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Sufficient Conditions and Bounding Properties for Control Functions Using Bernstein Expansion

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Abstract: Bernstein expansion of a polynomial function has linear and quadratic rates of convergence to the original function. In this paper, we extend a direct approximation method by the minimum and maximum Bernstein control points to multivariate polynomials and continuous rational functions over boxes. Furthermore, we explore the rate of convergence and properties of Bernstein basis and illustrate the advantages of this approach through its applications for positivity of nonlinear functions. To this end, sharpness, minimization, degree elevation and convergence properties of polynomials are extended to the multivariate rational polynomial Bernstein case. Subsequently, local and global positive values of control Bernstein points are computed. Finally, several valid optimization bounds for the degree of Bernstein basis and the width of a box are given.

Keywords: Bernstein basis, Optimization bounds, Polynomial rational functions, Positivity certificates, Range of functions.

1 Introduction

Verifying the positivity of a nonlinear function by means of its positive coefficients is a major issue in the formal verification of control systems [1]. Deciding whether the coefficients of multivariate polynomial rational functions over $n-$dimensional box are positive is proposed by several authors, see [2–6]. The same subject was addressed for particular functions in [7] and [8]. Positivity certificates (PC) for polynomial functions in the Bernstein basis were proposed in [8–10]. Furthermore, providing upper and lower bounds (minimization) for the minimum value of polynomial functions over triangles was extensively explored, see [11–14]. The Bernstein method can also be used for approximation of rational functions as in [2] and [15–18]. In [19], bounds for the range might be used to test if a polynomial f was positive over a given box Q. Many applications can be considered using Bernstein method for polynomials and rational functions, such as stability of control systems, interval computation and robotics, see [20–22]. In [3] and [14], the authors showed a rate of convergence and minimum bounds by subdivision of triangles; however, without investigating PC or optimizing the minimum lower bounds over boxes. In this paper, we extend PC to the multivariate polynomials and rational functions given in the Bernstein form. Subsequently, we extend properties under the Bernstein approach to the tensorial Bernstein cases with additional bounds for the rate of convergence. Our contributions are defined, as follows:

- Most of the previous PC studies have focused on the intervals and simplicial Bernstein basis, [14], [21, 22]. In this paper, we extend Bernstein PC to the multivariate polynomials and then rational functions over any given box.
- Moreover, we assert if the monomial form is positive and obtain PC in the rational Bernstein form by sharpness, raising the degree (global certificates) as well as width of a box and minimization methods.
- We give the main sufficient conditions for existence of a positive function of some maximum degree or width of a box through Bernstein basis approximations. Furthermore, we provide a bound that is independent of the number of the given dimensions.

This paper is organized, as follows: In Section 2, we recall the most important background of the tensorial Bernstein expansion. In Section 3, we provide the convergence rate between the bounds and multivariate polynomial functions. In Section 4, we extend the

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approach with minimization to rational functions. Bernstein PC and non-dimensional bounds of the rational case are presented in Section 5. Section 6 is devoted to conclusion.

2 Background

In this section, we present the tensorial Bernstein basis of polynomials in the state space, and introduce some important properties. First, we consider the Bernstein approach of a polynomial function f expanded over a general n-dimensional box Q in the real intervals set $\mathbb{I}(\mathbb{R})^n$,

$$Q = [\underline{q}_1, \overline{q}_1] \times .... \times [\underline{q}_n, \overline{q}_n]$$

with

$$\underline{q}_\mu \leq \overline{q}_\mu,\ \mu = 1,...,n.$$

The *width* of Q_μ is denoted by $w(Q_\mu)$,

$$w(Q_\mu) := \overline{q_\mu} - \underline{q_\mu}.$$

Let $||w(Q)||_\infty := \max\{|Q_1|,...,|Q_n|\}$ be the maximum width of a box, where $|Q_\mu| = \max\{|\underline{q}_\mu|, |\overline{q}_\mu|,\ \mu \in \{1,...,n\}\}$.

In this paper, the considered Bernstein form is called the *tensorial Bernstein form*. We define the arithmetic operations of *multiindices* $i = (i_1,...,i_n)$ as component-wise. For $x \in \mathbb{R}^n$ and a multiindex j, its monomial is $x^j := \prod_{\mu=1}^{n} x_\mu^{j_\mu}$. For $D = (D_1,...,D_n)$, we have $\sum_{j=0}^{D} := \prod_{\mu=1}^{n}\sum_{j_\mu=0}^{D_\mu}$ and $\binom{D}{i} := \prod_{\mu=1}^{n}\binom{D_\mu}{i_\mu}$. An n-variate polynomial function f is expressed in the *monomial form* as

$$f(x) = \sum_{j=0}^{d} c_j x^j,$$

where $d = (d_1,...,d_n)$, and can be expressed in the *Bernstein basis* by

$$f(x) = \sum_{i=0}^{D} C_i^{(D)}(f)\, S_i^{(D)}(x), \quad x \in Q. \tag{1}$$

We underline that 0 is the multiindex with all components equal to 0.

