Applied Mathematics & Information Sciences
An International Journal

Stability and boundedness for a kind of non-autonomous differential equations with constant delay

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Received: 2 Jan. 2012; Revised 8 May 2012; Accepted 17 Jul. 2012
Published online: 1 Jan. 2013

Abstract: We establish some new sufficient conditions to the uniform asymptotically stability and boundedness of the solutions for a kind of non-autonomous differential equations of third order with constant delay. By defining an appropriate Lyapunov functional, we prove two new theorems on the subject. Our results improve and form a complement to some known recent results in the literature.

Keywords: Uniform asymptotically stability; boundedness; nonlinear differential equation; third order; delay.

1. Introduction

Differential equations of higher order have proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find some applications such as nonlinear oscillations (Afuwape et al. [7], Andres [9] and Fridedrichs [22]), electronic theory Rauch [31], biological model and other models Cronin-Scanlon [12] and Gopalsamy [23], and etc.. Just as above, in the past few decades, there has been much attention paid to the discussion of the qualitative behaviors of the solutions of various nonlinear differential equations of third order without and with delay. For a comprehensive treatment of the subject, we refer the readers to the book of Reissig et al. [32] as a survey and the papers of Ademola and Arawomo ([1], [2]), Ademola et al. [3], Afuwape [5], Afuwape and Castellanos [6], Antiova [10], Chukwu [11], Ezeilo ([13]-[19]), Ezeilo and Tejumola ([20], [21]), Hara [24], Mehri and Shadman [25], Ogundare [26], Ogundare and Okecha [27], Omeike ([28]-[30]), Rauch [31], Swick ([33]-[35]), Tejumola ([36], [37]), Tunç ([38]-[55]), Tunç and Ateş ([56], [57]), Tunç and Ergören [58], Tunç and Tunç [59], Yao and Meng [60], Wu and Shi [62] and the references cited in these sources.

On the other hand, by a recent paper published in 2010, Ademola et al. [4] proved two new results on the boundedness and the uniform ultimate boundedness of the solutions of a nonlinear third order differential equation without delay ,

$$\dddot{x}(t)+f(\ddot{x}(t))+g(\dot{x}(t))+h(x(t)) = p(t, x(t), \dot{x}(t), \ddot{x}(t)). \tag{1}$$

In this paper, instead of Eq. (1), we consider the non-autonomous and nonlinear third order delay differential equations of the form

$$\begin{aligned}&\dddot{x}(t) + f(t, x(t), x(t-r), \dot{x}(t), \dot{x}(t-r), \ddot{x}(t), \ddot{x}(t-r))\\ &+g(\dot{x}(t-r)) + h(x(t-r))\\ &= p(t, x(t), x(t-r), \dot{x}(t), \dot{x}(t-r), \ddot{x}(t), \ddot{x}(t-r)).\end{aligned} \tag{2}$$

We write Eq. (2) in system form as follows

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= z,\\ \dot{z} &= -f(t, x, x(t-r), y, y(t-r), z, z(t-r)) - g(y)\\ &\quad -h(x) + \int_{t-r}^{t} g'(y(s))z(s)ds + \int_{t-r}^{t} h'(x(s))y(s)ds\\ &\quad +p(t, x, x(t-r), y, y(t-r), z, z(t-r)),\end{aligned} \tag{3}$$

in which the dots denote differentiation with respect to t, $t \in \Re_+$, $\Re_+ = [0, \infty)$; the functions f, g, h and p are

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continuous in their respective arguments on $\Re_+ \times \Re^6$, $\Re$, $\Re$ and $\Re_+ \times \Re^6$, respectively, and $\Re = (-\infty, \infty)$. It is also assumed that the functions g and h are differentiable.

This paper is motivated by the papers of Ademola et al. [4] and that mentioned above. We define a new Lyapunov functional for the results to be established here. Then, using that Lyapunov functional, we discuss the uniform asymptotically stability and boundedness of the solutions of Eq. (2) for the cases, $p(.) \equiv 0$ and $p(.) \neq 0$, respectively. Obviously, the equation discussed in [4], Eq. (1), is a particular case of our equation, Eq. (2). Here, by this work, we improve the boundedness result obtained in [4] obtained for ordinary nonlinear differential Eq. (1) without delay to the nonlinear differential Eq. (2) with delay. It should be noted that Ademola et al. [4] discussed the boundedness of the solutions of Eq. (1). In addition to the boundedness of the solutions, we also discuss the uniform asymptotically stability of the solutions. Our results will be also different from that mentioned above. It should be noted that the basic reason to investigate these topics here is that functional differential equations play a key role in applied sciences. However, we only study the theorical aspects of the topics here.

