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Compensator Design for Non-autonomous Linear Control Systems

Sutrima Sutrima[1,*], *Christiana Rini Indrati*[2] *and Lina Aryati*[2]

[1]Department of Mathematics, University of Sebelas Maret, Ir. Sutami, no.36 A Kentingan, 57126, Surakarta, Indonesia
[2]Department of Mathematics, University of Gadjah Mada, Sekip Utara Kotak Pos: BLS 21, 55281, Yogyakarta, Indonesia

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Abstract: This paper contributes to a design of stabilizing compensators for the stabilizable systems in the class. A strongly continuous quasi semigroup approach is implemented as a generalization of a strongly continuous semigroup for autonomous systems. Stability of the non-autonomous linear control system is identified by a uniformly exponential stability of a strongly continuous quasi semigroup on the state space. The results showed that in the infinite-dimensional state space, if the closed-loop non-autonomous linear control system was stabilizable and detectable, there existed an infinite-dimensional stabilizing compensators for the system. The assigned controller is given by $u = F\hat{x}$ where $\hat{x}$ is the Luenberger observer. In any non-autonomous Riesz-spectral system, there exists a finite-dimensional compensator for the system. The construction of the compensator is based on the separation of the unstable eigenvalues of the corresponding Riesz-spectral operator. The numbers of the unstable eigenvalues are defined to be an order of the compensator. An example of the non-autonomous heat equation is given to assert the theoretical results.

Keywords: Compensator, Luenberger observer, Non-autonomous system, Riesz-spectral system, Stabilizable

1 Introduction

This paper focuses on unstable infinite-dimensional non-autonomous linear control systems with state x, input u, and output y (see [1,2]):

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \quad t \geq 0, \quad x(0) = x_0, \\ y(t) &= C(t)x(t), \end{aligned} \tag{1}$$

where $A(t)$ is a linear closed operator in X with domain $\mathscr{D}(A(t)) = \mathscr{D}$ independent of t and dense in X, $B(t) : U \to X$ and $C(t) : X \to Y$ are bounded operators such that $B(\cdot) \in L_\infty(\mathbb{R}^+, \mathscr{L}(U,X))$ and $C(\cdot) \in L_\infty(\mathbb{R}^+, \mathscr{L}(X,Y))$, where X, U, Y are complex Banach spaces, and $\mathscr{L}(V,W)$ and $L_\infty(\Omega,W)$ denote the space of all bounded operators from V to W and the space of all bounded measurable functions from Ω to W provided with essential supremum norm, respectively. In sequel, we denote system (1) by $(A(t), B(t), C(t))$.

Recall that system (1) is considered stable if the family $\{A(t)\}$ is an infinitesimal generator of a strongly continu- ous quasi semigroup which is uniformly exponentially stable on X [3]. We often have problems of the instability as in system (1). This occurs when a control system is adjusted to improve the performance's system. To make the system behave as desired, we need to redesign the system and add a compensator, a device (artificial system) which compensates the deficient performance of the original system. The system conditioned by this way is considered stabilizable. Concretely, system (1) is stabilizable if there exists an admissible control $u(t)$ such that the corresponding solution $x(t)$ has some desired properties. If the stabilizability is identified by null controllability, system (1) is stabilizable if there exists a control $u(t) = F(t)x(t)$ such that the zero solution of the closed-loop system

$$\dot{x}(t) = [A(t) + B(t)F(t)]x(t), \quad t \geq 0,$$

is asymptotically stable in the Lyapunov sense. In this case, $u(t)$ is called the stabilizing feedback control.

If system (1) is a finite-dimensional stabilizable autonomous system, then construction of the stabilizing compensator can apply the Luenberger observer or Kalman filter, see [4–10]. The methods are also applicable for the finite-dimensional non-autonomous control system [11]. Similarly, if system (1) is the

* Corresponding author e-mail: sutrima@mipa.uns.ac.id

infinite-dimensional autonomous linear control system, then the Luenberger observer can also be generalized on the construction of the stabilizing compensator for the system, see [12–17]. However, there is no guarantee that a finite-dimensional controller can always produce the closed-loop exponential stability for arbitrary system. This is a fundamental question for the associated feedback control. The question is answered positively if the system is a Riesz-spectral system [14]. In fact, this system can always be exponentially stabilized by finite-dimensional controllers as long as the original system is stabilizable and detectable, see [18–20]. In other words, if the Riesz-spectral system is stabilizable and detectable, then there exists a finite-dimensional stabilizing compensator for the system, see [13, 21–23].

The references address the necessity of investigating the stabilizability of system (1) and characterize the finite-dimensional stabilizing compensator for the non-autonomous Riesz-spectral systems [24]. A strongly continuous quasi semigroup is chosen as an analytical approach because the quasi semigroups can be widely applied to the non-autonomous problems. Since it has been introduced by Leiva and Barcenas [25] in 1991, the research development of the strongly continuous quasi semigroups has increased and expanded in various applications, see [3, 24, 26–28].

This paper focuses on the stabilizing compensator for the stabilizable non-autonomous systems (1) using the strongly continuous quasi semigroup approach and the finite-dimensional stabilizing compensator for the non-autonomous Riesz-spectral systems. The organization of this paper is as follows: In Section 2, we provide some properties of the strongly continuous quasi semigroups and the uniformly exponential stability as identifying tools to the stabilizability and detectability. Application of the Lunberger observer to design the infinite-dimensional stabilizing compensator for system (1) is described in the Section 3. In Section 4, we design the finite-dimensional stabilizing compensator for the non-autonomous Riesz-spectral systems. The theoretical results are also attached by two examples.

2 Stabilizability and Detectability

A strongly continuous quasi semigroup is a major tool in this research. We recall the definition of the strongly continuous quasi semigroup which was initiated by Leiva and Barcenas [25]. The weaker definition that follows the definition of C_0-semigroup is provided.

Definition 1. Let $\mathscr{L}(X)$ be the set of all of bounded linear operators on a Banach space X. A two-parameter commutative family $\{R(t,s)\}_{s,t\geq 0}$ in $\mathscr{L}(X)$ is called a strongly continuous quasi semigroup (in short C_0-quasi semigroup) on X if:

(a) $R(t,0) = I$, the identity operator on X,
(b) $R(t,s+r) = R(t+r,s)R(t,r)$,
(c) $\lim_{s\to 0^+} \|R(t,s)x - x\| = 0$,
(d) there is a continuous increasing function $M : [0,\infty) \to [0,\infty)$ such that

$$\|R(t,s)\| \leq M(s), \tag{2}$$

for all $r,s,t \geq 0$ and $x \in X$.