In (1), the i^{th} Bernstein basis of degree $D \geq d$ is

$$S_i^{(D)}(x) = \binom{D}{i}(x - \underline{q})^i(\overline{q} - x)^{D-i} w(Q)^{-D}. \tag{2}$$

Moreover, the *Bernstein coefficients* $C_i^{(D)}(f)$ of degree D over Q are given by the formula

$$C_i^{(D)}(f) = \sum_{j=0}^{i} \frac{\binom{i}{j}}{\binom{D}{j}} s_j, \quad 0 \leq i \leq D, \tag{3}$$

where

$$s_j = w(Q)^j \sum_{\tau=j}^{d} \binom{\tau}{j} c_\tau \underline{q}^{\tau-j},\ c_j = 0 \text{ for } d < j. \tag{4}$$

Remark. The Bernstein basis polynomials are by construction non-negative for all $x \in Q$, i.e., $S_i^{(D)}(x) \geq 0$, $\forall i = 0,...,D$.

Without loss of generality, we can consider the domain of f to be the unit box $U = [0,1]^n$, since any non-empty box in $\mathbb{R}^n$ can be transformed thereupon by a linear transformation. Hence, the expression of f as (1) can be simplified with

$$S_i^{(D)}(x) = \binom{D}{i} x^i (1-x)^{D-i},\ x \in U, \tag{5}$$

and

$$C_i^{(D)}(f) = \sum_{j=0}^{i} \frac{\binom{i}{j}}{\binom{D}{j}} c_j,\ 0 \leq i \leq D. \tag{6}$$

We highlight two important properties of Bernstein polynomials, namely the *endpoint interpolation property*

$$C_{\hat{i}}^{(D)}(f) = f\left(\frac{\hat{i}}{D}\right), \tag{7}$$

for some $\hat{i}$, where $0 \leq \hat{i} \leq D$ satisfies $\hat{i}_\mu \in \{0, D_\mu\}$, $\mu = 1,...,n$, and the *enclosing property*

$$\min_{0 \leq i \leq D} C_i^{(D)}(f) \leq f(x) \leq \max_{0 \leq i \leq D} C_i^{(D)}(f),$$

for all $x \in U$.

Denote the enclosure bound of a polynomial by the interval

$$I^{(D)}(f,Q) := [\min_{0 \leq i \leq D} C_i^{(D)}(f), \max_{0 \leq i \leq F} C_i^{(D)}(f)],$$

and the range $G(Q) := [\min f(x), \max f(x)]$. Finally, we define h to be the Haussdorff distance between $I^{(D)}(f,Q)$ and $G(Q)$,

$$h(G(Q), I^{(D)}(f,Q)) =$$

$$\max\{|\min_{0 \leq i \leq D} C_i^{(D)}(f) - \min f(x)|, |\max_{0 \leq i \leq D} C_i^{(D)}(f) - \max f(x)|\}. \tag{8}$$

3 Polynomial Convergence Properties

In this section, we extend convergence properties between the range of a polynomial and its enclosure bound to the multivariate case over a box. Furthermore, we extend the sharpness property to multivariate polynomials in the Bernstein basis. Let a (multivariate) Bernstein polynomial of degree d be given over U. By application of raising the

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degree [23] to Bernstein basis, we conclude that we can compute Bernstein coefficients of degree D as linear convex combinations of the coefficients of degree d. It follows that

$$I^{(D)}(f,U) \subseteq I^{(d)}(f,U). \tag{9}$$

In the following subsection, we provide a linear convergence with respect to the maximum degree D' of Bernstein basis. Define

$$D' = \max\{D_1,...,D_n\}.$$

3.1 Linear Convergence

Linear convergence of the enclosure $I^{(D)}(f,Q)$ to the range $G(Q)$ with respect to raising the degree of Bernstein basis is extended from the bivariate case [12, Theorem 3] to the multivariate case over a box in the following theorem.

Theorem 31 *Let $f(x)$ be of degree $d \leq D$, the following overestimation of the range $G(Q)$ holds for f over Q. Precisely,*

$$h(G(Q),I^{(D)}(f,Q)) \leq \frac{K}{D'}, \tag{10}$$

where

$$K := \sum_{j=0}^{d}\sum_{\mu=1}^{n}[\max(0,j_\mu - 1)]^2|s_j|, \tag{11}$$

and the coefficients s_j are given by (4).