2. Preliminaries

Consider the functional differential equation

$$\dot{x}(t) = F(t, x_t), x_t = x(t+\theta), -r \leq \theta \leq 0, t \geq 0, \quad (4)$$

with $F : \Re_+ \times C_H \to \Re^n$ being continuous, $F(t, 0) = 0$, and we suppose that F takes closed bounded sets into bounded sets of $\Re^n$. Here $(C, \ \|.\|)$ is the Banach space of continuous function $\varphi : [-r, \ 0] \to \Re^n$ with supremum norm, $r > 0$; C_H is the open H -ball in C; $C_H := \{\varphi \in C([-r, 0], \ \Re^n) : \|\varphi\| < H\}$.

Theorem 1. (Yoshizawa [61]). Suppose that there exists a continuous Lyapunov functional $V(t, \varphi)$ defined on $t \in \Re_+$, $\|\varphi\| < H_1$, $0 < H_1 < H$, which satisfies the following conditions;

(i) $a(\|\varphi\|) \leq V(t, \varphi) \leq b(\|\varphi\|)$, where $a(r) \in CIP$ and $b(r) \in CIP$ (CIP denotes the families of continuous increasing, positive definite functions),

(ii) $\dot{V}(t, \varphi) \leq -c(\|\varphi\|)$, where $c(r)$ is continuous and positive for $r > 0$.

Then, the zero solution of (4) is uniform-asymptotically stable.

3. Main results

Let $p(.) = 0$.

Our first result is given by the following theorem.

Theorem 2. Suppose that there exist positive constants $a, \ b, \ b_1, \ c, \ \delta_0$ and L such that the following conditions hold:

(i)

$$h(0) = 0, \delta_0 \leq \frac{h(x)}{x}, (x \neq 0), h'(x) \leq c,$$

$$g(0) = 0, b \leq \frac{g(y)}{y} \leq b_1, (y \neq 0), |g'(y)| \leq L,$$

(ii)

$$a \leq \frac{f(t, x, x(t-r), y, y(t-r), z, z(t-r))}{z}, (z \neq 0).$$

Then, the zero solution of Eq. (2) is uniform-asymptotically stable provided that

$$r < \min\left\{\frac{a\delta_0}{\alpha(c+L)}, \frac{7(\alpha a+ab-c)}{4(c\alpha+2ac+c+aL)}, \frac{\alpha}{L(\alpha+a+2)+c}\right\}.$$

The proof of Theorem 2 and that of the subsequent result depend on some certain fundamental properties of a continuously differentiable Lyapunov functional $V = V(x_t, y_t, z_t)$ defined by

$$2V = 2a\int_0^x h(\xi)d\xi + 2\int_0^y g(\tau)d\tau + 2yh(x) + \alpha bx^2$$
$$+(\alpha + a^2)y^2 + z^2 + 2\alpha axy + 2\alpha xz$$
$$+2ayz + 2\lambda_1\int_{-r}^0\int_{t+s}^t y^2(\theta)d\theta ds + 2\lambda_2\int_{-r}^0\int_{t+s}^t z^2(\theta)d\theta ds,$$

where α a is positive fixed constant satisfying

$$0 < \alpha < b - ca^{-1}, \quad (5)$$

and λ_1 and λ_2 are some positive constants which will be determined later in the proof.

