right)^n, \tag{36}$$

and

$$\mathscr{K}_2=\frac{([p+2]_q-[p]_q)}{2e^{-m}([p+1]_q-[p]_q)}\left(\frac{[p]_q[p+2]_q}{([p+1]_q)^2}\right)^n.$$

The result is sharp.

Theorem 9. *Let the function f given by (1) belong to the class $\mathscr{G}_p^{n,q}(\zeta,\mathscr{P}_p^m,\Upsilon)$, with $\Upsilon(z)=1+\mathscr{B}_1z+\mathscr{B}_2z^2+\ldots$ satisfying the conditions of the Definition 3 and $\omega,\mathscr{B}_2\in\mathbb{R}$, and $\zeta>0$, then the following inequalities hold:*

(i) for $\eta_1^*<\omega\le\eta_3^*$, we have

$$\left|a_{p+2}-\omega a_{p+1}^2\right|+\frac{\left([p+1]_q-[p]_q\right)^2}{\zeta\mathscr{B}_1^2[p]_q([p+2]_q-[p]_q)}\left(\frac{[p+1]_q}{[p]_q}\right)^{2n}$$
$$\cdot\left[\mathscr{B}_1-\mathscr{B}_2-\left(1-\omega\frac{([p+2]_q-[p]_q)}{2e^{-m}([p+1]_q-[p]_q)}\left(\frac{[p]_q[p+2]_q}{\left([p+1]_q\right)^2}\right)^n\right)\right]$$
$$\cdot\left|a_{p+1}\right|^2$$
$$\le\frac{2\zeta\mathscr{B}_1[p]_q}{m^2e^{-m}([p+2]_q-[p]_q)}\left(\frac{[p]_q}{[p+2]_q}\right)^n$$

(ii) for $\eta_3^*\le\omega\le\eta_2^*$, we have

$$\left|a_{p+2}-\omega a_{p+1}^2\right|+\frac{\left([p+1]_q-[p]_q\right)^2}{\zeta\mathscr{B}_1^2[p]_q([p+2]_q-[p]_q)}\left(\frac{[p+1]_q}{[p]_q}\right)^{2n}$$
$$\cdot\left[\mathscr{B}_1-\mathscr{B}_2-\left(\omega\frac{([p+2]_q-[p]_q)}{2e^{-m}([p+1]_q-[p]_q)}\left(\frac{[p]_q[p+2]_q}{\left([p+1]_q\right)^2}\right)^n-1\right)\right],$$
$$\cdot\left|a_{p+1}\right|^2$$
$$\le\frac{2\zeta\mathscr{B}_1[p]_q}{m^2e^{-m}([p+2]_q-[p]_q)}\left(\frac{[p]_q}{[p+2]_q}\right)^n,$$

where η_1^* and η_2^* are defined by (35) and (36), respectively,

$$\eta_3^*=\frac{2e^{-m}\left([p+1]_q-[p]_q\right)\left[\mathscr{B}_2\left([p+1]_q-[p]_q\right)+\zeta\mathscr{B}_1^2[p]_q\right]}{\zeta\mathscr{B}_1^2[p]_q\left([p+2]_q-[p]_q\right)}\left(\frac{\left([p+1]_q\right)^2}{[p]_q[p+2]_q}\right)^n.$$

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Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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