Proof. For simplicity, we provide the proof for the uni-variate case. The multivariate case holds by using the same arguments and following Lemma 1 with Theorem 33, bellow. For $x \in Q$, assume that $\delta_i := f(\frac{i}{D}) - C_i^{(D)}(f)$, $i = 0,...,D$, and $S_{(D)}(f,x) := \sum_{i=0}^{D} f(\frac{i}{D})\binom{D}{i}x^i(1-x)^{D-i}$. Then, we deduce that

$$h(G(Q),I^{(D)}(f,Q)) \leq \max_{x\in Q}|S_{(D)}(f,x) - f(x)|$$

$$= \max_{x\in Q}\Big|\sum_{i=0}^{D}\delta_i S_i^{(D)}(x)\Big| \leq \max_{0\leq i\leq D}|\delta_i|.$$

Define $\delta_i(j) := (\frac{i}{D})^j - \frac{\binom{i}{j}}{\binom{D}{j}}$, it follows that $\delta_i = \sum_{j=0}^{i} s_j\delta_i(j)$. Since $\delta_i(0) = \delta_i(1) = 0$, then we assume that $j \geq 2$.

If $0 \leq i < j$, then we have

$$\delta_i(j) = (\frac{i}{D})^j \leq (\frac{j-1}{D})^j$$

$$\leq (\frac{j-1}{D})^2 \leq \frac{(j-1)^2}{D}.$$

If $2 \leq j \leq i$, then

$$\delta_i(j) = \left((\frac{i}{D})^j - \frac{i!(D-j)!}{(i-j)!D!}\right)$$

$$= \left((\frac{i}{D})^j - \frac{i(i-1)...(i-(j-1))}{D(D-1)...(D-(j-1))}\right)$$

$$= (\frac{i}{D})^j\left(1 - \frac{(1-\frac{1}{i})...(1-\frac{j-1}{i})}{(1-\frac{1}{D})...(1-\frac{j-1}{D})}\right)$$

$$\leq (\frac{i}{D})^j\left(1 - (1-\frac{1}{i})...(1-\frac{j-1}{i})\right)$$

$$\leq (\frac{i}{D})^j\left(1 - (1-\frac{j-1}{i})^{i-1}\right).$$

Applying the mean value theorem, we obtain

$$1 - (1-\frac{j-1}{i})^{i-1} \leq \frac{(j-1)^2}{i},$$

so

$$\frac{(j-1)^2}{D}(\frac{i}{D})^{j-1} \leq \frac{(j-1)^2}{D},$$

from which the proof follows.

3.2 Quadratic Convergence

In this subsection, we provide a quadratic convergence derived from the maximum width of a box Q.

Let

$$c_j^{(Q)} = \frac{f^j(\underline{Q})}{j!},$$

be the $j-$th Taylor coefficient of f. Hence,

$$f(Q) = \sum_{j=0}^{d} c_j^{(Q)}(x-\underline{Q})^j. \tag{12}$$

Thus,

$$f(x) = \sum_{j=0}^{d} c_j^{(Q)}\sum_{i=j}^{d}\frac{\binom{i}{j}}{\binom{D}{j}}w(Q)^j S_i^{(D,Q)},\ d \leq D,$$

where

$$S_i^{(D,Q)}(x) = \binom{D}{i}\frac{(x-\underline{Q})^i(\overline{Q}-x)^{D-i}}{w(Q)^D}.$$

The polynomial f in (12) has Bernstein coefficients given by

$$C_i^{(D,Q)}(f) = \sum_{j=0}^{i} c_j^{(Q)}\frac{\binom{i}{j}}{\binom{D}{j}}w(Q)^j. \tag{13}$$

The *enclosing property* over Q is given as

$$\min_{0\leq i\leq D} C_i^{(D,Q)}(f) \leq f(x) \leq \max_{0\leq i\leq D} C_i^{(D,Q)}(f),\ \text{ for all } x \in Q. \tag{14}$$

First, we show that the vertex condition (sharpness) property for the multivariate case holds. To this end, we apply the linear transformation which maps Q to the unit box U and to use the Bernstein coefficients (6) of the mapped polynomial.

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Proposition 32 *The Bernstein enclosure bound of a multivariate polynomial is sharp, i.e.*

$$\max_{0\le i\le D} C_i^{(D)}(f) = \max f(x)$$

if and only if

$$\max_{0\le i\le D} C_i^{(D)}(f) = C_{\hat{i}}^{(D)}(f) \text{ for some } \hat{i} \text{ satisfying } \hat{i}_\mu \in \{0, D_\mu\}.$$

A similar equality holds for the minimum value.

Proof. By the interpolation property, $C_{\hat{i}}^{(D)}(f)$ with $\hat{i}_\mu \in \{0, D_\mu\}$ is a value for f at a vertex of U. It follows that $\max_{0\le i\le D} C_i^{(D)}(f)$ is sharp if it appears at such a Bernstein value.

