Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/100214

Gharaibeh Distribution and its Applications

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Received: 10 Oct. 2020, Revised: 25 Nov. 2020, Accepted: 27 Nov. 2020
Published online: 1 Jul. 2021

Abstract: In this paper, we propose a new continuous one parameter distribution called Gharaibeh distribution. Several mathematical and statistical properties of the proposed distribution, such as the shapes of the distribution, moments, skewness, kurtosis, hazard rate function, mean residual life function, stochastic ordering, mean deviations, Bonferroni and Lorenz curves, order statistics, and Renyi entropy are investigated. The method of maximum likelihood estimation is suggested to estimate the parameter of the distribution. Applications to three real lifetime data sets indicated that the proposed distribution provides better fit and outperforms other existing distributions.

Keywords: Lifetime distribution, Moments, Maximum likelihood, Reliability, Renyi entropy, Order statistics, Bonferroni and Lorenz curves, Stochastic ordering, Mean deviations.

1 Introduction

Modeling lifetime data is an important issue that arises in various areas, such as engineering, economics, demography, biological studies, medical and environmental sciences. The exponential and Lindley distributions are popular distributions for modeling such data. However, these two classical distributions are inadequate for many lifetime data. Consequently, several distributions that can provide better fit to the available data have been introduced over the past years.

The idea of mixing distributions has been adopted by many authors to introduce new distributions. For example, [1] suggested Rama distribution as a two components mixture of exponential(θ) and Gamma($4,\theta$). Another two component mixture of the exponential(θ) and Gamma($3,\theta$) distributions called Ishita distribution was proposed by [2]. [3] suggested Aradhana distribution by mixing exponential(θ), Gamma($2,\theta$), and Gamma($3,\theta$) distributions with some mixing proportions. [4] introduced Akshaya distribution using a four component mixture of exponential(θ), Gamma($2,\theta$), Gamma($3,\theta$) and Gamma($4,\theta$) with mixing proportions $\frac{\theta^3}{\theta^3+3\theta^2+6\theta+6}$, $\frac{3\theta^2}{\theta^3+3\theta^2+6\theta+6}$, $\frac{6\theta}{\theta^3+3\theta^2+6\theta+6}$ and $\frac{6}{\theta^3+3\theta^2+6\theta+6}$, respectively.

Some other distributions that have been suggested in the literature include Akash distribution [5], Weibull distribution [6], gamma-normal distribution [7], transmuted Aradhana distribution [8], Rayleigh distribution [9], transmuted Ishita distribution [10], Shanker distribution [11], Burr XII modified Weibull distribution [12], Topp-Leone Mukherjee-Islam distribution [13], Darna distribution [14], power length-biased Suja distribution [15], transmuted Mukherjee-Islam distribution [16], Weibull Fréchet distribution [17], beta exponential Fréchet Distribution [18], among others.

In this research, we adopt the idea of mixing distributions to propose a new one parameter distribution. The proposed distribution (i.e. Gharaibeh distribution) shows its flexibility and superiority to fit some real lifetime data sets compared to some competing distributions.

The present paper is organized, as follows: In Section 2, we define the probability density function (pdf) and cumulative distribution function (cdf) of the Gharaibeh distribution. In Section 3, we address some mathematical properties, including the moment generating function, r^{th} moment, mean, variance, skewness, kurtosis, coefficient of variation. We discuss in Section 4 the reliability, hazard rate, cumulative hazard, reversed hazard, odds, and mean residual life functions of the Gharaibeh distribution. In Section 5, we explore the distributions of order statistics. We present Bonferroni and Lorenz Curves and Gini index in Section 6. Section 7 provides the stochastic ordering. The Renyi

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entropy is derived in Section 8. Mean deviations about mean and median are investigated in Section 9. Maximum likelihood estimate of the distribution parameter is given in Section 10. We present applications to three real lifetime data sets in Section 11. Conclusion is presented in Section 12.

2 Gharaibeh Distribution

In this section, we define the probability density function (pdf) and cumulative distribution function (cdf) of the suggested Gharaibeh distribution with an illustration of the distribution shapes.

