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Some Sequence Spaces of Sigma Means Defined by Orlicz Function

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Abstract: Ganie and Sheikh [12] have recently studied the space $\mathscr{V}_\infty(\theta)$ adopting the notion of sigma means and lacunary sequence $\theta=(k_r)$. In the present paper, we introduce and explore $\mathscr{V}_\infty(\mathscr{M},\theta)$ and $\mathscr{V}_\infty(\mathscr{M},p,\theta)$, where $\mathscr{M}$ is an Orlicz function. Some inclusion relations will be defined between the concerned spaces.

Keywords: Lacunary sequence, Orlicz function, Sigma means,.

1 Introduction

We represent the set of all sequences (real or complex) by Ω. Any subspace of Ω is called the sequence space. Following the authors in [1], [2], [3], [4], let $\mathbb{C}$ represent the complex field, $\mathbb{N}=\{0,1,2,\cdots\}$ and the set of real numbers be abbreviated by $\mathbb{R}$. Further, as in [5], [6], [7] and [8], we abbreviate the set of all bounded sequences by ℓ_∞ and c will represent the set of all convergent sequences.

Let $\mathscr{X}$ be a real or complex linear space, $\mathfrak{H}: X\to\mathbb{R}$. Then, paranormed space denotes the pair $(\mathscr{X},\mathscr{H})$ with paranorm $\mathfrak{H}$ if:
(i) $\mathfrak{H}(\theta)=0$, where θ is the zero entry of $\mathscr{X}$
(ii) $\mathfrak{H}(-u)=\mathfrak{H}(u)$,
(iii) $\mathfrak{H}(u+w)\le\mathfrak{H}(u)+\mathfrak{H}(w)$, and
(iv) scalar multiplication is continuous, means if (a_j) is a sequence of scalars with $a_j\to a$ as $j\to\infty$ and (u_n) is a sequence in $\mathscr{X}$ with $\mathfrak{H}(u_j-u)\to 0$ as $j\to\infty$ then $\mathfrak{H}(a_ju_j-au)\to 0$ as $j\to\infty$. Assume that (p_k) is a bounded sequence of strictly positive real numbers with $\mathscr{H}=max\{1,\sup_j p_j\}$.

Let σ be a mapping of the set of natural numbers into itself. By an invariant mean (or a σ mean), we call a functional $\mathfrak{S}$ on ℓ_∞ satisfying:

(i) $\mathfrak{S}(v)\ge 0$, when the sequence $v=(v_j)$ has $v_j\ge 0$, for all j,

(ii) $\mathfrak{S}(e)=0$, where $e=(1,1,\ldots)$,
and

(iii) $\mathfrak{S}(v_{\sigma(n)})=\mathfrak{S}(v)$ for all $v\in\ell_\infty$.

If we choose σ to be translation operator $j\to j+1$, then σ mean is often called as a Banach limit (see, [9]-[12], [13], [14], [15]). A invariant mean extends the limit functional on c in the notion that $\mathfrak{S}(v)=\lim_{v\to\infty} v$, for all $v\in c$, if and only if σ has no finite orbits, which means if and only if for all $j\ge 0$, $i\ge 1$, $\sigma^i(j)\ne j$ (see, [16], [17]-[19]).

A sequence $u\in\ell_\infty$ as a σ-convergent sequence if all its σ-means are the same and by $\mathscr{V}_\sigma$ we represent the set of all σ-convergent sequences. For $u=(u_n)$, we write $\mathscr{T}u=(\mathscr{T}u_j)=(u_{\sigma(j)})$, then (see, [12], [16], [20])

$$\mathscr{V}_\sigma=\left\{u\in\ell_\infty:\lim_{p\to\infty}s_{pn}(u)=\mathscr{L},\text{ uniformly in } j,\ \mathscr{L}=\sigma-\lim u\right\},$$

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where $s_{pj}(u) = \frac{1}{p+1} \sum_{m=0}^{p} u_{\sigma^m(j)}$.