Conversely, suppose that

$$\max_{0\le i\le D} C_i^{(D)}(f) = \max f(x) = f(\hat{x}), \text{ for some } \hat{x} \in U,$$

and

$$\max_{0\le i\le D} C_i^{(D)}(f) > C_{\hat{i}}^{(D)} \text{ with } \hat{i}_\mu \in \{0, D_\mu\}.$$

If $0 < \hat{x}_\mu < 1$, $\mu = 1,\dots,n$, then $0 < C_{i_\mu}(\hat{x}_\mu) < 1$ and

$$f(\hat{x}) = \sum_{i=0}^{D} C_i^{(D)}(f)\, S_i^{(D)}(\hat{x})$$

$$< \max C_i^{(D)}(f) \sum_{i=0}^{D} S_i^{(D)}(\hat{x})$$

$$= \max C_i^{(D)}(f),$$

a contradiction. The proof of other cases is analogous.

The following lemma will be used in the proof of quadratic convergence rate.

Lemma 1. *Let* $\delta_i(j,Q) := (\frac{i}{D}w(Q))^j - \frac{\binom{i}{j}}{\binom{D}{j}}w(Q)^j$, *and* $j, i = 0,\dots,D$. *Then,*

$$0 \le \delta_i(j,Q) \le \frac{D-1}{D^2}\sum_{\mu=1}^{n}[\max(0, j_\mu - 1)]^2 w(Q)^j. \quad (15)$$

Proof. Using the tensor product, the proof of this lemma follows using arguments similar to Stahl's proof for the univariate case [24].

Theorem 33 *Let $B \in \mathbb{IR}^n$ be fixed. Then, for all $Q \in \mathbb{IR}^n$ where $Q \subseteq B$, and $d \le D$ it holds that*

$$h(G(Q), I^{(D)}(f,Q)) \le K'||w(Q)||_\infty^2, \quad (16)$$

where K' is an explicit constant independent of Q.

Proof. For $x \in Q$, put $S_{(D)}(f,x) := \sum_{i=0}^{D} f(\frac{i}{D}w(Q) + \underline{Q})S_i^{(D,Q)}$. Then,

$$S_{(D)}(f,x) - f(x) = \sum_{i=0}^{D} f(\frac{i}{D}w(Q) + \underline{Q})S_i^{(D,Q)} - \sum_{i=0}^{D} C_i^{(D,Q)} S_i^{(D,Q)}$$

$$= \sum_{i=0}^{D}\sum_{j=0}^{d} c_j^{(Q)}(\frac{i}{D}w(Q))^j S_i^{(D,Q)} - \sum_{i=0}^{D} C_i^{(D,Q)} S_i^{(D,Q)}$$

$$= \sum_{i=0}^{D}\sum_{j=0}^{d} c_j^{(Q)}\delta_i(j,Q) S_i^{(D,Q)}.$$

Let $B \in \mathbb{IR}^n$ be fixed and $Q \subseteq B$, with $c_j^{(B)} = \max\{|c_j^{(Q)}|,\ Q \in \mathbb{IB}\}$. Then, it follows by Lemma 1, $\binom{i}{j} = 0$ if $i_{\mu_0} < j_{\mu_0}$, that

$$h(G(Q), I^{(D)}(f,Q))$$

$$\le \frac{D-1}{D^2}\sum_{j=0}^{d}\Big(\sum_{\mu=1}^{n}[\max(0, j_\mu - 1)]^2\Big)|c_j^{(B)}|w(Q)^j$$

$$\le \frac{D-1}{D^2}\sum_{j=0}^{d}\Big(\sum_{\mu=1}^{n}[\max(0, j_\mu - 1)]^2\Big)|c_j^{(B)}|\,||w(Q)||_\infty^j.$$

If $\sum_{\mu=1}^{n}[\max(0, j_\mu - 1)]^2 \ne 0$, then there exist at least a single μ_0 with $j_{\mu_0} \ge 0$. Then, $||w(Q)||_\infty^2$ can be taken. Evaluate the remaining degrees of $||w(Q)||_\infty$ by the respective degrees of $||w(B)||_\infty$, then the extracted constant value is independent of Q.

Remark. Let $\boldsymbol{x}_i^{(D)}$ be a grid point and the μth component is given by

$$\boldsymbol{x}_{i,\mu}^{(D_\mu)} = \underline{x}_\mu + \frac{i_\mu}{D_\mu}(\overline{x}_\mu - \underline{x}_\mu),\ \mu = 1,\dots,n. \quad (17)$$

Then, following the proof of Theorem 33, the absolute difference $|f(\boldsymbol{x}_i^{(D)}) - C_i^{(D,Q)}|$ can be optimized from above for all i, $0 \le i_\mu \le D_\mu$, by $K'||w(Q)||_\infty^2$.

4 Bernstein Form for Rational Controller

In this section, we assume that any rational (control) function $r := p/g$ is of the same degree d for both polynomials p and g. Otherwise, we can elevate the degree of Bernstein basis of either p or g, which is necessary to ensure that their Bernstein coefficients are of the same order $d \le D$. Let the range of r over U be defined by $R(U) := [\min r(x), \max r(x)] =: [\underline{r}, \overline{r}]$. The *tensorial rational Bernstein coefficients* of r of degree D are given by

$$C_i^{(D)}(r) = \frac{C_i^{(D)}(p)}{C_i^{(D)}(g)},\ 0 \le i \le D. \quad (18)$$

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Without loss of generality, we assume that $C_i^{(D)}(g) > 0$, $\forall\, 0 \le i \le D$.