Definition 1. *A random variable X is said to have a Gharaibeh distribution if its pdf is defined as*

$$f(x) = \frac{\beta^6}{120(\beta^6+\beta^4+\beta^2+1)}(x^5+20x^3+120x+120\beta)e^{-\beta x}; \qquad x>0; \beta>0. \tag{1}$$

This proposed distribution is a four component mixture of exponential(β), Gamma$(2,\beta)$, Gamma$(4,\beta)$ and Gamma$(6,\beta)$ with mixing proportions $\frac{\beta^6}{\beta^6+\beta^4+\beta^2+1}$, $\frac{\beta^4}{\beta^6+\beta^4+\beta^2+1}$, $\frac{\beta^2}{\beta^6+\beta^4+\beta^2+1}$ and $\frac{1}{\beta^6+\beta^4+\beta^2+1}$, respectively.

The corresponding cdf of (1) is given by

$$F(x) = 1-\left(\frac{\beta^5x^5+5\beta^4x^4+20(\beta^5+\beta^3)x^3+60(\beta^4+\beta^2)x^2+120(\beta^5+\beta^3+\beta)x}{120(\beta^6+\beta^4+\beta^2+1)}+1\right)e^{-\beta x}; x>0, \beta>0. \tag{2}$$

Figure 1 shows the pdf and cdf of the Gharaibeh distribution with different values of the distribution parameter. It can be seen that Gharaibeh distribution is right skewed.

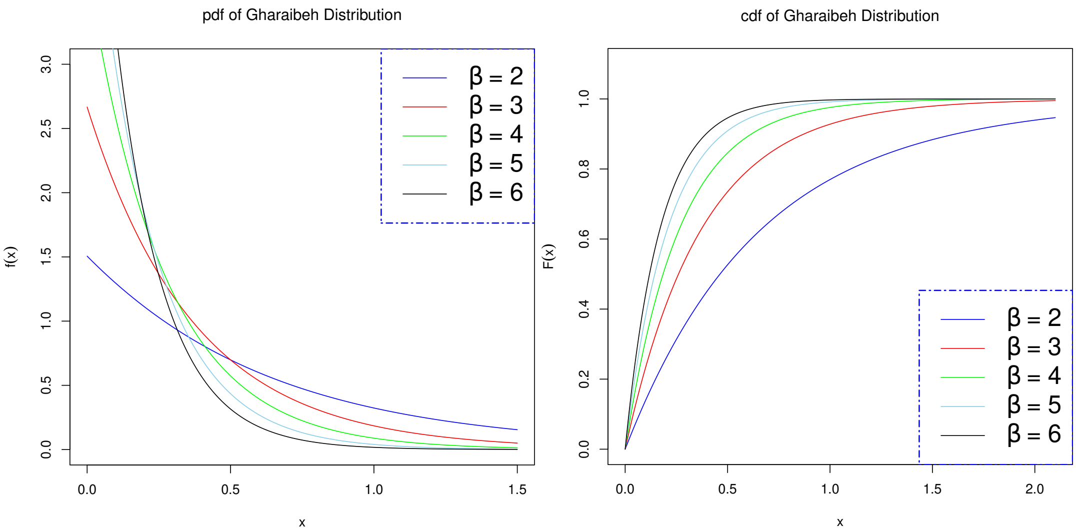

Fig. 1: The pdf and cdf of Gharaibeh distribution with different values of β

3 Moments and associated measures

In this section, the moment generating function and the k^{th} moment for the Gharaibeh distribution random variable are derived. Also, the mean, variance, coefficient of variation, kurtosis, and skewness are investigated.

Theorem 1. *The moment generating function of the Gharaibeh distribution random variable is given by*

$$M_X(t) = \sum_{j=0}^{\infty}\frac{(t/\beta)^j}{j!(\beta^6+\beta^4+\beta^2+1)}\left[\beta^6 j!+\beta^4(j+1)!+\frac{\beta^2(j+3)!}{6}+\frac{(j+5)!}{120}\right]. \tag{3}$$