An increasing sequence $\theta = (k_m)$ is called a lacunary sequence provided $k_0 = 0$ and $H_m = k_m - k_{m-1} \to \infty$. In this manuscript, the intervals computed by θ shall be abbreviated by $I_m = (k_{m-1}, k_m]$ and the fraction $\frac{k_m}{k_{m-1}}$ will be symbolized by q_m (see, [21,22,23]).

Hamid and Neyaz (see, [12]) and Mursaleen (see, [16]) introduced the space $\mathscr{V}_\infty(\theta)$, which is given by:

$$\mathscr{V}_\infty(\theta) = \left\{ u \in \ell_\infty : \sup_{m,j} |t_{mj}(u)| < \infty \right\},$$

where, $t_{mj}(u) = \frac{1}{h_m} \sum_{i \in I_m} u_{\sigma^i(j)}$.

As in [23], a continuous, convex and non-decreasing map $\mathscr{M} : [0,\infty) \to [0,\infty)$ satisfying $\mathscr{M}(0) = 0$, $\mathscr{M}(u) > 0 \, for \, u > 0$ and $\mathscr{M}(u) \to \infty \, as \, u \to \infty$ is called an Orlicz function. If the convex nature in it is replaced by subadditivity i.e. $\mathscr{M}(u+w) \leq \mathscr{M}(u) + \mathscr{M}(w)$, then the map $\mathscr{M}$ reduces to modulus function as is defined in [24]) and was further investigated in [16], [22], [25], [26], and other works.

Orlicz map $\mathscr{M}$ is known to attain Δ_2- condition, if we can find a constant $\mathscr{K} > 0$ in such a way that $\mathscr{M}(2v) \leq \mathscr{K}\mathscr{M}(v)$ $(v \geq 0)$ for each value of v. The Δ_2-condition is equivalent to $\mathscr{M}(lv) \leq \mathscr{K} l \mathscr{M}(v)$ for all values of v and for $l > 1$. Lindenstrauss and Tzafriri [25] used this notion and introduced the following space:

$$L_{\mathscr{M}} = \left\{ u \in \Omega : \Sigma_{r=0}^{\infty} \mathscr{M}\left(\frac{|v_r|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}.$$

The space $L_{\mathscr{M}}$ with norm

$$\|v\| = \inf \{ \rho > 0 : \sum_{k=1}^{\infty} \mathscr{M}\left(\frac{|v_k|}{\rho}\right) \leq 1 \}$$

yields a Banach space and is known as an Orlicz sequence space. For $\mathscr{M}(t) = t^p, 1 \leq p < \infty$, the space $L_{\mathscr{M}}$ coincides with the classical sequence spaces ℓ_p, i.e.

$$\ell_p = \left\{ v = (v_j) \in \Omega : \sum_{j=0}^{\infty} |v_j|^p < \infty \right\}.$$

2 Main results

This section addresses the new spaces $\mathscr{V}_\infty(\mathscr{M}, \theta)$ and $\mathscr{V}_\infty(\mathscr{M}, p, \theta)$, where $\mathscr{M}$ is an Orlicz function. We also establish some inclusion relations between these spaces.

There exist various ways of framing new spaces by virtue of the given sequence space. One of them relates to the field of convergence of any infinite matrix. Utilizing these notions, numerous interesting properties have been constructed for newly made spaces (see, 1-24]).

Let $\mathscr{M}$ be an Orlicz function, we define

$$\mathscr{V}_\infty(\mathscr{M}, \theta) = \left\{ u = (u_k) : \sup_{m,i} M\left(|t_{mi}(u)|\right) < \infty \right\},$$

and

$$\mathscr{V}_\infty(\mathscr{M}, p, \theta) = \left\{ u = (u_k) : \sup_{m,i} \left(\mathscr{M}|t_{mi}(u)|\right)^{p_m} < \infty \right\},$$

where, $t_{mi}(u) = \frac{1}{h_m} \sum_{r \in I_m} u_{\sigma^r(i)}$.