Proof. We observe that the above Lyapunov functional can be rewritten as follows

$$2V = V_1 + V_2 + 2\lambda_1\int_{-r}^0\int_{t+s}^t y^2(\theta)d\theta ds + 2\lambda_2\int_{-r}^0\int_{t+s}^t z^2(\theta)d\theta ds, \quad (6)$$

where

$$V_1 = 2a\int_0^x h(\xi)d\xi + 2\int_0^y g(\tau)d\tau + 2yh(x)$$

and

$$V_2 = \alpha bx^2 + (\alpha + a^2)y^2 + z^2 + 2\alpha axy + 2\alpha xz + 2ayz.$$

In view of assumption (i) of Theorem 2, we have $g(y) \geq by$ for all $y \neq 0$. Hence

$$2\int_0^y g(\tau)d\tau + 2yh(x) \geq 2\int_0^y b\tau d\tau + 2yh(x)$$
$$= (by + h(x))^2 b^{-1} - b^{-1}h^2(x)$$
$$\geq -b^{-1}h^2(x).$$

Moreover, assumption (i) of Theorem 2 implies

$$2a\int_0^x h(\xi)d\xi = 2b^{-1}\int_0^x (ab - h'(\xi))h(\xi)d\xi + b^{-1}h^2(x)$$
$$\geq (ab - c)b^{-1}\delta_0 x^2 + b^{-1}h^2(x).$$

Combining the above estimates into V_1, we obtain

$$V_1 \geq (ab-c)b^{-1}\delta_0 x^2 = k_1 x^2, \ \ (k_1 = (ab-c)b^{-1}\delta_0 > 0).$$

We can also rearrange V_2 as follows

$$V_2 = XQ_0X^T,$$

where $X = (x, y, z)$, $Q_0 = \begin{pmatrix} \alpha b & \alpha a & \alpha \\ \alpha a & \alpha + a^2 & a \\ \alpha & a & 1 \end{pmatrix}$ and $\det Q_0 = \alpha^2(b - \alpha) > 0$ since $b - \alpha > 0$. Hence, we get

$$V_2 \geq \alpha^2(x^2 + y^2 + z^2).$$

Gathering the estimates for V_1 and V_2 into (6), it follows that

$$2V \geq (\alpha^2 + k_1)x^2 + \alpha^2(y^2 + z^2)$$
$$+2\lambda_1 \int_{-r}^0 \int_{t+s}^t y^2(\theta)d\theta ds + 2\lambda_2 \int_{-r}^0 \int_{t+s}^t z^2(\theta)d\theta ds$$
$$\geq k_2(x^2 + y^2 + z^2)$$
$$+2\lambda_1 \int_{-r}^0 \int_{t+s}^t y^2(\theta)d\theta ds + 2\lambda_2 \int_{-r}^0 \int_{t+s}^t z^2(\theta)d\theta ds,$$

where $k_2 = \alpha^2$. On the other hand, by the assumptions of Theorem 2 and the estimate $2|m|\ |n| \leq m^2 + n^2$, it can be easily obtained for a positive constant k_3 that

$$2V \leq k_3(x^2 + y^2 + z^2)$$
$$+2\lambda_1 \int_{-r}^0 \int_{t+s}^t y^2(\theta)d\theta ds + 2\lambda_2 \int_{-r}^0 \int_{t+s}^t z^2(\theta)d\theta ds,$$

so that

$$\bar{k}_2(x^2 + y^2 + z^2) + \lambda_1 \int_{-r}^0 \int_{t+s}^t y^2(\theta)d\theta ds$$
$$+\lambda_2 \int_{-r}^0 \int_{t+s}^t z^2(\theta)d\theta ds \leq V$$
$$\leq \bar{k}_3(x^2 + y^2 + z^2) + \lambda_1 \int_{-r}^0 \int_{t+s}^t y^2(\theta)d\theta ds$$
$$+\lambda_2 \int_{-r}^0 \int_{t+s}^t z^2(\theta)d\theta ds,$$

where $\bar{k}_2 = 2^{-1}k_2$ and $\bar{k}_3 = 2^{-1}k_3$.

Thus, subject to the above discussion, it can be shown that condition (i) of Theorem 1 holds.

Let $(x, y, z) = (x(t), y(t), z(t))$ be a solution of (3). Along this solution, it follows from (6) and (3) that

$$\frac{dV}{dt} = -\alpha x h(x) - \{ayg(y) - y^2h'(x)\} - \alpha\{g(y) - by\}x$$
$$-(\alpha x + ay + z)\{f(.) - az\}$$
$$+(\alpha x + ay + z)\int_{t-r}^t h'(x(s))y(s)ds$$
$$+(\alpha x + ay + z)\int_{t-r}^t g'(y(s))z(s)ds + \lambda_1 y^2 r + \lambda_2 z^2 r$$
$$-\lambda_1 \int_{t-r}^t y^2(s)ds - \lambda_2 \int_{t-r}^t z^2(s)ds + \alpha YQ_1Y^T,$$

where $Y = (y, z)$, $Q_1 = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$ and $\det Q_1 = -1$.

Making use of the assumptions of Theorem 2, we get

$$\frac{dV}{dt} \leq -\tfrac{1}{2}\alpha\delta_0 x^2 - \tfrac{7}{8}(\alpha a + ab - c)y^2 - \tfrac{1}{2}\alpha z^2$$
$$+(\alpha x + ay + z)\int_{t-r}^t h'(x(s))y(s)ds$$
$$+(\alpha x + ay + z)\int_{t-r}^t g'(y(s))z(s)ds + \lambda_1 y^2 r$$
$$+\lambda_2 z^2 r - \lambda_1 \int_{t-r}^t y^2(s)ds - \lambda_2 \int_{t-r}^t z^2(s)ds - W_i,$$

$(i = 1,\ 2,\ 3)$, where

$$W_1 = \alpha\{\frac{1}{4}\delta_0 x^2 + (g(y) - by)x + \frac{1}{16\alpha}(\alpha a + ab - c)y^2\},$$

$$W_2 = \alpha\{\frac{1}{4}\delta_0 x^2 + (f(.) - az)x + \frac{1}{4}z^2\},$$

$$W_3 = a\{\frac{1}{16a}(\alpha a + ab - c)y^2 + (f(.) - az)y + \frac{\alpha}{4a}z^2\}.$$

Using the estimates

$$\{g(y) - by\}^2 < \frac{\delta_0(\alpha a + ab - c)}{16\alpha}y^2,$$

$$\{f(.) - az\}^2 < \frac{\delta_0}{4\alpha^2}z^2,$$

$$\{f(.) - az\}^2 < \frac{\alpha(\alpha + ab - c)}{16a^2}z^2,$$

respectively, we conclude

$$W_1 \geq \frac{\alpha}{16}\left(2\sqrt{\delta_0}\,|x| - \sqrt{\frac{\alpha a + ab - c}{\alpha}}\,|y|\right)^2 \geq 0,$$

$$W_2 \geq \frac{\alpha}{4}(\sqrt{\delta_0}\,|x| - |z|)^2 \geq 0,$$

$$W_3 \geq \frac{a}{16}\left(\sqrt{\frac{\alpha a + ab - c}{a}}\,|y| - 2\sqrt{\frac{\alpha}{a}}\,|z|\right)^2 \geq 0.$$

By noting the assumptions $0 < h'(x) \leq c$, $|g'(y)| \leq L$ and the estimate $2\,|m|\;\;|n| \leq m^2 + n^2$, it is followed that

$$(\alpha x + ay + z)\int_{t-r}^{t} h'(x(s))y(s)ds \leq$$

$$\tfrac{r}{2}(\alpha c x^2 + acy^2 + cz^2) + \tfrac{c}{2}(\alpha + a + 1)\int_{t-r}^{t} y^2(s)ds,$$

$$(\alpha x + ay + z)\int_{t-r}^{t} g'(y(s))z(s)ds \leq$$

$$\tfrac{rL}{2}(\alpha x^2 + ay^2 + z^2) + \tfrac{L}{2}(\alpha + a + 1)\int_{t-r}^{t} z^2(s)ds.$$

In view of the above estimates, we get

$$\begin{aligned}\frac{dV}{dt} &\leq -2^{-1}\{\alpha\delta_0 - (\alpha c + \alpha L)r\}x^2\\ &-\{\tfrac{7}{8}(\alpha a + ab - c) - \tfrac{1}{2}(2\lambda_1 + ac + aL)r\}y^2\\ &-2^{-1}\{\alpha - (2\lambda_2 + c + L)r\}z^2\\ &-\{\lambda_1 - 2^{-1}c(\alpha + a + 1)\}\int_{t-r}^{t} y^2(s)ds\\ &-\{\lambda_2 - 2^{-1}L(\alpha + a + 1)\}\int_{t-r}^{t} z^2(s)ds.\end{aligned}$$

Let

$$\lambda_1 = \frac{1}{2}c(\alpha + a + 1)$$

and

$$\lambda_2 = \frac{1}{2}L(\alpha + a + 1).$$

Then, we get

$$\begin{aligned}\frac{dV}{dt} &\leq -\tfrac{1}{2}\{\alpha\delta_0 - (\alpha c + \alpha L)r\}x^2\\ &-\{\tfrac{7}{8}(\alpha a + ab - c) - \tfrac{1}{2}(c\alpha + 2ac + c + aL)r\}y^2\\ &-\tfrac{1}{2}\{\alpha - [L(\alpha + a + 2) + c]r\}z^2.\end{aligned}$$

Hence, we conclude

$$\frac{d}{dt}V(x_t, y_t, z_t) \leq -k_4x^2 - k_5y^2 - k_6z^2 \leq 0$$

for some positive constants k_4, k_5 and k_6 provided that

$$r < \min\left\{\tfrac{a\delta_0}{\alpha(c+L)},\ \tfrac{7(\alpha a + ab - c)}{4(c\alpha + 2ac + c + aL)},\ \tfrac{\alpha}{L(\alpha + a + 2) + c}\right\}.$$

This completes the proof of Theorem 2.

Let $p(.) \neq 0$.

Our second result is given by the following theorem.

Theorem 3. We assume that all the assumptions of Theorem 2 and

$$|p(t, x, x(t-r), y, y(t-r), z, z(t-r))| \leq |q(t)|,$$

$$\int_0^t |q(s)|\,ds \leq P_0 < \infty$$

hold, where P_0 is a positive constant.

Then, there exists a positive constant M such that the solution $x(t)$ of Eq. (2) defined by the initial function

$$x(t) = \phi(t), x'(t) = \phi'(t), x''(t) = \phi''(t)$$

satisfies

$$|x(t)| \leq M, |x'(t)| \leq M, |x''(t)| \leq M$$

for all $t \geq t_0 \geq 0$, where $\phi \in C^2([t_0 - r, t_0],\ \Re)$, provided that

$$r < \min\left\{\tfrac{a\delta_0}{\alpha(c+L)},\ \tfrac{7(\alpha a + ab - c)}{4(c\alpha + 2ac + c + aL)},\ \tfrac{\alpha}{L(\alpha + a + 2) + c}\right\}.$$

Proof. Let $(x(t), y(t), z(t))$ be any solution of (3). Then, by an easy calculation, it is obtained that

$$\begin{aligned}\frac{dV}{dt} &\leq -k_4x^2 - k_5y^2 - k_6z^2 + (\alpha x + ay + z)\\ &\times p(t, x, x(t-r), y, y(t-r), z, z(t-r))\\ &\leq \max(\alpha,\ a,\ 1)(|x| + |y| + |z|)\,|q(t)|\\ &\leq \delta_1(|x| + |y| + |z|)\,|q(t)|\\ &\leq \delta_1(3 + x^2 + y^2 + z^2)\,|q(t)|\,,\end{aligned}$$

where $\delta_1 \equiv \max(\alpha,\ a,\ 1)$. Making use of the estimate $x^2 + y^2 + z^2 \leq \bar{k}_2^{-1}V(x_t, y_t, z_t)$, we get

$$\frac{d}{dt}V(x_t, y_t, z_t) \leq \delta_2\,|q(t)| + \delta_2 V(x_t, y_t, z_t)\,|q(t)|\,,$$

where $\delta_2 = \max(3\delta_1,\ \delta_1\bar{k}_2^{-1})$. Integrating the above estimate from 0 to t, it follows that

$$V(x_t, y_t, z_t) \leq V(x_0, y_0, z_0) + \delta_2\int_0^t |q(s)|\,ds$$

$$+\delta_2\int_0^t V(x_s, y_s, z_s)\,|q(s)|\,ds.$$

Using the Gronwall-Reid-Bellman inequality, (see Ahmad and Rama Mohana Rao [8]), and the assumption

$$\int_0^t |q(s)|\,ds \leq P_0 < \infty,$$

we can conclude the result of Theorem 3.

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