By [17, Theorem 3.1], the *range enclosure bound* for a rational polynomial function is given as

$$L^{(D)} := \min C_i^{(D)}(r) \le r(x) \le \max C_i^{(D)}(r) =: M^{(D)}. \quad (19)$$

Denote the enclosure bound of r by $I^{(D)}(r,U) := [L^{(D)}, M^{(D)}]$.

Remark. From (19), the enclosure bound optimizes the range of a rational function,

$$R(U) \subseteq I^{(D)}(r,U).$$

By applying (18) to (19), the following theorem provides the sharpness property of r with respect to its enclosure bound.

Proposition 41 *[15, Proposition 3] For $x \in \mathbb{R}^n$, it holds that $L^{(D)} = \underline{r}$ $(M^{(D)} = \overline{r})$ if and only if $L^{(D)}$ $(M^{(D)}) = C_{\hat{i}}^{(D)}(r)$ with some $\hat{i}$ satisfying $\hat{i}_\mu \in \{0, D_\mu\}$.*

4.1 Minimum Bound

By the expansion of a rational function onto Bernstein form, the minimum Bernstein coefficient optimizes the minimum range of r over a box. Choosing $D = d$ and considering the unit box U as a domain of r, the enclosure improves by subdividing U into subdomains and computing enclosures for r over each subdomain. At subdivision level $1 \le l$, we can repeat bisection of $U^{(0,1)} := U$ in all n coordinate directions in subboxes $U^{(l,v)}$ of edge length 2^{-l}, $v = 1, \ldots, 2^{nl}$, see [12], [25]. An n-dimensional polynomial rational function $r = p/g$ can be represented as

$$r(x) = \sum_{i=0}^{d} C_i^{(d,v)}(r)\, S_i^{(d,U^{(l,v)})}(x), \text{ for } x \in U^{(l,v)}, \quad (20)$$

where $C_i^{(d,v)}(r)$ denote the rational *Bernstein coefficients* of r of degree d over $U^{(l,v)} = [\overline{q}_{(l,v)}, \underline{q}_{(l,v)}]$.

The method in the following remark can be used to search for a subdomain where the minimum Bernstein value appears.

Remark.(cut-off-test) Let U' be a subbox of U, and r^* an upper bound on $\underline{r}$ over U. If $\min C_i^{(d,U')}(r) > r^*$, then $\underline{r}$ can not occur in U'. Hence, U' can be deleted from the set of subboxes to be subdivided.

Remark. Assume that $L^{(d)} = C_{i_0}^{(d)}(r)$, for some $0 \le i_0 \le d$, is attained over some $U^{(l,v_0)}$, $1 \le v_0 \le 2^{nl}$, and the corresponding grid point $\boldsymbol{x}_{i_0}^{(d,U^{(l,v_0)})}$ in $U^{(l,v_0)}$. For all $\mu = 1, \ldots, n$, define the value L^* by

$$L^* = \min\{r(\boldsymbol{x}_{i_0}^{(d,U^{(l,v_0)})}),\ C_{\hat{i}}^{(U^{(l,v_0)})}(r),\ \ \hat{i}_\mu \in \{0, d_\mu\}\}.$$

Then, by the interpolation property (7) and the enclosure bound (19), we have

$$\min_{\substack{0\le i\le d,\\ 1\le v\le 2^{nl}}} C_i^{(l,v)}(r) \le \min_{x\in U} r(x) \le L^*. \quad (21)$$

The following theorem provides a quadratic convergence with respect to subdivision.

Theorem 42 *[15] For each $1 \le l$, it holds that*

$$\min_{\substack{0\le i\le l,\\ 1\le v\le 2^{nd}}} C_i^{(l,v)}(r) - \underline{r} \le T(2^{-l})^2$$

where T is an explicit constant independent of l.

Theorem 43 *Let $\varepsilon > 0$ be a real number and satisfying*

$$\frac{1}{(2^{-l})^2} > \frac{T}{\varepsilon}.$$

Then,

$$L^* - \min_{\substack{0\le i\le d,\\ 1\le v\le 2^{nl}}} C_i^{(l,v)}(r) < \varepsilon. \quad (22)$$

Proof. Let $\boldsymbol{x}_{i_0}^{(d,U^{(l,v_0)})}$ be a grid point in $U^{(l,v_0)}$ and $C_{i_0}^{(l,v_0)}(r)$ is the corresponding Bernstein coefficient. Then,

$$L^* - \min_{\substack{0\le i\le d,\\ 1\le v\le 2^{nl}}} C_i^{(l,v)}(r) \le |r(\boldsymbol{x}_{i_0}^{(d,U^{(l,v_0)})}) - \min_{\substack{0\le i\le d,\\ 1\le v\le 2^{nl}}} C_i^{(l,v)}(r)|$$

$$= |r(\boldsymbol{x}_{i_0}^{(d,U^{(l,v_0)})}) - C_{i_0}^{(l,v_0)}(r)|$$

$$\le T(2^{-l})^2,$$

where the last inequality follows by Theorem 42.

Corollary 1 *Given $r = p/g$ holds with the same assumptions of Theorem 43, it follows that r of degree D has only positive Bernstein coefficients if $\underline{r} \ge \varepsilon$.*

5 Positivity of Rational Functions

It may be the case where we have positive rational functions over a box in the monomial form, but they have non-positive Bernstein coefficients as shown in the following example.

Example 51 *Let $p(x) = 7x^2 - 5x + 1$ and $g(x) = x^2 - 2x + 7$ be in the monomial form of degree 2. It can be immediate to find that $r = p/g$ is positive over $[-1,1]$. However, note that $\min C_i^{(2)}(r) = -1$ is negative in the Bernstein form.*

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It follows by Remark 2 that the (univariate) Bernstein basis of f of degree D over $[\underline{q},\overline{q}]$

$$S_i^{(D)}(x) = \binom{D}{i}\frac{(\overline{q}-x)^{D-i}(x-\underline{q})^i}{w(Q)^D},\ i=0,\dots,D,$$

is positive over $(\underline{q},\overline{q})$. For $r = p/g$, the Bernstein coefficient $C_0^{(D)}$ is the value of r at $\underline{q}$, and $C_D^{(D)}$ is the value at $\overline{q}$. Hence, if the minimum Bernstein coefficient of $r(x)$ is positive, then r satisfies the positivity certificates (PC) over a given domain. $O^{(D)}(r)$ denotes the ordered list of rational Bernstein coefficients of r over U. Furthermore, we define PC in the Bernstein basis by $PC(O^{(D)}(r))$,

$$PC(O^{(D)}(r)) : \begin{cases} C_i^{(D)}(r) \geq 0 & \text{for all } 0 \leq i \leq D \\ C_{\hat{i}}^{(D)}(r) > 0 & \text{for } \hat{i},\ \hat{i}_\mu \in \{0, D_\mu\}. \end{cases}$$

5.1 Sharpness for Positivity

The sharpness property can be used to assert the positivity of rational functions. In other words, if the sharpness property in Proposition 41 holds, r satisfies PC.

Proposition 51 *Let r be positive in the monomial form over U. If $\min C_i^{(D)}(r) = C_{\hat{i}}(r)$ for some $\hat{i}$, with $\hat{i}_\mu \in \{0, D_\mu\}$. Then r satisfies PC.*

Proof. Suppose that $\min C_i^{(D)}(r) = L^{(D)}$. From Proposition 41, we have $L^{(D)} = \underline{r}$ if and only if $L^{(D)} = C_{\hat{i}}^{(D)}(r)$ for some $\hat{i}$ satisfying $\hat{i}_\mu \in \{0, D_\mu\}$. Using the interpolation property (7), it follows that $L^{(D)}$ is positive if $r(x)$ in the monomial form is positive.

5.2 Degree Elevation for Positivity

Here, we study PC for rational functions if the degree $d \leq D$ is elevated. Specifically, we certify that $r = p/g$ has global PC of some degree D over U. To this end, a linear rate of convergence for $I^{(D)}(r,U)$ to $R(U)$ with respect to raising the degree is given in the following theorem.

Theorem 52 *[15, Theorem 5] For $d \leq D$ is the Bernstein degree of $r(x)$, it holds that*

$$h(R(U), I^{(D)}(r,U)) \leq \frac{A}{D'}, \tag{23}$$

where A is an explicit constant independent of the total degree D'.

Remark. By raising the degree D' high enough, from Theorem 52, the minimum Bernstein bound of r converges to the minimum value of r, and consequently satisfies $PC(O^{(D)}(r))$.

The degree of Bernstein form can be bounded in the following proposition.

Proposition 53 *Assume that $r(x)$ of degree d is a positive in the monomial form over U. If*

$$D' > \frac{A}{\underline{r}},$$

where A is the constant (23), then r satisfies the global $PC(O^{(D)}(r))$.

Proof. Elevate $D \geq d$, so

$$\underline{r} - L^{(D)} \leq \underline{r}.$$

Then, $C_i^{(D)}(r)$ are nonnegative. Therefore, Theorem 52 implies that

$$\underline{r} - L^{(D)} \leq \frac{A}{D'},$$

and the interpolation property shows that $C_{\hat{i}}$, $0 \leq \hat{i} \leq D$, are positive.