Now, we begin with the following result:

Theorem 2.1: The space $\mathscr{V}_\infty(\mathscr{M}, \theta)$ is a Banach space normed by

$$\|u\| = \inf \left\{ \rho > 0 : \sup_{m,i} \mathscr{M}\left(\frac{|t_{m,i}(u)|}{\rho}\right) \leq 1 \right\}.$$

Proof: It is obvious that $\mathscr{V}_\infty(\mathscr{M}, \theta)$ is a linear space under coordinate-wise addition and scalar multiplication over $\mathbb{C}$. It is also normed space equipped with the following norm $\|u\| = \inf\left\{ \rho > 0 : \sup_{m,i} \mathscr{M}\left(\frac{|t_{m,i}(u)|}{\rho}\right) \leq 1 \right\}$.

Let $u = (u^j)$ be any Cauchy sequence in $\mathscr{V}_\infty(\mathscr{M}, \theta)$, where $u^j = (u_k^j)_k$ with $k = 1, 2, 3, \cdots$. For ε small and positive, choose $v, u_0 > \varepsilon$ as fixed in such a manner that $\mathscr{M}\left(\frac{vu_0}{2}\right) \geq 1$. Then, for each $\frac{\varepsilon}{u_0 v} > 0$, we can find a natural number K such that for all $r, s \geq K$, we have

$$\|u^r - u^s\| \leq \frac{\varepsilon}{u_0 v}.$$

Consequently, aforementioned norm and for all $r, s \geq K$, we see that

$$\sup_{m,i} M\left(\frac{|t_{m,i}(u^r - u^s)|}{\|u^r - u^s\|}\right) \leq 1,$$

as $\|u^r - u^s\|$ is positive, so plugin ρ for $\left\|u^i - u^j\right\|$. Thus, we have for all $m \geq 0$ and for all $r, s \geq K$ that

$$\mathscr{M}\left(\frac{|t_{m,i}(u^r - u^s)|}{\|u^r - u^s\|}\right) \leq 1.$$

Also, since $\mathscr{M}\left(\frac{vu_0}{2}\right) \geq 1$, we have

$$\mathscr{M}\left(\frac{|t_{m,i}(u^r - u^s)|}{\|u^r - u^s\|}\right) \leq \mathscr{M}\left(\frac{vu_0}{2}\right) \forall \, m, i.$$

This shows that

$$|t_{m,i}(u^r - u^s)| \leq \frac{vu_0}{2}\frac{\varepsilon}{vu_0} = \frac{\varepsilon}{2}\ \forall\, i.$$

Thus, for each ε, we can find a natural number K in such a way that

$$|t_{m,i}(u^r - u^s)| < \varepsilon\ \forall\, r,s \geq K \text{ and for all } i,m.$$

Since $\mathscr{M}$ is continuous and letting $s \to \infty$, we see that

$$\sup_{m\geq K} \mathscr{M}\left(\frac{|t_{m,i}(u^r - u)|}{\rho}\right) \leq 1.$$

On taking infimum over such ρ's, we get the following for all i that

$$\inf\left\{\rho > 0 \,:\, \sup_{m\geq K} \mathscr{M}\left(\frac{|t_{m,i}(u^r - u)|}{\rho}\right) \leq 1\right\} < \varepsilon$$

for every $r \geq K$. However, $\mathscr{M}$ is Orlicz function with $u^r \in \mathscr{V}_\infty(\mathscr{M},\theta)$, we conclude that $u \in \mathscr{V}_\infty(\mathscr{M},\theta)$, which completes the proof.

We state the following result without proof, which can be established using standard technique.