For each $t \geq 0$ we define an operator $A(t)$ on $\mathscr{D}$ by

$$A(t)x = \lim_{s\to 0^+} \frac{R(t,s)x - x}{s},$$

where $\mathscr{D}$ is a set of all $x \in X$ such that the following limits exist

$$\lim_{s\to 0^+} \frac{R(t,s)x - x}{s}, \quad t \geq 0.$$

The family $\{A(t)\}$ is called an infinitesimal generator of the C_0-quasi semigroup $\{R(t,s)\}$. Some examples and properties of C_0-quasi semigroups can be founded in [25], [26], and [27]. In the sequel, we write the quasi semigroup $\{R(t,s)\}$ and the infinitesimal generator $\{A(t)\}$ by $R(t,s)$ and $A(t)$, respectively.

In general $A(t)$ does not need to be closed and $\mathscr{D}$ does not need to be dense in X, as shown in Example 3.2 and Example 3.3 of [27]. However, in this paper we always assume that each $A(t)$ is a closed operator and the domain $\mathscr{D}$ is a dense set in X. As the first auxiliary result and using a similar principle we can construct a new C_0-quasi semigroup from any C_0-quasi semigroup.

Lemma 1. *Let $R_1(t,s)$ be a C_0-quasi semigroup on a Banach space X_1 with the infinitesimal generator $A_1(t)$. If there is an $H \in \mathscr{L}(X_1,X_2)$ such that $H^{-1} \in \mathscr{L}(X_2,X_1)$ and $HH^{-1} = I_{X_2}$ and $H^{-1}H \in \mathscr{L}(X_2,X_1)$ for some Banach space X_2, then $R_2(t,s)x_2 := HR_1(t,s)H^{-1}x_2$ for all $x_2 \in X_2$ is a C_0-quasi semigroup on a Banach space X_2 with the infinitesimal generator $A_2(t) = HA_1(t)H^{-1}$ on domain $\mathscr{D}(A_2(t)) = \{x_2 \in X_2 : H^{-1}x_2 \in \mathscr{D}(A_1(t))\}$.*

Proof. From the definition of $R_2(t,s)$ we can easily verify that it satisfies Definition 1. Moreover, by Theorem 3.2 of [27] and differentiating $R_2(t,s)$ respect to s then evaluating for $s = 0$ we obtain $A_2(t)$. □

Stabilizability and detectability of the non-autonomous linear control systems are characterized by the uniformly exponential stability of the associated C_0-quasi semigroup.

Definition 2. A C_0-quasi semigroup $R(t,s)$ is considered uniformly exponentially stable on a Banach space X if there exist constants $\alpha > 0$ and $N \geq 1$ such that

$$\|R(t,s)x\| \leq Ne^{-\alpha s}\|x\|, \quad x \in X, \quad t,s \geq 0. \tag{3}$$

The definitions of stabilizability and detectability of the non-autonomous linear control systems follow the similar definitions for the autonomous systems developed by Curtain and Zwart [14].

Definition 3. The non-autonomous linear control system $(A(t),B(t),C(t))$ is considered:

(a) stabilizable if there exsits an $F \in L_\infty(\mathbb{R}^+,\mathscr{L}(X,U))$ such that $A(t)+B(t)F(t)$ is an infinitesimal generator of a uniformly exponentially stable C_0-quasi semigroup $R_F(t,s)$. The operator F is called a stabilizing feedback operator;

(b) detectable if there exists an $L \in L_\infty(\mathbb{R}^+,\mathscr{L}(Y,X))$ such that $A(t)+L(t)C(t)$ is an infinitesimal generator of a uniformly exponentially stable C_0-quasi semigroup $R_L(t,s)$. L is called an output injection operator.

α in (3) is called the decay rate and the supremum over all possible values of α is called the stability margin of $R(t,s)$. Indeed, the stability margin is minus its uniform growth bound $\omega_0(R)$ defined by

$$\omega_0(R) = \inf_{t\geq 0} \omega_0(t),$$

where $\omega_0(t) = \inf_{s>0}\left(\frac{1}{s}\log\|R(t,s)\|\right)$.

Theorem 1. *Let $R(t,s)$ be a C_0-quasi semigroup on the Banach space X. $R(t,s)$ is uniformly exponentially stable on X if and only if $\omega_0(R)<0$.*

Proof. Let $R(t,s)$ be uniformly exponentially stable on X. There exist constants $\alpha>0$ and $N\geq 1$ such that

$$e^{\alpha s}\|R(t,s)x\| \leq N\|x\|,$$

for all $t,s\geq 0$ and $x\in X$. This provides

$$\|R(t,s)\| \leq Ne^{-\alpha s} \quad \text{or} \quad \frac{1}{s}\log\|R(t,s)\| \leq \frac{\log N}{s} - \alpha,$$

for all $t\geq 0$ and $s>0$. This gives that $\omega_0(R)\leq -\alpha<0$.

Conversely, let $\omega_0(R)=-\alpha$ for some $\alpha>0$. From Definition 1, there exist $s_0,s_1>0$ and $n\in\mathbb{N}$ such that

$$\|R(t,s)\| \leq M(s_0)^{n+1} \leq e^{-\alpha s}(e^{-\alpha k s_0 s_1}M(s_0)^{n+1}),$$

for all $t\geq 0$ and $s>s_0$. For $0\leq s\leq s_0$ we have

$$\|R(t,s)\| \leq M(s) \leq M(s_0) \leq e^{-\alpha s}(e^{\alpha s_0}M(s_0)),$$

for all $t\geq 0$. From both it can be chosen an $N>1$ such that

$$\|R(t,s)\| \leq Ne^{-\alpha s}, \quad t,s\geq 0.$$

Thus, $R(t,s)$ is uniformly exponentially stable on X. □

The following theorem gives a method to construct a quasi semigroup from two quasi semigroups and the relationship of their growth bounds.

Theorem 2. *Let $R_1(t,s)$ and $R_2(t,s)$ be the C_0-quasi semigroups on Banach spaces X_1 and X_2 with the infinitesimal generator $A_1(t)$ and $A_2(t)$, respectively. If $D(\cdot)\in L_\infty(\mathbb{R}^+,\mathscr{L}(X_1,X_2))$ and $\|R_i(t,s)\|\leq M_i(s)$, for all $t,s\geq 0$, $i=1,2$, then $A(t)=\begin{bmatrix} A_1(t) & 0 \\ D(t) & A_2(t)\end{bmatrix}$ with domain $\mathscr{D}=\mathscr{D}(A_1(t))\times\mathscr{D}(A_2(t))$ is the infinitesimal generator of the C_0-quasi semigroup $R(t,s)$ on $X=X_1\times X_2$ given by*

$$R(t,s) = \begin{bmatrix} R_1(t,s) & 0 \\ S(t,s) & R_2(t,s)\end{bmatrix} \tag{4}$$

where $S(t,s)x_1 = \int_0^s R_2(t+\tau,s-\tau)D(t+\tau)R_1(t,\tau)x_1\,d\tau$, for all $x_1\in X_1$. Moreover, there exists a positive continuous function M such that

$$\|R(t,s)\| \leq M(s), \quad t,s\geq 0.$$

Proof. We see that the matrix operator $\begin{bmatrix} A_1(t) & 0 \\ 0 & A_2(t)\end{bmatrix}$ is the infinitesimal generator of a C_0-quasi semigroup $\begin{bmatrix} R_1(t,s) & 0 \\ 0 & R_2(t,s)\end{bmatrix}$ on X. Since $\begin{bmatrix} 0 & 0 \\ D(t) & 0\end{bmatrix}$ is bounded on X, Theorem 3 of [3] implies that $A(t)=\begin{bmatrix} A_1(t) & 0 \\ D(t) & A_2(t)\end{bmatrix}$ is the infinitesimal generator of a C_0-quasi semigroup.