Example 52 *Let $p(x) = 5x^2 - 3x + 1$ and $g(x) = x^2 + 1$ be of degree 2. Note that $r = p/g$ is positive over $[0,1]$. From (18), $\min C_i^{(2)}(r) = -0.5$ is negative. By raising the degree of r, the coefficients of degree 3 can be computed using (6) and (18), $\min C_i^{(3)}(r) = 0$, $C_0^{(3)}(r) = 1$ and $C_3^{(3)}(r) = 1.5$. Then, $r(x)$ has global PC at $D = 3$.*

5.3 Positivity over a Box

Bounding the range of a rational function r can be used to test if the coefficients of r are positive over Q. Quadratic convergence of $I^{(D)}(r,Q)$ to $R(Q)$ is given in the following theorem.

Theorem 54 *[15, Theorem 6] Assume that B is fixed, so $B \in \mathbb{I}(\mathbb{R})^n$. Then, for all $Q \in \mathbb{I}(\mathbb{R})^n$ where $Q \subseteq B$, and $d \leq D$,*

$$h(R(Q), I^{(D)}(r,Q)) \leq A'\, ||w(Q)||_\infty^2, \tag{24}$$

where A' is an explicit constant independent of Q.

The immediate results of Theorem 54 and the interpolation property are given in the next corollary

Corollary 2 *Given $r(x)$ is positive in the monomial rational form of degree d over Q. Let $\underline{r}$ be the minimum range of r. If r is given in the Bernstein form and*

$$||w(Q)||_\infty^2 < \frac{\underline{r}}{A'},$$

then r satisfies $PC(O^{(D)}(r))$ with respect to the width of a box.

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5.4 Independent Rational Bounds for PC

A tight bound in this section is given with no relation to the number of dimensions of $r = p/g$. Precisely, we extend independent bounds of polynomials from [10] and [26] to the rational case.

Theorem 55 *[8, Theorem 3] Given f is a positive polynomial of maximum degree d'. Let $\underline{f}$ be the minimum of f on U. Then, for*

$$D' > \frac{d'(d'-1)}{2} \frac{\max|C_i^{(d)}(f)|}{\underline{f}}, \tag{25}$$

the Bernstein coefficients for f of degree D are positive.

This bound for the maximum degree D' can be extended to the tensorial rational Bernstein form.

Corollary 3 *Let $r = p/g$ be given in the monomial rational form of maximum degree d', positive over U. If*

$$D' > \frac{d'(d'-1)}{2} \frac{\max|C_i^{(d)}(p)|}{\underline{p}},$$

then r satisfies $PC(O^{(D)}(r))$.

Proof. Elevate the degree D of r high enough and assume, in the monomial form, that $\frac{p(x)}{g(x)} > 0 =: e$. Therefore, $t(x) := p(x) - e \cdot g(x) > 0$ satisfies from Theorem 55 the PC for $t(x)$ in the Bernstein form. Thus, if

$$D' > \frac{d'(d'-1)}{2} \frac{\max|C_i^{(d)}(t)|}{\underline{t}},$$

then all $C_i^{(D)}(p) - e \cdot C_i^{(D)}(g)$ are positive. Hence, the Bernstein coefficients $C_i^{(D)}(p)/C_i^{(D)}(g)$ are positive, $\forall\, 0 \le i \le D$.

Corollary 4 *Let $J_1 = \frac{A}{\min r(x)}$, where A is the explicit constant (23), and let $J_2 = \frac{d'(d'-1)}{2} \frac{\max|C_i^{(d)}(p)|}{\min t(x)}$. Then, the rational monomial function $r = p/g$ satisfies $PC(O^{(D)}(r))$ over U if $D' > \max\{J_1, J_2\}$.*

6 Conclusion

In this paper, we investigated (multivariate) polynomials and rational functions in the monomial form and expanded them onto the Bernstein form over a box. The minimum and maximum Bernstein control points optimized the range of these functions over the whole domain. We proved that these bounds would converge linearly to the original range if the degree was raised. Furthermore, we proved the quadratic convergence between these bounds and their range with respect to the maximum width of a box. Minimization of the minimum range was also achieved with respect to subdivision of the given domain. Subsequently, we optimized the degree of Bernstein basis and width of a box by computed tight bounds, and applied them to positivity certificates of rational polynomial functions in the tensorial Bernstein basis. Moreover, we provided important properties for the tensorial Bernstein basis named sharpness and monotonicity of the bound. Finally, several valid optimization bounds for the degree of Bernstein bases and the width of a box were given.

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Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

References

[1] P. Tabuada, *Verification and control of hybrid systems: a symbolic approach.* No. 1, Springer-Verlag US, 2009.
[2] T. Hamadneh, N. Athanasopoulos, and M. Ali, “Minimization and positivity of the tensorial rational Bernstein form,” in *2019 IEEE Jordan International Joint Conference on Electrical Engineering and Information Technology (JEEIT)*, pp. 474–479, IEEE, 2019.
[3] T. Hamadneh, N. Athanasopoulos, and R. Wisniewski, “Control design and lyapunov functions via bernstein approximation: Exact results,” in *21st Ifac World Congress*, Elsevier, to appear, 2020.
[4] A. Straub, “Positivity of szegö’s rational function,” *Advances in Applied Mathematics*, vol. 41, no. 2, pp. 255–264, 2008.
[5] A. Straub and W. Zudilin, “Positivity of rational functions and their diagonals,” *Journal of Approximation Theory*, vol. 195, pp. 57–69, 2015.
[6] G. Szegö, “Über gewisse potenzreihen mit lauter positiven koeffizienten,” *Mathematische Zeitschrift*, vol. 37, no. 1, pp. 674–688, 1933.
[7] R. Askey, “Certain rational functions whose power series have positive coefficients. ii,” *SIAM Journal on Mathematical Analysis*, vol. 5, no. 1, pp. 53–57, 1974.
[8] V. Powers and B. Reznick, “A new bound for pólya’s theorem with applications to polynomials positive on polyhedra,” *Journal of Pure and Applied Algebra*, vol. 164, no. 1, pp. 221–229, 2001.
[9] F. Boudaoud, F. Caruso, and M.-F. Roy, “Certificates of positivity in the Bernstein basis,” *Discrete & Computational Geometry*, vol. 39, no. 4, pp. 639–655, 2008.

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[10] R. Leroy, *Certificats de positivité et minimisation polynomiale dans la base de Bernstein multivariée*. PhD thesis, Université Rennes 1, France, 2008.
[11] E. De Klerk, D. Den Hertog, and G. Elabwabi, "On the complexity of optimization over the standard simplex," *European Journal of Operational Research*, vol. 191, no. 3, pp. 773–785, 2008.
[12] J. Garloff, "Convergent bounds for the range of multivariate polynomials," in *Interval Mathematics 1985*, vol. 347, pp. 37–56, Springer, Berlin, 1986.
[13] T. Hamadneh, H. Al-Zoubi, and S. A. Alomari, "Fast computation of polynomial data points over simplicial face values," *Journal of Information & Knowledge Management*, vol. 19, p. 204, 2020.
[14] R. Leroy, "Convergence under subdivision and complexity of polynomial minimization in the simplicial Bernstein basis," *Reliable Computing*, vol. 17, pp. 11–21, Springer, 2012.
[15] J. Garloff and T. Hamadneh, "Convergence and inclusion isotonicity of the tensorial rational Bernstein form," in *Scientific Computing, Computer Arithmetic, and Validated Numerics*, vol. 9553, pp. 171–179, Springer, Cham, 2015.
[16] T. Hamadneh, M. Ali, and H. Al-Zoubi, "Linear optimization of polynomial rational functions: Applications for positivity analysis," *Mathematics*, vol. 8 (2), p. 283, 2020.
[17] A. Narkawicz, J. Garloff, A. P. Smith, and C. A. Munoz, "Bounding the range of a rational functiom over a box," *Reliable Computing*, vol. 17(2012), pp. 34–39, 2012.
[18] J. Titi, T. Hamadneh, and J. Garloff, "Convergence of the simplicial rational Bernstein form," in *Modelling, Computation and Optimization in Information Systems and Management Sciences*, vol. 359, pp. 433–441, Springer, Cham, 2015.
[19] J. M. Lane and R. F. Riesenfeld, "Bounds on a polynomial," *BIT Numerical Mathematics*, vol. 21, no. 1, pp. 112–117, 1981.
[20] S. Foufou and D. Michelucci, "The Bernstein basis and its applications in solving geometric constraint systems.," *Reliable Computing*, vol. 17, no. 2, pp. 192–208, 2012.
[21] T. Hamadneh and R. Wisniewski, "Algorithm for Bernstein polynomial control design," in *6th IFAC Conference on Analysis and Design of Hybrid Systems ADHS 2018 Oxford*, vol. 51, pp. 283–289, Elsevier, 2018.
[22] T. Hamadneh and R. Wisniewski, "The Barycentric Bernstein form for control design," in *2018 Annual American Control Conference (ACC)*, pp. 3738–3743, IEEE, 2018.
[23] R. T. Farouki and V. Rajan, "Algorithms for polynomials in Bernstein form," *Computer Aided Geometric Design*, vol. 5, no. 1, pp. 1–26, 1988.
[24] V. Stahl, *Interval methods for bounding the range of polynomials and solving systems of nonlinear equations*. PhD thesis, Johannes Kepler Universität, Linz, 1995.
[25] H. C. Fischer, "Range computation and applications," *Contributions to Computer Arithmetic and Self-Validating Numerical Methods*, vol. 1, pp. 197–211, 1990.
[26] V. Powers and B. Reznick, "Polynomials positive on unbounded rectangles," in *Positive polynomials in control*, pp. 151–163, Springer, 2005.

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