Theorem 2.2: The space $\mathscr{V}_\infty(\mathscr{M},p,\theta)$ is a paranormed space with

$$\|\mathscr{G}(u)\| = \inf\left\{\rho^{\frac{p_n}{\mathscr{H}}} \,:\, \left\{\sup_{m,i}\mathscr{M}\left(\frac{|t_{m,i}(u)|}{\rho}\right)^{p_m}\right\}^{\frac{1}{\mathscr{H}}} \leq 1\right\}.$$

Theorem 2.3: If (p_j) and (q_j) are any two real sequences such that $0 < p_j \leq q_j < \infty$ for each $j \in \mathbb{N}$, then

$$\mathscr{V}_\infty(\mathscr{M},p,\theta) \subseteq \mathscr{V}_\infty(\mathscr{M},q,\theta)$$

Proof: Let $u \in \mathscr{V}_\infty(\mathscr{M},p,\theta)$. Then,

$$\sup_{m,i}\left(\mathscr{M}|t_{mi}(u)|\right)^{p_m} < \infty.$$

This gives

$$\mathscr{M}\left(|t_{mi}(u)|\right) < \infty.$$

Since $\mathscr{M}$ is increasing, we have

$$\sup_{m,i}\left(\mathscr{M}|t_{mi}(u)|\right)^{q_m} \leq \sup_{m,i}\left(\mathscr{M}|t_{mi}(u)|\right)^{p_m} < \infty.$$

Thus, we conclude that $u \in \mathscr{V}_\infty(\mathscr{M},q,\theta)$ and the proof follows.

Theorem 2.4: For (p_j) to be any sequence of real numbers. Then,

(i) For $0 < \inf p_j \leq 1$ for each $j \in \mathbb{N}$, we have

$$\mathscr{V}_\infty(\mathscr{M},p,\theta) \subseteq \mathscr{V}_\infty(\mathscr{M},\theta).$$

(ii) For $1 < p_j \leq \sup p_k \leq \infty$ for each $j \in \mathbb{N}$, we have

$$\mathscr{V}_\infty(\mathscr{M},\theta) \subseteq \mathscr{V}_\infty(\mathscr{M},p,\theta).$$

Proof: (i) Let $u \in V_\infty(M,p,\theta)$. Then,

$$\sup_{m,i}\left(M|t_{mi}(u)|\right)^{p_m} < \infty.$$

Now, since $0 < \inf p_j \leq 1$ for each $j \in \mathbb{N}$, we have

$$\mathscr{M}\left(|t_{mi}(u)|\right) \leq \sup_{m,i}\left(\mathscr{M}|t_{mi}(u)|\right)^{p_m} < \infty.$$

Thus, we conclude that $u \in \mathscr{V}_\infty(\mathscr{M},\theta)$.

(ii) Let $u \in \mathscr{V}_\infty(\mathscr{M},\theta)$. Then, for each $\varepsilon > 0$, there exists a natural number K such that

$$\left(\mathscr{M}|t_{mi}(u)|\right) < \varepsilon,\ \forall m \geq K \text{ for all } i \in \mathbb{N}.$$

Since $1 < p_j \leq \sup p_j < \infty$, we have

$$\sup_{m,i}\left(\mathscr{M}|t_{mi}(u)|\right)^{p_m} \leq \left(\mathscr{M}|t_{mi}(u)|\right) < \varepsilon.$$

Since ε is arbitrary small positive number, we conclude that $u \in \mathscr{V}_\infty(\mathscr{M},p,\theta)$. Hence, the proof of the theorem is complete.

3 Conclusion

Sequence spaces have been addressed by various authors as cited in the paper. In this paper we have introduced the new approach of sequences by the combination of sigma means with lacunary sequences and orlicz function. Some basic properties have been presented to best benefit the readers.

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Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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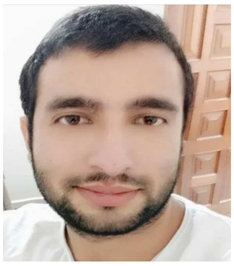

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