By the definition of $S(t,s)$ and the transformation of variable $\upsilon=r+\tau$ we have

$$\begin{aligned}
&S(t+r,s)R_1(t,r)x_1 + R_2(t+r,s)S(t,r)x_1 = \\
&\quad \int_0^s R_2(t+r+\tau,s-\tau)D(t+r+\tau)R_1(t,r+\tau)x_1\,d\tau + \\
&\quad \int_0^r R_2(t+\upsilon,r+s-\upsilon)D(t+\upsilon)R_1(t,\upsilon)x_1\,d\upsilon \\
&= \int_0^{r+s} R_2(t+\upsilon,r+s-\upsilon)D(t+\upsilon)R_1(t,\upsilon)x_1\,d\upsilon \\
&= S(t,r+s)x_1.
\end{aligned}$$

Therefore, the operator $R(t,s)$ in (4) satisfies

$$\begin{aligned}
R(t,r+s)x &= \begin{bmatrix} R_1(t,r+s) & 0 \\ S(t,r+s) & R_2(t,r+s)\end{bmatrix}\begin{bmatrix} x_1 \\ x_2\end{bmatrix} \\
&= \begin{bmatrix} R_1(t+r,s) & 0 \\ S(t+r,s) & R_2(t+r,s)\end{bmatrix}\begin{bmatrix} R_1(t,r) & 0 \\ S(t,r) & R_2(t,r)\end{bmatrix}\begin{bmatrix} x_1 \\ x_2\end{bmatrix} \\
&= R(t+r,s)R(t,r)x, \quad x\in X.
\end{aligned}$$

This gives that $R(t,s)$ is a C_0-quasi semigroup with the infinitesimal generator $A(t)$.

Next, for $x=\begin{bmatrix} x_1 \\ x_2\end{bmatrix}$, we have

$$R(t,s)x = \begin{bmatrix} R_1(t,s)x_1 \\ Q\end{bmatrix},$$

$$Q = \int_0^s R_2(t+\tau,s-\tau)D(t+\tau)R_1(t,\tau)x_1\,d\tau + R_2(t,s)x_2.$$

Therefore,

$$\begin{aligned}
\|R(t,s)x\|_X \leq \max\{&M_1(s)\|x_1\|, M_1(s)M_2(s)s\|D(\cdot)\|\|x_1\| \\
&+ M_2(s)\|x_2\|\} \leq M(s)\|x\|_X,
\end{aligned}$$

where

$M(s) = \max\{M_1(s), M_1(s)M_2(s)s\|D(\cdot)\| + M_2(s)\}$. □

3 Compensator Design

When we consider the stabilizing by state feedback $u(t) = F(t)x(t)$, we hope that the whole state can be measured. However, it is impossible for an infinite-dimensional system. A more realistic assumption is that can only measure an output that involves information about a part of the state. A problem that arises naturally is how to stabilize the system using only partial information about the state, as schematically shown in Figure 1. The second system in Figure 1 is an artificial system which is called a compensator, whose input and output are the output and input of the original system, respectively. The whole system in Figure 1 (modified from [14]) is called a closed-loop system.

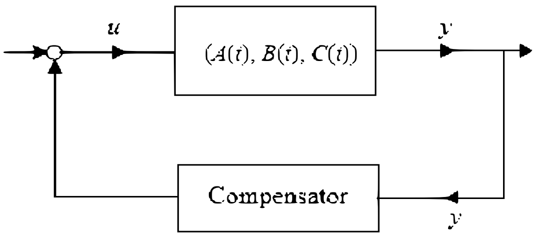

Figure 1: Closed-loop system

A basic question is how to design the compensator. In the finite-dimensional systems, we have two approaches, i.e. by Luenberger observer or by Kalman filter. Since the designed controller uses the partial information of the state to estimate the whole state and to apply the state feedback on the estimated state, we use Luenberger observer.

Definition 4. Luenberger observer for the non-autonomous linear control system $(A(t),B(t),C(t))$ (1) is defined by

$$\begin{aligned}\dot{\hat{x}}(t) &= A(t)\hat{x}(t)+B(t)u(t)+L(t)(\hat{y}(t)-y(t)),\\ \hat{y}(t) &= C(t)\hat{x}(t),\end{aligned} \tag{5}$$

where $L(\cdot)\in L_\infty(\mathbb{R}^+,\mathscr{L}(Y,X))$.

For the non-autonomous linear control system (1), how to design an observer of (5) such that $\hat{x}$ is the best estimation of the state x. This is possible if the system $(A(t),B(t),C(t))$ is detectable.

Lemma 2. *Let $(A(t),B(t),C(t))$ be a non-autonomous linear control system with the associated Luenberger observer (5). If $A(t)+L(t)C(t)$ is the infinitesimal generator of a uniformly exponential C_0-quasi semigroup where $L(\cdot)\in L_\infty(\mathbb{R}^+,\mathscr{L}(Y,X))$, then approximated error $e(t) := \hat{x}(t)-x(t)$ converges to 0 exponentially as $t\to\infty$.*

Proof. Let $R_L(t,s)$ be a uniformly exponential C_0-quasi semigroup generated by the family $A(t)+L(t)C(t)$. For $y(t) = C(t)x(t)$, the mild solution of (5) is

$$\begin{aligned}\hat{x}(t) =& R_L(0,t)\hat{x}_0+\int_0^t R_L(s,t-s)B(s)u(s)ds\\ &-\int_0^t R_L(s,t-s)L(s)C(s)x(s)ds.\end{aligned} \tag{6}$$

By Theorem 3 of [29] $R_L(t,s)$ verifies the integral equation

$$\begin{aligned}R_L(r,t)x = R(r,t)x_0+\int_0^t R_L(r+s,t-s)L(r+s)\cdot\\ C(r+s)R(r,s)x_0ds.\end{aligned}$$