*Proof.*Using $f(x)$ in (1) and the binomial series, the moment generating function of the Gharaibeh distribution can be proved as

$$
\begin{aligned}
M_X(t) &= E(e^{tX}) = \int_0^\infty e^{tx} f(x)dx \\
&= \frac{\beta^6}{120(\beta^6+\beta^4+\beta^2+1)} \int_0^\infty (x^5+20x^3+120x+120\beta)e^{-x(\beta-t)}dx \\
&= \frac{\beta^6}{120(\beta^6+\beta^4+\beta^2+1)} \left[\frac{120}{(\beta-t)^6}+\frac{120}{(\beta-t)^4}+\frac{120}{(\beta-t)^2}+\frac{120\beta}{(\beta-t)}\right] \\
&= \frac{1}{(\beta^6+\beta^4+\beta^2+1)} \left[\frac{\beta^6}{(1-\frac{t}{\beta})}+\frac{\beta^4}{(1-\frac{t}{\beta})^2}+\frac{\beta^2}{(1-\frac{t}{\beta})^4}+\frac{1}{(1-\frac{t}{\beta})^6}\right] \\
&= \frac{1}{(\beta^6+\beta^4+\beta^2+1)} \left[\beta^6 \sum_{j=0}^{\infty} \binom{j}{j}(t/\beta)^j + \beta^4 \sum_{j=0}^{\infty} \binom{j+1}{j}(t/\beta)^j \right. \\
&\qquad \left. +\beta^2 \sum_{j=0}^{\infty} \binom{j+3}{j}(t/\beta)^j + \sum_{j=0}^{\infty} \binom{j+5}{j}(t/\beta)^j\right] \\
&= \sum_{j=0}^{\infty} \frac{(t/\beta)^j}{j!(\beta^6+\beta^4+\beta^2+1)} \left[\beta^6 j! + \beta^4(j+1)! + \frac{\beta^2(j+3)!}{6} + \frac{(j+5)!}{120}\right].
\end{aligned}
$$

The k^{th} moment about the origin of the Garaibeh distribution random variable can be obtained as the coefficient of $\frac{t^k}{k!}$ in the moment generating function $M_X(t)$ in (3). That is

$$
E(X^k) = \frac{\beta^6 k! + \beta^4(k+1)! + \frac{\beta^2(k+3)!}{6} + \frac{(k+5)!}{120}}{\beta^k(\beta^6+\beta^4+\beta^2+1)}; \qquad k = 1,2,3,..... \tag{4}
$$

Using (4), the first four moments about the origin of the Gharaibeh distribution are

$$
\begin{aligned}
\mu = E(X) &= \frac{\beta^6+2\beta^4+4\beta^2+6}{\beta(\beta^6+\beta^4+\beta^2+1)}, \qquad E(X^2) = \frac{2\beta^6+6\beta^4+20\beta^2+42}{\beta^2(\beta^6+\beta^4+\beta^2+1)}, \\
E(X^3) &= \frac{6\beta^6+24\beta^4+120\beta^2+336}{\beta^3(\beta^6+\beta^4+\beta^2+1)}, \qquad E(X^4) = \frac{24\beta^6+120\beta^4+840\beta^2+3024}{\beta^4(\beta^6+\beta^4+\beta^2+1)}.
\end{aligned} \tag{5}
$$

Based on these moments, the variance, the coefficient of variation $(C.V)$, the skewness and the kurtosis of the Gharaibeh distribution random variable are , respectively, defined as:

$$
\begin{aligned}
\sigma^2 = Var(X) = E(X^2) - \mu^2 &= \frac{\beta^{12}+4\beta^{10}+16\beta^8+42\beta^6+28\beta^4+14\beta^2+6}{\beta^2(\beta^6+\beta^4+\beta^2+1)^2} \\
C.V &= \frac{\sigma}{\mu} = \frac{(\beta^{12}+4\beta^{10}+16\beta^8+42\beta^6+28\beta^4+14\beta^2+6)^{1/2}}{\beta^6+2\beta^4+4\beta^2+6}, \\
sk(X) &= \frac{E(X^3)-3\mu E(X^2)+2\mu^3}{\sigma^3} \\
&= \frac{2(\beta^{18}+6\beta^{16}+39\beta^{14}+155\beta^{12}+189\beta^{10}+129\beta^8+52\beta^6+30\beta^4+18\beta^2+6)}{(\beta^{12}+4\beta^{10}+16\beta^8+42\beta^6+28\beta^4+14\beta^2+6)^{3/2}}, \\
ku(X) &= \frac{E(X^4)-4\mu E(X^3)+6\mu^2 E(X^2)-3\mu^4}{\sigma^4} \\
&= \frac{\begin{bmatrix} 9\beta^{24}+72\beta^{22}+600\beta^{20}+3156\beta^{18}+6384\beta^{16}+8580\beta^{14} \\ +9180\beta^{12}+8352\beta^{10}+6792\beta^8+4512\beta^6+1968\beta^4+648\beta^2+144 \end{bmatrix}}{(\beta^{12}+4\beta^{10}+16\beta^8+42\beta^6+28\beta^4+14\beta^2+6)^2}.
\end{aligned}
$$

Table 1 provides the mean, variance, coefficient of variation, skewness, and kurtosis of the Gharaibeh distribution random variable with variant values of β.

Table 1 indicates that the values of mean and variance decrease as the value of the parameter β increases. Also, the positive values of skewness indicate that the Gharaibeh distribution is skewed to the right as shown in Figure 1.

Table 1: The mean, variance, coefficient of variation, skewness, and kurtosis of the Gharaibeh distribution with different values of the parameter β.

β	***Mean*** (μ)	***Variance*** (σ^2)	***C.V***	$sk(X)$	$ku(X)$
0.1	59.7979	602.0707	0.4103	0.8121	3.9932
0.5	10.7529	27.1515	0.4845	0.6717	3.7465
0.8	5.2132	11.8770	0.6610	0.7564	3.5367
1	3.2500	6.9375	0.8104	1.0688	4.0903
2	0.6941	0.5358	1.0546	2.2417	10.4944
3	0.3792	0.1496	1.0200	2.1528	10.2931
4	0.2676	0.0727	1.0075	2.0604	9.5414
5	0.2086	0.0437	1.0031	2.0215	9.2426
10	0.1010	0.0102	0.9999	1.9217	9.3446

4 Reliability analysis

The reliability or survival function, $R(x)$, is the probability of an item that will survive after a time x. Using (2), the reliability function of the Gharaibeh distribution is defined as

$$R(x)=1-F(x)=\left(\frac{\beta^5x^5+5\beta^4x^4+20(\beta^5+\beta^3)x^3+60(\beta^4+\beta^2)x^2+120(\beta^5+\beta^3+\beta)x}{120(\beta^6+\beta^4+\beta^2+1)}+1\right)e^{-\beta x}$$

Based on (1) and (2), the hazard function, $h(x)$, cumulative hazard function, $H(x)$, reversed hazard rate function, $rh(x)$, and odds function, $O(x)$, of the Gharaibeh distribution are , respectively, defined as

$$h(x)=\frac{f(x)}{1-F(x)}=\frac{\beta^6(x^5+20x^3+120x+120\beta)}{\begin{bmatrix}\beta^5x^5+5\beta^4x^4+20(\beta^5+\beta^3)x^3+60(\beta^4+\beta^2)x^2\\+120(\beta^5+\beta^3+\beta)x+120(\beta^6+\beta^4+\beta^2+1)\end{bmatrix}}$$

$$\begin{aligned}H(x)&=-ln(1-F(x))\\&=\beta x-ln\left(\frac{\beta^5x^5+5\beta^4x^4+20(\beta^5+\beta^3)x^3+60(\beta^4+\beta^2)x^2+120(\beta^5+\beta^3+\beta)x}{120(\beta^6+\beta^4+\beta^2+1)}+1\right)\end{aligned}$$

$$rh(x)=\frac{f(x)}{F(x)}=\frac{\beta^6(x^5+20x^3+120x+120\beta)}{\begin{bmatrix}120(\beta^6+\beta^4+\beta^2+1)(e^{\beta x}-1)-\beta^5x^5-5\beta^4x^4\\-20(\beta^5+\beta^3)x^3-60(\beta^4+\beta^2)x^2-120(\beta^5+\beta^3+\beta)x\end{bmatrix}}$$

and

$$O(x)=\frac{F(x)}{1-F(x)}=\frac{e^{\beta x}}{\frac{\beta^5x^5+5\beta^4x^4+20(\beta^5+\beta^3)x^3+60(\beta^4+\beta^2)x^2+120(\beta^5+\beta^3+\beta)x}{120(\beta^6+\beta^4+\beta^2+1)}+1}-1$$

The mean residual life function, $MRL(x)$, is defined as the expected value of the remaining lifetimes after a time x. That is,

$$MRL(x)=E(X-x|X>x)=\frac{1}{1-F(x)}\int_x^\infty(1-F(t))\,dt \tag{6}$$

Using (2) and (6), the mean residual life function of the Gharaibeh distribution can be obtained as

$$MRL(x)=\frac{\begin{bmatrix}120\beta^6+(x^5+20x^3+120x)\beta^5+(10x^4+120x^2+240)\beta^4\\+(60x^3+360x)\beta^3+(240x^2+480)\beta^2+600x\beta+720\end{bmatrix}}{\begin{bmatrix}\beta(120\beta^6+(x^5+20x^3+120x)\beta^5+(5x^4+60x^2+120)\beta^4\\+(20x^3+120x)\beta^3+(60x^2+120)\beta^2+120x\beta+120\end{bmatrix}}$$

It is obvious that $MRL(0)=\frac{\beta^6+2\beta^4+4\beta^2+6}{\beta(\beta^6+\beta^4+\beta^2+1)}=E(X)=\mu$.

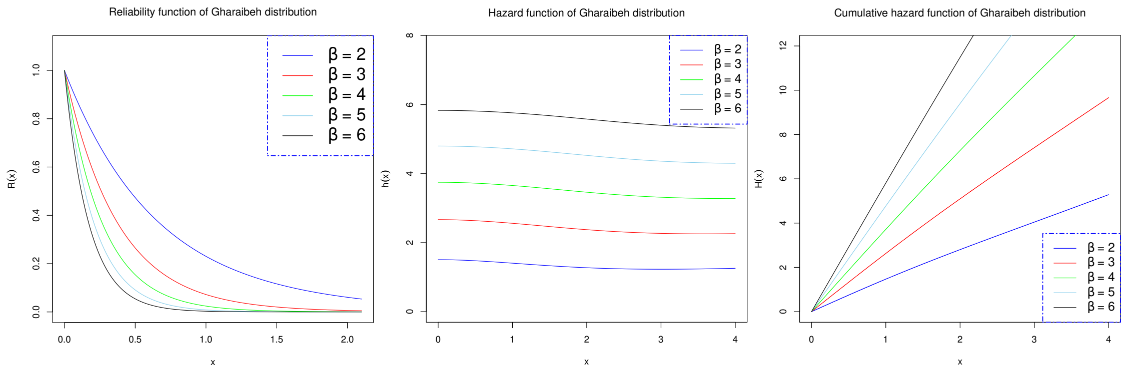

Fig. 2: The reliability, hazard, and cumulative hazard functons of Gharaibeh distribution with variant values of β

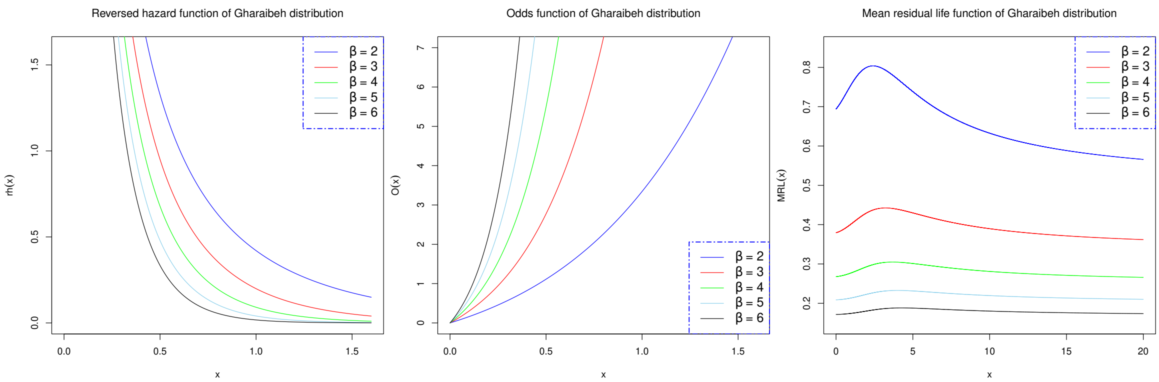

Fig. 3: The reversed hazard, odds, and mean residual life functions of Gharaibeh distribution with variant values of β

The graphs of reliability, hazard, cumulative hazard, reversed hazard, odds and mean residual life functions of the Gharaibeh distribution for different values of the parameter β are shown in Figures 2 and 3. It is noticeable that the hazard, cumulative hazard, and odds are increasing functions of β, while the reliability, reversed hazard and mean residual life are decreasing functions of β.

5 Order Statistics

In this section, we derive the distributions of order statistics from the Gharaibeh distribution. Let $X_{(1)}, X_{(2)}, \ldots X_{(n)}$ be the order statistics of the random sample $X_1, X_2, \ldots X_n$ selected from Gharaibeh distribution with pdf $f(x)$ in (1) and cdf $F(x)$ in (2). Then, the pdf of the i^{th} order statistic, $X_{(i)}$, (see [19]) is defined as

$$f_{(i)}(x) = i\binom{n}{i} f(x)[F(x)]^{i-1}[1-F(x)]^{n-i} \tag{7}$$

Plugging (1) and (2) in (7) by using binomial series, we have

$$f_{(i)}(x)=i\binom{n}{i}\frac{\beta^6}{120(\beta^6+\beta^4+\beta^2+1)}(x^5+20x^3+120x+120\beta)e^{-\beta x}\left[\sum_{k=0}^{i-1}\binom{i-1}{k}(-1)^k\right.$$
$$\left.\times\left[\left(\frac{\beta^5x^5+5\beta^4x^4+20(\beta^5+\beta^3)x^3+60(\beta^4+\beta^2)x^2+120(\beta^5+\beta^3+\beta)x}{120(\beta^6+\beta^4+\beta^2+1)}+1\right)e^{-\beta x}\right]^{n+k-i}\right]$$
$$=i\binom{n}{i}\beta^6(x^5+20x^3+120x+120\beta)\left[\sum_{k=0}^{i-1}\binom{i-1}{k}(-1)^k e^{-\beta x(n+k+1-i)}\right.$$
$$\left.\times\sum_{l=0}^{n+k-i}\binom{n+k-i}{l}\frac{(\beta^5x^5+5\beta^4x^4+20(\beta^5+\beta^3)x^3+60(\beta^4+\beta^2)x^2+120(\beta^5+\beta^3+\beta)x)^l}{(120(\beta^6+\beta^4+\beta^2+1))^{l+1}}\right]$$

Therefore, the pdfs of the minimum and maximum order statistics are, respectively, given by

$$f_{(1)}(x)=n\beta^6(x^5+20x^3+120x+120\beta)e^{-n\beta x}$$
$$\times\sum_{l=0}^{n-1}\binom{n-1}{l}\frac{(\beta^5x^5+5\beta^4x^4+20(\beta^5+\beta^3)x^3+60(\beta^4+\beta^2)x^2+120(\beta^5+\beta^3+\beta)x)^l}{(120(\beta^6+\beta^4+\beta^2+1))^{l+1}},$$

and

$$f_{(n)}(x)=n\beta^6(x^5+20x^3+120x+120\beta)\left[\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^k e^{-\beta x(k+1)}\right.$$
$$\left.\times\sum_{l=0}^{k}\binom{k}{l}\frac{(\beta^5x^5+5\beta^4x^4+20(\beta^5+\beta^3)x^3+60(\beta^4+\beta^2)x^2+120(\beta^5+\beta^3+\beta)x)^l}{(120(\beta^6+\beta^4+\beta^2+1))^{l+1}}\right].$$

6 Bonferroni and Lorenz Curves and Gini index

The Bonferroni and Lorenz curves and Gini index have applications in many fields, such as economics, insurance, demography, reliability and medicine. The Bonferroni and Lorenz curves for a random variable X are, respectively, defined as

$$B(p)=\frac{1}{p\mu}\int_0^q xf(x)dx, \qquad L(p)=\frac{1}{\mu}\int_0^q xf(x)dx, \tag{8}$$

where $q=F^{-1}(p); p\in(0,1]$ and $\mu=E(X)$. Using (1) and (5) in (8), the Bonferroni and Lorenz curves for Gharaibeh distribution are, respectively, obtained as

$$B(p)=\frac{1}{p}-\frac{\begin{bmatrix}120q\beta^7+\left(q^6+20q^4+120q^2+120\right)\beta^6+\left(6q^5+80q^3+240q\right)\beta^5+\\(30q^4+240q^2+240)\beta^4+(120q^3+480q)\beta^3+(360q^2+480)\beta^2+720q\beta+720\end{bmatrix}e^{-q\beta}}{120p\left(\beta^6+2\beta^4+4\beta^2+6\right)},$$

$$L(p)=1-\frac{\begin{bmatrix}120q\beta^7+\left(q^6+20q^4+120q^2+120\right)\beta^6+\left(6q^5+80q^3+240q\right)\beta^5+\\(30q^4+240q^2+240)\beta^4+(120q^3+480q)\beta^3+(360q^2+480)\beta^2+720q\beta+720\end{bmatrix}e^{-q\beta}}{120\left(\beta^6+2\beta^4+4\beta^2+6\right)}$$

The Gini index is defined as

$$G=1-2\int_0^1 L(p)dp$$

Therefore, the Gini index for Gharaibeh distribution is

$$G=\frac{2\begin{bmatrix}120q\beta^7+\left(q^6+20q^4+120q^2+120\right)\beta^6+\left(6q^5+80q^3+240q\right)\beta^5+\\(30q^4+240q^2+240)\beta^4+(120q^3+480q)\beta^3+(360q^2+480)\beta^2+720q\beta+720\end{bmatrix}e^{-q\beta}}{120\left(\beta^6+2\beta^4+4\beta^2+6\right)}-1$$

7 Stochastic Ordering

Stochastic ordering is a useful tool to compare two positive continuous distributions. A random variable X is smaller than a random variable Y in
1- Likelihood ratio order $(X \leq_{LR} Y)$ if $\frac{f_X(x)}{f_Y(x)}$ decreases in x.
2- Stochastic order $(X \leq_{ST} Y)$ if $F_X(x) \geq F_Y(x)$ for all x.
3- Hazard rate order $(X \leq_{HR} Y)$ if $h_X(x) \geq h_Y(x)$ for all x.
4- Mean residual life order $(X \leq_{MRL} Y)$ if $MRL_X(x) \leq MRL_Y(x)$ for all x.
[20] showed that

$$\begin{array}{ccccc} X \leq_{LR} Y & \Rightarrow & X \leq_{HR} Y & \Rightarrow & X \leq_{MRL} Y \\ & & \Downarrow & & \\ & & X \leq_{ST} Y & & \end{array}$$

The following theorem shows that Gharaibeh distribution satisfies the strongest ordering (likelihood ratio ordering).

Theorem 2. *Let $X \sim Gharaibeh(\beta_1)$ and $Y \sim Gharaibeh(\beta_2)$. If $\beta_1 > \beta_2$, then $X \leq_{LR} Y$ and thus $X \leq_{HR} Y, X \leq_{MRL} Y$ and $X \leq_{ST} Y$.*

Proof. Using the pdf of Gharaibeh distribution (1), we have

$$\frac{f_X(x;\beta_1)}{f_Y(x;\beta_2)} = \left[\frac{\beta_1^6(\beta_2^6+\beta_2^4+\beta_2^2+1)}{\beta_2^6(\beta_1^6+\beta_1^4+\beta_1^2+1)}\right]\left[\frac{x^5+20x^3+120x+120\beta_1}{x^5+20x^3+120x+120\beta_2}\right]e^{-x(\beta_1-\beta_2)}$$

Therefore,

$$log\frac{f_X(x;\beta_1)}{f_Y(x;\beta_2)} = log\left[\frac{\beta_1^6(\beta_2^6+\beta_2^4+\beta_2^2+1)}{\beta_2^6(\beta_1^6+\beta_1^4+\beta_1^2+1)}\right] + log(x^5+20x^3+120x+120\beta_1)$$
$$-log(x^5+20x^3+120x+120\beta_2) - x(\beta_1-\beta_2)$$

and

$$\frac{\partial}{\partial x}\left[log\frac{f_X(x;\beta_1)}{f_Y(x;\beta_2)}\right] = \frac{120(\beta_2-\beta 1)(5x^4+60x^2+120)}{(x^5+20x^3+120x+120\beta_1)(x^5+20x^3+120x+120\beta_2)} + (\beta_2-\beta_1)$$
$$= (\beta_2-\beta_1)\left[\frac{120(5x^4+60x^2+120)}{(x^5+20x^3+120x+120\beta_1)(x^5+20x^3+120x+120\beta_2)} + 1\right]$$

Hence, if $\beta_1 > \beta_2$, $\frac{\partial}{\partial x}\left[log\frac{f_X(x;\beta_1)}{f_Y(x;\beta_2)}\right] < 0$. Consequently, $X \leq_{LR} