Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/090205

Semilinear Abstract Cauchy Problem of Conformable Type

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Received: 2 Jun. 2021, Revised: 18 Sep. 2021, Accepted: 23 Sep. 2021
Published online: 1 Apr. 2023

Abstract: In this paper, we discuss the solution of the semilinear conformable abstract Cauchy problem. We study the existence of mild solution using the fractional semigroups. We establish the global and local mild solutions using the concept of contraction principle. The stability and regularity of mild solutions are studied. Applications illustrating our main abstract results are also given.

Keywords: $\alpha-$semigroup of operators, conformable derivative, semilinear Cauchy problem.

1 Introduction

Fractional differential equations have been proved to be one of the most effective tools in the modeling of many phenomena in various fields of physics, mechanics, chemistry, engineering, etc. They have a great number of applications in nonlinear oscillation of earthquakes, many physical phenomena such as seepage flow in porous media and in the fluid dynamic traffic model. For more details on this theory and its applications we refer to [1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20].

Recently, Khalil et al. in [12] introduced a new differential operator, called the conformable derivative. Such a fractional order derivative satisfies many well-known properties of the integer order derivative including linearity, product rule and division rule. In addition, Rolle's Theorem and Mean Value Theorem are also applicable, see [12]. In 2015, Abdeljawad in [1] made extensive results for the conformable fractional derivative. Khalil et al. in [21] presented a geometric meaning of the conformable derivative via fractional cords. Based on the conformable fractional derivative, Ableljawad et al. in [22] introduced the so called $C_0-\alpha-$semigroup $(T_\alpha(t))_{t\geq 0}$ which is a generalization of the classical strongly continuous semigroup with its infinitesimal generator. In the last few years many research articles, using conformable derivative, were published, see [1,6,23,24,25,26,27,28,29,30,31,32,33,34,35,36,37,38,39,40,41,42].

In this paper, we will study the following semilinear conformable fractional initial value problem:

$$\begin{cases} \frac{d^\alpha u(t)}{dt^\alpha}+Au(t)=f(t,u(t)) \quad t>t_0 \\ u(t_0)=u_0, \end{cases} \tag{1}$$

where $\frac{d^\alpha u(t)}{dt^\alpha}$ is the conformable fractional derivative of order $\alpha\in(0,1]$, A is a closed operator on a Banach space $(X,\|.\|)$, $u_0\in X$, and $f:[t_0,S]\times X\longrightarrow X$ is a function that satisfies some conditions. We denote by $(C([t_0,S]:X),\|.\|_\infty)$ the Banach space of continuous functions from $[t_0,S]$ into X with the norm $\|g\|_\infty=\sup_{t\in[t_0,S]}\|g(t)\|$.

This paper is organized as follows, in section 2 we introduce an important analysis of conformable fractional calculus and fractional semigroups. In section 3, we study the existence, uniqueness, stability and regularity of the mild solution of problem (1). In section 4 we give applications illustrating our abstract results.

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2 Preliminaries

In this section we introduce an important analysis of conformable calculus and conformable semigroups. Let us begin by defining the Banach space $L^p_\alpha(0,S;X)$. Let $p\geq 1$, we define the Banach space $L^p_\alpha(0,S;X)$ by

$$L^p_\alpha(0,S;X)=\{f:[0,S]\rightarrow X \text{ is measurable X valued function and } \int_0^S \frac{1}{t^{1-\alpha}}\|f(t)\|^p dt<\infty\},$$

under the norm $\|f\|_p=(\int_0^S \frac{1}{t^{1-\alpha}}\|f(t)\|^p dt)^{\frac{1}{p}}$.

Definition 1. *Let $u:[0,\infty)\longrightarrow X$ be an X valued function. The conformable derivative of u of order $\alpha\in(0,1]$, at $t>0$ is defined by*

$$\frac{d^\alpha u(t)}{dt^\alpha}=\lim_{\varepsilon\rightarrow 0}\frac{u(t+\varepsilon t^{1-\alpha})-u(t)}{\varepsilon},$$

where the limit is taken in the norm of X.
When the limit exists, we say that u is $\alpha-$differentiable at t.
If u is $\alpha-$differentiable in some $(0,a]$, $a>0$ and $\lim_{t\rightarrow 0^+}u^{(\alpha)}(t)$ exists in X, then we define $u^{(\alpha)}(0)=\lim_{t\rightarrow 0^+}u^{(\alpha)}(t)$.
The $\alpha-$fractional integral of a function $u\in L^1_\alpha(0,S;X)$ is given by

$$I^a_\alpha(u)(t)=\int_a^t \frac{1}{s^{1-\alpha}}u(s)\,ds$$

Theorem 1. *If a function $u:[0,\infty)\longrightarrow X$ is $\alpha-$differentiable at $t>0$, $\alpha\in(0,1]$, then u is continuous at t. If, in addition, u is differentiable, then $\frac{d^\alpha u(t)}{dt^\alpha}=t^{1-\alpha}\frac{du(t)}{dt}$.*

Lemma 1. *Let $u:[0,\infty)\longrightarrow X$ be differentiable and $\alpha\in(0,1]$. Then, for all $t>0$ we have*

$$I^0_\alpha(\frac{d^\alpha u}{dt^\alpha})(t)=u(t)-u(0).$$

Therefore, if u is continuous then

$$\frac{d^\alpha I^0_\alpha(u)(t)}{dt^\alpha}=u(t).$$

Definition 2. *(see [22]) Let $\alpha\in(0,a]$ for any $a>0$. For a Banach space X, a family $\{T_\alpha(t)\}_{t\geq 0}\subseteq\mathscr{L}(X,X)$ is called a fractional $\alpha-$semigroup (or $\alpha-$semigroup) of operators if*

$(i)T_\alpha(0)=I$,
$(ii)T_\alpha(t+s)^{\frac{1}{\alpha}}=T_\alpha(t^{\frac{1}{\alpha}})T_\alpha(s^{\frac{1}{\alpha}})$ for all $t,s\in[0,\infty)$.

Clearly, if $\alpha=1$, then $1-$semigroups are just the usual semigroups.

Definition 3. *(see [22]) An $\alpha-$semigroup $T_\alpha(t)$ is called a C_0-semigroup, if for each $x\in X$, $T_\alpha(t)x\rightarrow x$ as $t\rightarrow 0^+$.*
The conformable $\alpha-$derivative of $T_\alpha(t)$ at $t=0$ is called the $\alpha-$infinitesimal generator of the fractional $\alpha-$semigroup $T_\alpha(t)$, with domain equals:

$$\{\,x\in X,\ \lim_{t\rightarrow 0^+}(T_\alpha)^{(\alpha)}(t)x \text{ exists }\}.$$

We will write A for such a generator.

Example 1. (see [22])

(i)For a bounded linear operator A, define $T_{\frac{1}{2}}(t)=e^{2\sqrt{t}A}$. Then $(T_{\frac{1}{2}}(t))_{t\geq 0}$ is $\frac{1}{2}-$semigroup.
(ii)Let $X=C([0,\infty):\mathbb{R})$ be the Banach space of bounded uniformly continuous functions on $[0,\infty)$ with supremum norm. For $f\in X$ we define $(T_{\frac{1}{2}}(t)f)(s)=f(s+\sqrt{t})$. It is easy to check that $T_{\frac{1}{2}}(t)$ is a $C_0-\frac{1}{2}-$semigroup of operators.

Theorem 2. *(see [43]) Let $T_\alpha(t)$ be a $C_0-\alpha-$semigroup where $\alpha\in(0,1]$. There exist constants $\omega\geq 0$ and $M\geq 1$ such that*

$$\|T_\alpha(t)\|\leq Me^{\omega t^\alpha}\quad \text{for}\quad 0\leq t\leq\infty.$$

Corollary 1.*(see [43]) If $T_\alpha(t)$ is a $C_0-\alpha-$semigroup, then for every $x \in X$, $t \to T_\alpha(t)x$ is a continuous function from $\mathbb{R}_0^+$ (the nonnegative real line) into X.*

Theorem 3.*(see [43]) Let $T_\alpha(t)$ be a $C_0-\alpha-$semigroup where $\alpha \in (0,1]$ and A be its $\alpha-$infinitesimal generator. Then*

a)For $x \in X$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon t^{1-\alpha}} \frac{1}{s^{1-\alpha}} T_\alpha(s)x\,ds = T_\alpha(t)x \quad \text{for every} \quad t > 0.$$

b)For $x \in X$, $\int_0^t \frac{1}{s^{1-\alpha}} T_\alpha(s)x\,ds \in D(A)$ and

$$A\left(\int_0^t \frac{1}{s^{1-\alpha}} T_\alpha(s)x\,ds\right) = T_\alpha(t)x - x.$$

c)For $x \in D(A)$, $T_\alpha(t)x \in D(A)$ and

$$\frac{d^\alpha}{dt^\alpha} T_\alpha(t)x = AT_\alpha(t)x = T_\alpha(t)Ax \tag{2}$$

d)For $x \in D(A)$

$$T_\alpha(t)x - T_\alpha(s)x = \int_s^t \frac{1}{u^{1-\alpha}} T_\alpha(u)Ax\,du = \int_s^t \frac{1}{u^{1-\alpha}} AT_\alpha(u)x\,du$$

Lemma 2.*(see [43]) Let A be the $\alpha-$infinitesimal generator of a $C_0-\alpha-$semigroup $T_\alpha(t)$. Then A is closed and densely defined linear operator.*

Proposition 1.*(Gronwall's inequality)(see [1])*
Let r be a continuous, nonnegative function on $[t_0,S]$ and δ, k be nonnegative constants such that

$$r(t) \le \delta + k \int_{t_0}^t \frac{1}{s^{1-\alpha}} r(s)\,ds.$$

Then for all $t \in [t_0,S]$

$$r(t) \le \delta e^{\frac{k}{\alpha}(t^\alpha - t_0^\alpha)}$$

3 Main results

In this section we prove existence and uniqueness of mild solution of the abstract Cauchy problem (1) under certain conditions. Before presenting our main results, we introduce the following assumptions

(H_1)The function $f(t,.) : X \to X$ is uniformly Lipschitz continuous on X if there exist $L > 0$ such that

$$\|f(t,y) - f(t,z)\| \le L\|y - z\|$$

for all $t \in [t_0,S]$ and all $y,z \in X$.

(H_2)The function $f(.,y) : [t_0,S] \to X$ is continuous for all $y \in X$

3.1 Existence And Uniqueness Of The Mild Solution

If u is the solution of (1), then the X valued function $g(s) = T_\alpha(t^\alpha - s^\alpha)^{\frac{1}{\alpha}} u(s)$ is $\alpha-$deffrientiable for $t_0 < s < t$ and

$$\begin{aligned}\frac{d^\alpha g(s)}{ds^\alpha} &= AT_\alpha(t^\alpha - s^\alpha)^{\frac{1}{\alpha}} u(s) + T_\alpha(t^\alpha - s^\alpha)^{\frac{1}{\alpha}} \frac{d^\alpha u(s)}{ds^\alpha} \\ &= AT_\alpha(t^\alpha - s^\alpha)^{\frac{1}{\alpha}} u(s) + T_\alpha(t^\alpha - s^\alpha)^{\frac{1}{\alpha}} [-Au(s) + f(s,u(s))] \\ &= T_\alpha(t^\alpha - s^\alpha)^{\frac{1}{\alpha}} f(s,u(s)).\end{aligned} \tag{3}$$

From assumptions, (H_1) and (H_2) on f, the X valued function $s \to T(t^\alpha - s^\alpha)^{\frac{1}{\alpha}} f(s,u(s)) \in L^1_\alpha(0,S;X)$ and

$$I_\alpha^{t_0}\left(\frac{d^\alpha g(s)}{ds^\alpha}\right)(t) = T_\alpha(t^\alpha - t^\alpha)^{\frac{1}{\alpha}} u(t) - T_\alpha(t^\alpha - t_0^\alpha)^{\frac{1}{\alpha}} u(t_0) = \int_{t_0}^t \frac{1}{s^{1-\alpha}} T_\alpha(t^\alpha - s^\alpha)^{\frac{1}{\alpha}} f(s,u(s))\,ds.$$

So

$$u(t) = T_\alpha(t^\alpha - t_0^\alpha)^{\frac{1}{\alpha}} u_0 + \int_{t_0}^{t} \frac{1}{s^{1-\alpha}} T_\alpha(t^\alpha - s^\alpha)^{\frac{1}{\alpha}} f(s,u(s))\,ds. \tag{4}$$

Definition 4. *Let $T_\alpha(t)$ be a $C_0-\alpha-$semigroup with its generator $-A$. If $u_0 \in X$, then the function $u \in C([t_0,S]:X)$ satisfying the integral equation (4) is called the mild solution of the initial value problem (1).*

Theorem 4. *Let $-A$ be the generators of a $C_0-\alpha-$semigroup $T_\alpha(t)$, $t \geq 0$ on X. If assumptions, (H_1) and (H_2) hold, then for every $u_0 \in X$ the initial value problem (1) has a unique mild solution $u \in C([t_0,S]:X)$.*

Proof. We define the mapping $F : C([t_0,S]:X) \to C([t_0,S]:X)$ by

$$(Fu)(t) = T_\alpha(t^\alpha - t_0^\alpha)^{\frac{1}{\alpha}} u_0 + \int_{t_0}^{t} \frac{1}{s^{1-\alpha}} T_\alpha(t^\alpha - s^\alpha)^{\frac{1}{\alpha}} f(s,u(s))\,ds. \tag{5}$$

where $t_0 \leq t \leq S$.

Claim: F has a unique fixed point.
Let $u,v \in C([t_0,S]:X)$. We have

$$\begin{aligned}\|(Fu)(t)-(Fv)(t)\| &\leq \int_{t_0}^{t} \frac{1}{s^{1-\alpha}} T_\alpha(t^\alpha - s^\alpha)^{\frac{1}{\alpha}} \|f(s,u(s)) - f(s,v(s))\|\,ds \\ &\leq ML\|u-v\|_\infty \int_{t_0}^{t} s^{\alpha-1}\,ds \\ &\leq \frac{ML}{\alpha}(t^\alpha - t_0^\alpha)\|u-v\|_\infty, \end{aligned} \tag{6}$$

where M is a bound of $T(t)$ on $[t_0,S]$. Inductively by using (5) and (6) it follows that

$$\|(F^n u)(t) - (F^n v)(t)\| \leq \frac{(ML(t^\alpha - t_0^\alpha))^n}{\alpha^n n!}\|u-v\|_\infty$$

where $(F^n \equiv F \circ F \circ \ldots \circ F)$. Hence

$$\|F^n u - F^n v\|_\infty \leq \frac{(MLS^\alpha)^n}{\alpha^n n!}\|u-v\|_\infty.$$

For n large enough $\frac{(MLS^\alpha)^n}{\alpha^n n!} < 1$. Hence by using the Banach contraction principle, F has a unique fixed point $u \in C([t_0,S]:X)$, which is the mild solution of the conformable initial value problem (1).

The uniqueness and the continuous dependence of the mild solution are consequences of the following result.

Theorem 5. *(Stability of solution)*
Assume that (H_1) and (H_2) are satisfied. Let $u_0, v_0 \in X$ and denote by u and v the mild solutions of (1) associated to the initial conditions u_0 and v_0, respectively. Then, we have the following estimate

$$\|u-v\|_\infty \leq M e^{\frac{ML}{\alpha}(S^\alpha - t_0^\alpha)}\|u_0 - v_0\|$$

Proof. For $t \in [t_0,S]$ we have

$$\begin{aligned}\|u(t)-v(t)\| &\leq \|T_\alpha(t^\alpha - t_0^\alpha)^{\frac{1}{\alpha}} u_0 - T_\alpha(t^\alpha - t_0^\alpha)^{\frac{1}{\alpha}} v_0\| \\ &\quad + \int_{t_0}^{t} \frac{1}{s^{1-\alpha}} \|T_\alpha(t^\alpha - s^\alpha)^{\frac{1}{\alpha}}\| \|f(s,u(s)) - f(s,v(s))\|\,ds \\ &\leq M\|u_0 - v_0\| + ML\int_{t_0}^{t} \frac{1}{s^{1-\alpha}}\|u(s)-v(s)\|\,ds. \end{aligned}$$

Using Gronwall's inequality in proposition 1 it follows that

$$\|u(t)-v(t)\| \leq M e^{\frac{ML}{\alpha}(S^\alpha - t_0^\alpha)}\|u_0 - v_0\|.$$

Thus

$$\|u-v\|_\infty \leq M e^{\frac{ML}{\alpha}(S^\alpha - t_0^\alpha)}\|u_0 - v_0\|.$$

This yields both uniqueness of u and Lipschitz continuous of the map $u_0 \to u$.

From the proof of Theorem 4 we obtain a more general result:

Corollary 2.*If A and f satisfy the conditions of Theorem 4, then for all* $g \in C([t_0,S];X)$ *the integral equation*

$$v(t) = g(t) + \int_{t_0}^{t} \frac{1}{s^{1-\alpha}} T_\alpha (t^\alpha - s^\alpha)^{\frac{1}{\alpha}} f(s,v(s))\,ds$$

has a unique solution $v \in C([t_0,S];X)$.

From Theorem 4, the uniform Lipschitz condition of the function f assures the existence of a global mild solution of (1).

Now, let us assume that $f : [0,\infty[\times X \longrightarrow X$ satisfies only the following local Lipschitz condition:

$(H_3)$for every $\tau \geq 0$ and constant $k \geq 0$ there is a constant $L(k,\tau)$ such that for all $y,z \in X$ with $\|y\| \leq \delta$, $\|z\| \leq \delta$ and $t \in [0,\tau]$

$$\|f(t,y) - f(t,z)\| \leq L(k,\tau)\|y - z\|. \tag{7}$$

Under this condition we have the following local version of Theorem 4.

Theorem 6.*Let* $-A$ *be the generator of a* $C_0-\alpha-$*semigroup* $T_\alpha(t)$ *on* X. *If assumptions* (H_2) *and* (H_3) *hold, then for every* $u_0 \in X$ *there is a* $t_{max} \leq \infty$ *such that the initial value problem*

$$\begin{cases} \frac{d^\alpha u(t)}{dt^\alpha} + Au(t) = f(t,u(t)), & t \geq 0, \\ u(0) = u_0 \end{cases} \tag{8}$$

has a unique mild solution u on $[0,t_{max}[$. *Moreover, if* $t_{max} < \infty$, *then* $\lim_{t \to t_{max}} \|u(t)\| = \infty$.

*Proof.*First, we want to show that for every $t_0 \geq 0$, $u_0 \in X$ the conformable initial value problem (1) has under our assumptions, a unique mild solution u on interval $[t_0,t_1]$ such that $t_1 = (t_0^\alpha + \delta(t_0,\|u_0\|))^{\frac{1}{\alpha}}$ by choosing

$$\delta(t_0,\|u_0\|) = \min\{\alpha, \frac{\alpha\|u_0\|}{K(t_0)L(K(t_0),(t_0^\alpha + \alpha)^{\frac{1}{\alpha}}) + N(t_0)}\} \tag{9}$$

where $L(k,t)$ is the local Lipschitz constant of f as defined by (7), $M(t_0) = \sup\{\|T_\alpha(t)\| : 0 \leq t \leq (t_0^\alpha + \alpha)^{\frac{1}{\alpha}}\}$, $K(t_0) = 2\|u_0\|M(t_0)$ and $N(t_0) = \max\{\|f(t,0)\| : 0 \leq t \leq (t_0^\alpha + \alpha)^{\frac{1}{\alpha}}\}$. Indeed, we define the ball $B(K(t_0)) = \{u \in C([t_0,t_1] : X) : \|u\| \leq K(t_0)\}$.

Claim: The mapping F defined by (5) maps the ball $B(K(t_0))$ into itself.

For $t \in [t_0,t_1]$ we have the following estimate

$$\begin{aligned} \|(Fu)(t)\| &\leq \|T_\alpha(t^\alpha - t_0^\alpha)^{\frac{1}{\alpha}} u_0\| + \int_{t_0}^{t} \frac{1}{s^{1-\alpha}} \|T_\alpha(t^\alpha - s^\alpha)^{\frac{1}{\alpha}}\| (\|f(s,u(s)) - f(s,0)\| + \|f(s,0)\|)\,ds \\ &\leq M(t_0)\|u_0\| + (M(t_0)K(t_0)L(K(t_0),(t_0^\alpha + \alpha)^{\frac{1}{\alpha}}) + M(t_0)N(t_0)) \int_{t_0}^{t} \frac{1}{s^{1-\alpha}}\,ds \\ &\leq M(t_0)[\|u_0\| + (K(t_0)L(K(t_0),(t_0^\alpha + \alpha)^{\frac{1}{\alpha}}) + N(t_0))(\frac{t^\alpha}{\alpha} - \frac{t_0^\alpha}{\alpha})] \end{aligned}$$

From the definition of t_1 we get

$$\|(Fu)(t)\| \leq 2M(t_0)\|u_0\| = K(t_0).$$

In this ball, assumptions (H_1) and (H_2) hold, with the constant $L(K(t_0),(t_0^\alpha + \alpha)^{\frac{1}{\alpha}})$. By the same argument in proof of Theorem 4, F has a unique fixed point in this ball which is the mild solution of (1) on the interval $[t_0,t_1]$.

Now, if u is a mild solution of (8) on the interval $[0,\tau]$, then it can be extended to the interval $[0,(\tau^\alpha + \delta)^{\frac{1}{\alpha}}]$ with $\delta > 0$ by defining $u(t) = v(t)$ on $[\tau,(\tau^\alpha + \delta)^{\frac{1}{\alpha}}]$, where $v(t)$ is the solution of the integral equation

$$v(t) = T_\alpha(t^\alpha - t_0^\alpha)^{\frac{1}{\alpha}} u(\tau) + \int_{t_0}^{t} \frac{1}{s^{1-\alpha}} T_\alpha(t^\alpha - s^\alpha)^{\frac{1}{\alpha}} f(s,v(s))\,ds, \quad \tau \leq t \leq (\tau^\alpha + \delta)^{\frac{1}{\alpha}}.$$

Moreover, δ depends only on $\|u(\tau)\|$, $K(\tau)$ and $N(\tau)$.

Let $[0,t_{max}[$ be the interval of existence of the mild solution u of (8). Assume that $\lim_{t \to t_{max}} \|u(t)\| \neq \infty$. Then there is a sequence $t_n \nearrow t_{max}$ such that for all n, $\|u(t_n)\| < C$. If we choose t_n close to t_{max} then the solution u defined on $[0,t_n]$ can

be extended to $[0,(t_n^{\alpha}+\delta)^{\frac{1}{\alpha}}]$ where $\delta>0$ is independent of t_n. Hence u can be extended beyond t_{max} which contradicts the definition of t_{max}. So $\lim_{t\to t_{max}}\|u(t)\|=\infty$.
For the uniqueness of the local mild solution of (8), if u and v are two mild solutions of (8), then on every closed interval $[0,t_0]$ where both u and v exist, they coincide by the uniqueness argument given in the proof of Theorem 5. Therefore, both u and v have the same t_{max} and $u\equiv v$ on $[0,t_{max}[$.

Let $-A$ be the infinitesimal generator of a $C-\alpha-$semigroup $T_\alpha(t)$ on X. For all $x\in D(A)$ we define $|x|_A=\|x\|+\|Ax\|$. It is not difficult to show that $D(A)$ with the norm $|.|_A$ is a Banach space which we denote by Y. Clearly $Y\subset X$ and since $T_\alpha(t):D(A)\to D(A)$, $T_\alpha(t)$, $t\geq 0$ is an $\alpha-$semigroup on Y which is easily seen to be a $C_0-\alpha-$semigroup on Y.
In general, if f satisfies the condition in Theorem 4 or Theorem 6, the mild solution of (1) need not be continuously $\alpha-$differentiable. A sufficient condition for the mild solution of (1) to be continuously $\alpha-$differentiable solution is given in the next theorem.

Theorem 7.*Let $f:[t_0,S]\times Y\longrightarrow Y$ be uniformly Lipschitz in Y and for all $y\in Y$ the function $f(t,y)$ is continuous from $[t_0,S]$ into Y. Then for every $u_0\in D(A)$, the initial value problem (1) has a unique solution (continuously $\alpha-$differentiable) on $[t_0,S]$.*

*Proof.*From Theorem 4, there exists a unique mild solution $u\in C([t_0,S]:Y)$ satisfying the integral equation

$$u(t)=T_\alpha(t^\alpha-t_0^\alpha)^{\frac{1}{\alpha}}u_0+\int_{t_0}^{t}\frac{1}{s^{1-\alpha}}T_\alpha(t^\alpha-s^\alpha)^{\frac{1}{\alpha}}f(s,u(s))\,ds.$$

Let $g(s)=f(s,u(s))$. From our assumption it follows that $g(s)\in D(A)$ for $s\in[t_0,S]$. Therefore $g\in C([t_0,S]:Y)$. This implies that $s\to g(s)$ and $s\to Ag(s)$ are continuous on X. Then from Corollary 4.3 (see [43]), the initial value problem

$$\begin{cases}\frac{d^\alpha v}{dt}+Av=g(t)\\ v(t_0)=u_0\end{cases}\tag{10}$$

has a unique solution (continuously $\alpha-$defferentiable) v on $[t_0,S]$. This solution is clearly a mild solution of (10) and therefore

$$\begin{aligned}v(t)&=T_\alpha(t^\alpha-t_0^\alpha)^{\frac{1}{\alpha}}u_0+\int_{t_0}^{t}\frac{1}{s^{1-\alpha}}T_\alpha(t^\alpha-s^\alpha)^{\frac{1}{\alpha}}g(s)\,ds\\&=T_\alpha(t^\alpha-t_0^\alpha)^{\frac{1}{\alpha}}u_0+\int_{t_0}^{t}\frac{1}{s^{1-\alpha}}T_\alpha(t^\alpha-s^\alpha)^{\frac{1}{\alpha}}f(s,u(s))\,ds\\&=u(t).\end{aligned}$$

Thus u is a continuously $\alpha-$differentiable solution of (1) on $[t_0,S]$.

Corollary 3.*Let $f:[t_0,S]\times Y\longrightarrow Y$ be locally Lipschitz continuous in Y uniformly in $[t_0,S]$. Then for every $u_0\in D(A)$, the initial value problem (1) has a unique solution (continuously $\alpha-$differentiable) on a maximal interval $[t_0,t_{max}[$. If $t_{max}\leq S$ then*

$$\lim_{t\to t_{max}}(\|u(t)\|+\|Au(t)\|)=\infty.$$

In this situation, it may be that $\|u(t)\|$ is bounded on $[t_0,t_{max}[$ and only $\|Au(t)\|\to\infty$ as $t\to t_{max}$.

4 Applications

*Example 2.*Consider the semilinear conformable heat equation:

$$\begin{cases}\frac{\partial^\alpha v(t,x)}{\partial t^\alpha}=\frac{1}{2}\frac{\partial^2 v(t,x)}{\partial x2}+a(t)\cos(v(t,x)) \quad t>0 \quad -\infty<x<+\infty,\\ v(0,x)=g(x)\end{cases}\tag{11}$$

where $\alpha\in(0,1)$ and h is bounded continuous function on $[0,\infty)$, g is continuous function on $\mathbb{R}$.
Let $X=C[-\infty,+\infty]$ be the Banach space of bounded uniformly continuous functions with usual supremum norm $\|f\|_\infty=\sup_{-\infty<x<+\infty}\|f(x)\|$. Define the operator A by:

$$A=\frac{\partial^2}{\partial x^2}(.),\quad D(A)=\{\psi\in X:\psi',\psi''\in X\}.$$

One can easily show that A is a generator of $C_0 - \alpha-$semigroup which is just a modified heat semigroup defined by

$$(T_\alpha(t)f)(x) = \begin{cases} \int_{\mathbb{R}} N_t(x-y)f(y)\,dy, & t > 0 \\ f(x), & t = 0 \end{cases}$$

where $N_t(y) = \sqrt{\frac{\alpha}{2\pi t^\alpha}} e^{-\frac{\alpha y^2}{2t^\alpha}}$.
The $\alpha-$semigroup property follows from the fact that

$$\sqrt{\frac{\alpha}{2\pi(t+t')}} e^{-\frac{\alpha x^2}{2(t+t')}} = \sqrt{\frac{\alpha}{2\pi t}} \sqrt{\frac{\alpha}{2\pi t'}} \int_{\mathbb{R}} e^{-\frac{\alpha(x^2-y^2)}{2t} - \frac{\alpha y^2}{2t'}} dy$$

(using Fubini's theorem).From the strong continuity of the classical heat C_0-semigroup T_1 and the remark that $T_\alpha(t) = T_1(\frac{t^\alpha}{\alpha})$ we can show that T_α is a $C_0 - \alpha-$semigroup. Since A is generator of T_1 (in the sense of natural derivative) then it follows that for all $t > 0$:

$$\frac{d^\alpha}{dt^\alpha} T_\alpha(t) = \frac{d^\alpha}{dt^\alpha} T_1(\frac{t^\alpha}{\alpha}) = t^{1-\alpha} \frac{d}{dt} T_1(\frac{t^\alpha}{\alpha}) = t^{1-\alpha} t^{\alpha-1} \dot{T}_1(\frac{t^\alpha}{\alpha}) = \dot{T}_1(\frac{t^\alpha}{\alpha}).$$

So for $f \in D(A)$

$$\lim_{t\to 0} \frac{d^\alpha}{dt^\alpha} T_\alpha(t) f = \lim_{t\to 0} \dot{T}_1(\frac{t^\alpha}{\alpha}) = Af = \frac{\partial^2}{\partial x^2} f.$$

Let $u : [0,+\infty) \to X$ defined by $u(t)(x) = v(t,x)$ and $u(0)(x) = g(x)$. Then the semilinear conformable heat equation (11) takes the following abstract form :

$$\begin{cases} \frac{d^\alpha u(t)}{dt^\alpha} = Au(t) + a(t)\cos(u(t)) & t > t_0 \\ u(t_0) = g. \end{cases}$$

Let $g \in D(A)$. From Theorem 7 and since $f(t,u) = a(t)\cos(u)$ is uniformly Lipschitz for all $u \in D(A)$ and continuous on t, the above conformable initial value problem has a unique continuously $\alpha-$differentiable solution.

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