Therefore, the mild solution of $(A(t),B(t),C(t))$ can be formulated to be

$$\begin{aligned}x(t) &= R(0,t)x_0+\int_0^t R(s,t-s)B(s)u(s)ds\\ &= R_L(0,t)x_0+\int_0^t R(s,t-s)B(s)u(s)ds\\ &\quad -\int_0^t R_L(s,t-s)L(s)C(s)[R(0,s)x_0+x(s)\\ &\quad -R(0,s)x_0]ds\\ &= R_L(0,t)x_0+\int_0^t R(s,t-s)B(s)u(s)ds\\ &\quad -\int_0^t R_L(s,t-s)L(s)C(s)x(s)ds.\end{aligned} \tag{7}$$

By subtracting (7) by (6) we have

$$e(t) = \hat{x}(t)-x(t) = R_L(0,t)(\hat{x}_0-x_0) = R_L(0,t)e_0,$$

where $e_0 = \hat{x}_0-x_0$. Since $R_L(0,t)$ converges to 0, then $e(t)$ converges to 0 exponentially. □

Lemma 2 states that the Luenberger observer (5) is a good estimator for the state of $(A(t),B(t),C(t))$ whenever $R_L(t,s)$ is uniformly exponential. If $x(t)$ is the state, then to stabilize the system we have to choose a feed back operator $u(t) = F(t)x(t)$ where $F(\cdot)\in(\mathbb{R}^+,\mathscr{L}(X,U))$ such that $A(t)+B(t)F(t)$ is the infinitesimal generator of a uniformly exponential C_0-quasi semigroup. However, the observation vias $y(t) = C(t)x(t)$ that gets a part information of the state $x(t)$.

The following theorem shows that the feed back $u(t) = F(t)\hat{x}(t)$ which is based to the estimated state having same influence, whenever the error of estimator converges to 0 as $t\to\infty$.

Theorem 3. *Let $(A(t),B(t),C(t))$ be the stabilizable and detectable non-autonomous linear control system. If $F(\cdot)\in L_\infty(\mathbb{R}^+,\mathscr{L}(X,U))$ and $L(\cdot)\in L_\infty(\mathbb{R}^+,\mathscr{L}(Y,X))$ such that $A(t)+B(t)F(t)$ and $A(t)+L(t)C(t)$ are the infinitesimal generator of uniformly exponential C_0-quasi semigroups, respectively, then the control $u = F\hat{x}$ where $\hat{x}$ is the Luenberger observer with output injection L stabilizes the closed-loop system, and the stabilizing compensator is given by*

$$\begin{aligned}\dot{\hat{x}}(t) &= [A(t)+L(t)C(t)]\hat{x}(t)+B(t)u(t)-L(t)y(t)),\\ u(t) &= F(t)\hat{x}(t),\end{aligned} \tag{8}$$

which is depicted in Figure 2.

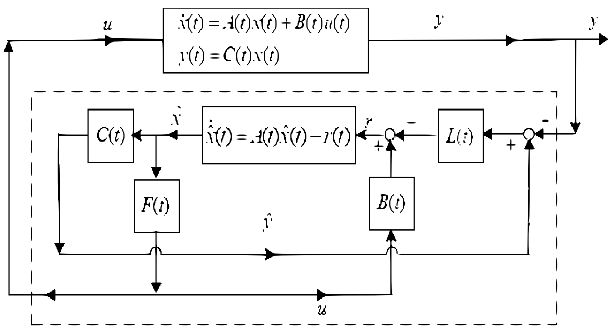

Figure 2: $(A(t), B(t), C(t))$ with compensator (8)

Proof. By hypothesis there exist the exponentially stable C_0-quasi semigroups $R_F(t,s)$ and $R_L(t,s)$ which are generated by $A(t) + B(t)F(t)$ and $A(t) + L(t)C(t)$, respectively. We notice the closed-loop system

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} = \mathfrak{A}(t) \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}, \quad t \geq 0, \tag{9}$$

where $\mathfrak{A}(t) = \begin{bmatrix} A(t) & B(t)F(t) \\ -L(t)C(t) & A(t) + B(t)F(t) + L(t)C(t) \end{bmatrix}$. Since $\mathfrak{A}(t)$ is a bounded perturbation of the infinitesimal generator $\begin{bmatrix} A(t) & 0 \\ 0 & A(t) \end{bmatrix}$, by Theorem 3 of [3] $\mathfrak{A}(t)$ is the infinitesimal generator of a C_0-quasi semigroup $R_{\mathfrak{A}}(t,s)$.

We must show that $R_{\mathfrak{A}}(t,s)$ is uniformly exponentially stable. On $\mathscr{D} \oplus \mathscr{D}$ we have the identity

$$\begin{bmatrix} A(t) + L(t)C(t) & 0 \\ -L(t)C(t) & A(t) + B(t)F(t) \end{bmatrix} = \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix} \mathfrak{A}(t) \begin{bmatrix} I & I \\ 0 & I \end{bmatrix}. \tag{10}$$

Let $R_0(t,s)$ be a C_0-quasi semigroup generated by the operator on the right-side of (10). Lemma 1 guarantees that $R_{\mathfrak{A}}(t,s)$ and $R_0(t,s)$ have the same growth constant. On other hand, Theorem 2 gives that the growth bound of $R_0(t,s)$ is the maximum of the growth bounds of $R_F(t,s)$ and $R_L(t,s)$, which are negative. Therefore, Theorem 1 asserts that the C_0-quasi semigroup $R_{\mathfrak{A}}(t,s)$ is uniformly exponentially stable. □

Since the compensator has the same state of the system $(A(t), B(t), C(t))$ which is infinite-dimensional, the compensator has infinite dimension. The following example illustrates how to design a compensator for the non-autonomous linear control systems.

Example 1. Consider the non-autonomous equation of the temperature of a heated rod, which can be measured via some device on interval $[0,1]$,

$$\begin{aligned} x_t(\xi,t) &= a(t)\pi^{-2}x_{\xi\xi}(\xi,t) + b(\xi)u(t), \\ x_\xi(0,t) &= x_\xi(1,t) = 0, \quad x(\xi,0) = x_0(\xi), \\ y(t) &= \int_0^1 c(\xi)x(\xi,t)d\xi, \end{aligned} \tag{11}$$

where $x(t,\xi) \in \mathbb{R}$ is the temperature at time t and at point $\xi \in [0,1]$, $u(t) \in \mathbb{R}$ is the control input, $a(t)$ is a time-dependent thermal (a positively bounded uniformly continuous function), b is the actuator function, and c is the sensor function.

We formulate system (11) in a Hilbert space $X = L_2(0,1)$, where $L_2(0,1)$ is the usual L_2-space with inner product $\langle \phi, \varphi \rangle := \int_0^1 \phi(\xi)\varphi(\xi)d\xi$. We define an operator $A : \mathscr{D}(A) \subset X \to X$ as

$$Ah = \frac{1}{\pi^2}\frac{d^2h}{d\xi^2}, \quad h \in \mathscr{D}(A),$$
$$\mathscr{D}(A) = \{h \in H_2(0,1) : h'(0) = 0 = h'(1)\}.$$

The A is a self-adjoint operator in X and its eigenvalues and eigenfunctions are $\lambda_0 = 0$, $\lambda_j = -j^2$ and $\phi_0(\xi) = 1$, $\phi_j(\xi) = \sqrt{2}\cos(j\pi\xi)$, $j = 1,2,\ldots$, respectively, where $\{\phi_j\}$ forms a complete orthogonal system in X. We also define $B(t)u = Bu = bu$ and $C(t)x = Cx = \langle c, x \rangle$.

Setting $A(t) = a(t)A$, we see that $A(t)$ and A have common eigenfunctions even though they have different eigenvalues. We can also verify that $A(t)$ satisfies the spectrum decomposition assumption for any β. Following Example 13 of [29], if we choose the stability margin $\beta = -0.5$, then we have the stabilizing feedback and stabilizing output given by $F(t)x = -\langle x, \phi_0 \rangle$ and $L(t)y = -y\phi_0$, respectively. Theorem 3 provides that the stabilizing compensator is given by

$$\begin{aligned} \hat{x}_t(\xi,t) &= a(t)\pi^{-2}\hat{x}_{\xi\xi}(\xi,t) - \int_0^1 c(\xi)\hat{x}(t,\xi)d\xi \\ &\quad + b(\xi)u(t) + y(t), \quad \hat{x}(\xi,0) = \hat{x}_0(\xi), \\ \hat{x}_\xi(0,t) &= \hat{x}_\xi(1,t) = 0, \\ u(t) &= -\int_0^1 \hat{x}(\xi,t)d\xi. \end{aligned} \tag{12}$$

Therefore, the system is stabilizable with the compensator instead of a state feedback.

4 Finite-Dimensional Compensator

Although in theory as shown in Example 1, we can construct an appropriate compensator, for a wide class of interesting systems, an implementation of state feedback or controllers of infinite order is often impossible. This section aims to find sufficient conditions under which system (1) can be stabilized by a finite-dimensional compensator. In particular, we focus on system (1) of a type of non-autonomous Riesz-spectral systems (see [24] for details). Recall that the family $A(t)$ in (1) generates a C_0-quasi semigroup $R(t,s)$ on X.

Refers to [24] and simplicity, we assume $A(t) = a(t)A$ where A is a self-adjoint Riesz-spectral operator, and $B(t) \in \mathscr{L}(\mathbb{R}^l, X)$ and $C(t) \in \mathscr{L}(X, \mathbb{R}^m)$ for all $t \geq 0$. In this case, we have

$$Ax = \sum_{j=1}^{\infty} \lambda_j \langle x, \phi_j \rangle \phi_j, \quad x \in \mathscr{D}(A),$$

where $\mathscr{D}(A) = \{x \in X : \sum_{j=1}^{\infty} |\lambda_j|^2 |\langle x, \phi_j \rangle|^2 < +\infty\}$, λ_j are the eigenvalues of A such that

$$\operatorname{Re}\lambda_1 \geq \operatorname{Re}\lambda_2 \geq \operatorname{Re}\lambda_3 \cdots$$

and ϕ_j are the corresponding eigenfunctions forming a Riesz basis for X.

The approach developed in this section generalizes the approaches of Schumacher [18] and Curtain and Salamon [13] to the non-autonomous systems. To make the output operator is well-defined for the solutions of system (1), we need the following hypothesis.

(H1) Operator $B(\cdot)$ is an admissible input for C_0-quasi semigroup $R(t,s)$ i.e for every $\tau > 0$ there exists a constant $\beta > 0$ such that $\int_0^\tau R(s, \tau - s)B(s)u(s)ds \in \mathscr{D}$ and

$$\left\| \int_0^\tau R(s, \tau - s)B(s)u(s)ds \right\| \leq \beta \|u\|_{L_p([0,\tau],\mathbb{R}^l)}$$

for every $u \in L_p([0,\tau],\mathbb{R}^l)$ for $1 \leq p < \infty$.

Hypothesis (H1) implies that for $F \in \mathscr{L}(X,\mathbb{R}^l)$, there exists a C_0-quasi semigroup $R_F(t,s)$ generated by $A(t) + B(t)F$ satisfies the equation (see Theorem 2.3 of [3]):

$$R_F(r,t)x = R(r,t)x + \int_0^t R(r+s,t-s)B(r+s)FR_F(r,s)xds. \quad (13)$$

As finite-dimensional model of system (1), we devote a finite-dimensional compensator of the form (see [13]):

$$\begin{aligned} \dot{w}(t) &= Mw(t) - Hy(t), \quad w(0) = w_0, \\ u(t) &= Kw(t), \end{aligned} \quad (14)$$

where $M \in \mathbb{R}^{n\times n}$, $H \in \mathbb{R}^{n\times m}$, $K \in \mathbb{R}^{l\times n}$ are suitably chosen matrices. The following result gives the well-posedness for the connected system (1), (14).

Theorem 4. *If (H1) is satisfied, then for all $x_0 \in \mathscr{D}$, $w_0 \in \mathbb{R}^n$, and $v \in L_{p,loc}([0,\infty),\mathbb{R}^l)$ there exists a unique solution pair $x(t)$ and $w(t)$ of system (1) and system (14), respectively. In other words, $x(t)$ is continuous in $\mathscr{D}$ and absolutely continuous in X and satisfies the first equation of (1) where $u(t)$ given in (14) and $w(t)$ is continuously differentiable and satisfies the first equation of (14) where $y(t)$ is given in (1).*

Proof. Set the spaces $\mathscr{D}_e = \mathscr{D} \times \mathbb{R}^n$, $X_e = X \times \mathbb{R}^n$, $U_e = \mathbb{R}^l \times \mathbb{R}^n$ and the operators $R_e(t,s) \in \mathscr{L}(\mathscr{D}_e)$, $B_e(t) \in \mathscr{L}(U_e, X_e)$, $F_e \in \mathscr{L}(\mathscr{D}_e, U_e)$ by

$$R_e(t,s) = \begin{bmatrix} R(t,s) & 0 \\ 0 & e^{Ms} \end{bmatrix}, \; B_e = \begin{bmatrix} B & 0 \\ 0 & -H \end{bmatrix}, \; F_e = \begin{bmatrix} 0 & K \\ C & 0 \end{bmatrix}.$$

Hypothesis (H1) is still valid if $\mathscr{D}, X, R(t,s), B(\cdot)$ is replaced by $\mathscr{D}_e, X_e, R_e(t,s), B_e(\cdot)$, respectively. Moreover, $x(t) \in \mathscr{D}$ and $w(t) \in \mathbb{R}^n$ satisfy (1) and (14) in the above sense, respectively, if and only if the following equation holds for every $t \geq 0$,

$$\begin{bmatrix} x(t) \\ w(t) \end{bmatrix} = R_e(0,t) \begin{bmatrix} x_0 \\ w_0 \end{bmatrix} + \int_0^t R_e(s,t-s)B_e(s) \left(F_e \begin{bmatrix} x(s) \\ w(s) \end{bmatrix} + \begin{bmatrix} v(s) \\ 0 \end{bmatrix} \right) ds$$

This proves the assertions. □

To complete the sufficiency for the existence of a stabilizing finite-dimensional compensator for system (1), we need the following hypothesis.

(H2) Assume that there exist operators $F \in \mathscr{L}(X,\mathbb{R}^l)$, $L \in \mathscr{L}(\mathbb{R}^m, X)$ and a finite dimensional subspace $W \subset X$ such that the following conditions are satisfied.

(1)The feedback quasi semigroup $R_F(t,s)$ satisfying (13) is uniformly exponentially stable.
(2)The observer quasi semigroup $R_L(t,s)$ generated by $A(t) + LC(t)$ is uniformly exponentially stable.
(3)$R_F(t,s)W \subset W$ for all $t,s \geq 0$.
(4)$\operatorname{ran} L \subset W$, where $\operatorname{ran} L$ denotes the range of L.

If (H2) is satisfied and $\dim W = n$, then there exist linear maps $\iota : \mathbb{R}^n \to X$ and $\pi : X \to \mathbb{R}^n$ satisfying

$$\pi\iota = I_{\mathbb{R}^n}, \quad \iota\pi x = x, \quad x \in W. \quad (15)$$

Moreover, $W \subset \mathscr{D}(A_F(t))$ and $\pi A_F(t)\iota$ is a well-defined linear map on $\mathbb{R}^n$, where $A_F(t) = A(t) + B(t)F$. Now, we consider system (14) in the form:

$$\begin{aligned} \dot{w}(t) &= \pi[A_F(t) + LC(t)]\iota w(t) - \iota L y(t), \\ u(t) &= F\iota w(t), \quad w(0) = w_0. \end{aligned} \quad (16)$$

Theorem 5. *If conditions of (H1), (H2) and (16) are satisfied, then the closed loop system (1),(16) is uniformly exponentially stable.*

Proof. By Theorem 4, the system (1),(16) is well-posed. Let $x(t) \in X$ and $w(t) \in \mathbb{R}^n$ be any solutions of (1) and (16), respectively. Define

$$z(t) = \iota w(t) - x(t) \in X, \quad t \geq 0. \quad (17)$$

Taking into account the second equation of (1), (17) and the first equation of (16), we have

$$\dot{w}(t) = \iota A_F(t)\pi w(t) + \pi LC(t)z(t). \quad (18)$$

We verify that $\pi A_F(t)\iota$ generates the C_0-quasi semigroup $\pi R_F(t,s)\iota$ on $\mathbb{R}^n$. Thus, the mild solution of (18) is given by

$$w(t) = \pi R_F(0,t)\iota w_0 + \int_0^t \pi R_F(s,t-s)LC(s)z(s)\,ds. \quad (19)$$

Inserting (13) with $r = 0$ into (19), (17) gives that

$$\begin{aligned} z(t) &= \iota\pi R_F(0,t)\iota w_0 + \\ &\quad \int_0^t \iota\pi R_F(s,t-s)LC(s)z(s)\,ds - x(t) \\ &= R(0,t)\iota w_0 + \\ &\quad \int_0^t R(s,t-s)[B(s)F\iota w(s) + LC(s)z(s)]ds \\ &\quad - R(0,t)x_0 - \int_0^t R(s,t-s)B(s)u(s)\,ds \\ &= R(0,t)z(0) + \int_0^t R(s,t-s)LC(s)z(s)\,ds. \end{aligned} \tag{20}$$

We can verify that $z(t)$ in (20) is a classical solution of $\dot{z}(t) = [A(t) + LC(t)]z(t)$. This gives $z(t) = R_L(0,t)z(0)$. This together with (18) gives the uniformly exponential stability of the pair $z(t)$, $w(t)$. The uniformly exponential stability of the pair $x(t)$, $z(t)$ follows from (17). □

Hypothesis (H2) is often difficult to check in concrete examples. Schumacher [18] and Curtain and Salamon [13] have proposed the equivalent conditions. The basic idea is to approximate L by generalized eigenvectors of A_F and to show that if A has a complete set of generalized eigenvectors which is stabilizable through B, there exists a stabilizing feedback operator F which does not affect the completeness property of A. More precisely, we require the following assumptions on A.

(H3) The resolvent operator of A is compact and the set $\Delta = \{\lambda \in P\sigma(A) : \mathrm{Re}\,\lambda \geq 0\}$ is finite.

(H3) guarantees that the orthogonal projection:

$$P_\Delta x = \frac{1}{2\pi i}\int_\Gamma (\lambda I - A)^{-1} x\,d\lambda, \quad x \in X,$$

is well-defined, where Γ is a simple rectifiable curve enclosing Δ but no other eigenvalue of A. This gives

$$X = X_\Delta \oplus X^\Delta,$$

where $X_\Delta = \mathrm{ran}\,P_\Delta$ and $X^\Delta = \ker P_\Delta$. We verify that these subspaces are invariant under $R(t,s)$. Furthermore, if $\dim X_\Delta = n_\Delta$, we have two maps

$$\iota_\Delta : \mathbb{R}^{n_\Delta} \to X_\Delta, \quad \pi_\Delta : X \to \mathbb{R}^{n_\Delta},$$

with the properties

$$\pi_\Delta \iota_\Delta = I, \quad \iota_\Delta \pi_\Delta = P_\Delta. \tag{21}$$

The projection $x_\Delta(t) = \pi_\Delta x(t)$ of the solution to system (1) satisfies the finite-dimensional differential equation

$$\begin{aligned} &\dot{x}_\Delta(t) = A_\Delta(t)x_\Delta(t) + B_\Delta(t)u(t), \quad x_\Delta(t) = \pi_\Delta x_0, \\ &y_\Delta(t) = C_\Delta(t)x_\Delta(t), \end{aligned} \tag{22}$$

where

$$A_\Delta(t) = \pi_\Delta A(t)\iota_\Delta,\ B_\Delta(t) = \pi_\Delta B(t),\ C_\Delta(t) = C(t)\iota_\Delta.$$

*Remark.*Proposition 2.6 of [13] implies that if (H3) is satisfied, system (22) is controllable and observable, the generalized eigenvectors of A is complete in X, and $R(t,s)$ is exponentially bounded on X_Δ, then (H2) is satisfied.

*Example 2.*We will find a stabilizing finite-dimensional compensator for system (11) in Example 1. We assume that a heating-cooling device is located over $[0.1, 0.2]$ and a temperature measuring device is over $[0.8, 0.9]$.

By the assumptions, we have $b(\xi) = 10\chi_{[0.1,0.2]}(\xi)$ and $c(\xi) = 10\chi_{[0.8,0.9]}(\xi)$, where $\chi_{[\cdot,\cdot]}$ is the characteristic function. Since the operators $A(t)$ and B given by $Bu = bu$ satisfy hypothesis (H1), system (11) is well- posed in X. Let $X_\Delta = \{\alpha\phi_0 : \alpha \in \mathbb{R}\}$ be the eigenspace of A corresponding to the unstable spectrum $\Delta = \{0\}$. The spectral projection of X onto X_Δ is given by

$$P_\Delta \phi(\xi) = \int_0^1 \phi(\xi)\,d\xi, \quad 0 \leq \xi \leq 1.$$

Choosing $\{\phi_0\}$ as a basis of X_Δ (21) gives $P_\Delta = \iota_\Delta \pi_\Delta$, where $\pi_\Delta : X \to \mathbb{R}$ and $\iota_\Delta : \mathbb{R} \to X$ are given by

$$\pi_\Delta \phi = \int_0^1 \phi(\xi)\,d\xi, \quad \iota_\Delta x_\Delta(\xi) = x_\Delta, \quad 0 \leq \xi \leq 1.$$

The reduced finite-dimensional system (22) is given by

$$A_\Delta(t) = 0, \quad B_\Delta = 1, \quad C_\Delta = 1.$$

This implies that the system is controllable and observable. As in Example 1, if we set the stability margin $\beta = -0.5$ with $F_\Delta = -1$ and $L_\Delta = -1$, then $A_\Delta(t)(t) + B_\Delta F_\Delta = -1$ and $A_\Delta(t)(t) + L_\Delta C_\Delta = -1$. Therefore, the operators $A_\Delta(t)$ with $F = F_\Delta \pi_\Delta : X \to \mathbb{R}$ is given by

$$A_F(t)h = \frac{a(t)}{\pi^2}\frac{\partial^2 h}{\partial \xi^2},$$

$$\mathscr{D}(A_F(t)) = \Big\{h \in H_2(0,1) : h'(1) = 0, \quad h'(0) = \pi^2 \int_0^1 h(\xi)\,d\xi\Big\}.$$

We verify that the eigenvectors and eigenvalues of $A_F = A + BF$ coincide with those of A except for $\lambda_0 = 0$ which is now replaced by $\lambda_F = -1$. The corresponding normalized eigenvector is given by

$$\phi_F(\xi) = \sqrt{2}\sin(\pi\xi), \quad 0 \leq \xi \leq 1.$$

We note that $A_F(t)$ and A_F have common eigenvectors even though they have different eigenvalues.

Setting $W = \mathrm{span}\{\phi_F\}$ and the maps

$$(\iota_F w)(\xi) = \phi_F(\xi)w, \quad 0 < \xi < 1, \quad w \in \mathbb{R},$$

$$(\pi_F \phi)(\xi) = \int_0^1 \phi_F(\xi)\phi(\xi), \quad \phi \in X,$$

we obtain that the map $\iota_F \pi_F : X \to W$ is an orthogonal projection onto W and $\iota_F \pi_F = 1$.

Since $L_\Delta = -1$, the operator $L : \mathbb{R} :\to X$ is given by

$$[Ly](\xi) = [\iota_\Delta L_\Delta y](\xi) = -y, \quad 0 \le \xi \le 1.$$

This implies

$$[\hat{L}y](\xi) = [\iota_F \pi_F L y](\xi) = -\tfrac{2\sqrt{2}}{\pi} y \phi_F(\xi), \quad 0 \le \xi \le 1,$$

whose range is in W. Moreover, since the perturbed operator $A(t) + \hat{L}C$ generates a uniformly exponentially stable C_0-quasi semigroup, it satisfies the spectrum determined growth assumption at $\beta = -0.5$. Thus, the operators F and $\hat{L}$ satisfy hypothesis (H2) with the one dimensional subspace $W = \text{span}\{\phi_F\}$. Therefore, the matrices in the compensator (14) can be counted as

$$\begin{aligned} M &= \pi_F(A_F(t) + \hat{L}C)\iota_F = -a(t) + \tfrac{2\sqrt{2}}{\pi}\delta, \\ H &= \pi_F \hat{L} = -\tfrac{2\sqrt{2}}{\pi}, \\ K &= F_\Delta \pi_\Delta \iota_F = -\tfrac{2\sqrt{2}}{\pi}, \end{aligned}$$

where $\delta = \cos(0.9\pi) - \cos(0.8\pi)$. These give that the first order system:

$$\begin{aligned} \dot{w}(t) &= \left[-a(t) + \tfrac{2\sqrt{2}}{\pi}\delta\right] w(t) + \tfrac{2\sqrt{2}}{\pi} y(t), \\ u(t) &= -\tfrac{2\sqrt{2}}{\pi} w(t), \end{aligned} \tag{23}$$

defines a stabilizing compensator for system (11).

If in system (11) we locate the heating-cooling device over $[0.1, 0.2]$ and the temperature measuring device over $[0.8, 0.9]$ as well as add the finite-dimensional compensator (23), system (11) is stabilizable i.e the deficient performance of the system can be compensated. In this case, the compensator (23) is more simple than the compensator (12) because it only works on the one-dimensional state space W generated by ϕ_F.

Remark.(1) We note that the one-dimensional space W is generated by the eigenfunction corresponding to the unstable eigenvalue of A. In this context, the compensator has an order 1. The order of a compensator denotes the numbers of unstable eigenvalues which are replaced with stable ones.

(2) All results of Example 2 can be generalized to arbitrary stabilizable-detectable non-autonomous Riesz-spectral system.

(3) The compensator design implies that it is possible to investigate disturbance decoupling problem:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) + D(t)q(t), \quad x(0) = x_0, \\ y(t) &= C(t)x(t), \quad t \ge 0. \end{aligned} \tag{24}$$

The disturbance decoupling problem is to find, if possible, a feedback system $u(t) = F(t)x(t)$ for the system (24) such that in the closed-loop system the disturbance input $q(\cdot)$ has no influence on $x(\cdot)$ for all disturbance signals.

5 Conclusion

The C_0-quasi semigroup approach was applied to design the stabilizing compensators for the infinite-dimensional stabilizable non-autonomous linear control systems as a generalization of the C_0-semigroup for autonomous ones. We used the uniformly exponential stability of the C_0-quasi semigroup on the state space to identify the stability of the non-autonomous systems. For the infinite-dimensional stabilizing compensator design, we used the Luenberger observer. In the stabilizable-detectable non-autonomous Riesz-spectral system, there existed a finite-dimensional stabilizing compensator for the system. The constructed compensator was based on the separation of the unstable eigenvalues of the corresponding Riesz-spectral operator. The numbers of the unstable eigenvalues was defined to be an order of the compensator. This compensator is more realistic to be applied to the real problems of the infinite-dimensional non-autonomous control systems.

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Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

References

[1] C. Chicone, Y. Latushkin, *Evolution semigroups in dyamical systems and differential equations*, Americans Mathematical Society, Rhode Island, 132-136 (1999).
[2] A. Bensoussan, G. Da Prato, M.C. Delfour, S.K. Mitter, *Representation and control of infinite dimensional systems*, Birkhaäuser, Boston, 414-429 (2007).
[3] S. Sutrima, C.R. Indrati, L. Aryati, Exact null controllability, stabilizability, and detectability of linear non-autonomous control systems: A quasi semigroup approach, *Abstr. Appl. Anal.*, **3791609**, 1-12 (2018).
[4] R.E. Kalman, Y.C. Ho, K.S. Narenda, Controllability of linear dynamical systems, *Contrib.Diff. Equ.*, **1**, 189-213 (1963).
[5] C.C. Tsui, On obust observer compensator design, *Automatica*, **24**, 687-692 (1988).
[6] A.J Pritchard, S. Townley, Robust compensator design via structured stability radii, *Systems Control Lett.*, **11**, 33-37 (1988).
[7] S.Y Zhang, Input-output feedback compensator design, *Automatica*, **23**, 255-257 (2005).

[8] A. Savoia, M. Mengoni, L. Zarri, D. Casadei, *A Nonlinear Luenberger observer for sensorless vector control of induction motors*, in Proc. Acemp-Electromotion IEEE 2011, 544-549 (2011).
[9] Z. Horváth, G. Molnárka, Design Luenberger observer for an electromechanical actuator, *Acta Tech.*, **7**, 328-343 (2014).
[10] T. Roubache, S. Chaouch, M.S Nat-Sad, Comparative study between Luenberger observer and extended Kalman filter for fault-tolerant control of induction motor drives, *Adv. Model. Anal. C*, **73**, 29-36 (2018).
[11] D.P Garg, A compensator design technique for control systems with time-varying nonlinear elements, *J. Franklin Inst.*, **292**, 35-44 (1971).
[12] N. Fujii, Feedback stabilization of distributed parameter systems by a functional observer, *SIAM J. Control Optim.*, **18**, 108-120 (1980).
[13] R.F. Curtain, D. Salamon, Finite-dimensional compensator for infinite-dimensional systems with unbounded input operators, *SIAM J. Control Optim.*, **24**, 797-816 (1986).
[14] R.F. Curtain, H.J. Zwart, *Introduction to Infinite-Dimensional Linear Systems Theory*, Springer, New York, 141-251 (1995).
[15] X. Yu, C.S. Hsu, Luenberger observer-based H_∞-compensator design for nonstandard problems, *IEEE Trans. Automat. Contr.*, **41**, 1364-1367 (1996).
[16] B. Abouzaid, M.E. Achhab, V. Wertz, Feedback stabilization of infinite-dimensional linear systems with constraints on control and its rateg, *Eur. J. Control*, **2**, 183-190 (2011).
[17] M.E. Achhab, On obsevers and compensators for infinite-dimensional semilinear systems, *Evol. Equ. Control Theory*, **4**, 131-142 (2015).
[18] J.M Schumacher, A direct approach to compensator design for distributed parameter systems, *SIAM J. Control Optim.*, **21**, 823–836 (1983).
[19] Y. Sakawa, Feedback Stabilization of linear diffusion systems, *SIAM J. Control Optim.*, **21**, 667–676 (1983).
[20] H.J. Balas, Finite-dimensional controllers for linear distributed parameter systems: exponential stability using residual mode filters, *J. Math. Anal. Appl.*, **133**, 283-296 (1988).
[21] R.F. Curtain, Compensators for infinite-dimensional linear systems, *J. Franklin Inst.*, *315*, 331-346 (1983).
[22] J. Deutscher, Finite-dimensional dual state feedback control of linear boundary control systems, *Internat. J. Control*, **86**, 41-53 (2013).
[23] H. Sano, Low order stabilizing controllers for a class of distributed parameter systems, *Automatica*, **92**, 49-55 (2018).
[24] S. Sutrima, C.R. Indrati, L. Aryati, Controllability and observability of non-autonomous Riesz-spectral systems, *Abstr. Appl. Anal.*, **4210135**, 1-10 (2018).
[25] H. Leiva, D. Barcenas, Quasi-semigroups, evolution equation and controllability, *Notas de Matematicas*, **109**, 1-29 (1991).
[26] D. Barcenas, H. Leiva, A.T. Moya, The dual quasi semi-group and controllability of evolution equations, *J. Math. Anal. Appl.*, **320**, 691-702 (2006).
[27] Sutrima, C.R. Indrati, L. Aryati, Mardiyana, The fundamental properties of quasi-semigroups, *J. Phys. Conf.*, **855**, 1-10 (2017).
[28] Sutrima, C.R. Indrati, L. Aryati, Stability of C_0-quasi semigroups in Banach spaces, *J. Phys. Conf.*, **943**, 1-13 (2017).
[29] Sutrima, C.R. Indrati, L. Aryati, Transfer function, stabilizability, and detectability of non-autonomous Riesz-spectral systems, *Telkomnika*, **16**, 3024–3033 (2018).

Sutrima Sutrima received the PhD degree in Mathematics at University of Gadjah Mada of Yogyakarta. His research interests are in the areas of functional analysis and operator semigroups including the evolution equations. He has published research articles in reputed international journals of mathematics and sciences. He is also a referee of mathematical journal.

Christiana Rini Indrati is Professor of Mathematics at University of Gadjah Mada of Yogyakarta. She received the PhD degree in Mathematics at Gadjah Mada University (Indonesia). She is referee and Editor of several international journals in the frame of pure and applied mathematics. She has published research articles in reputed international journals of mathematics and sciences. Her main research interests are integral theory, differential equations, and functional analysis.

Lina Aryati is Associate Professor of Mathematics at University of Gadjah Mada of Yogyakarta. She received the PhD degree in Applied Mathematics at University of Graz (Austria). She has published research articles in reputed international journals of mathematics and sciences. Her main research interests are biological mathematics and perturbation theory.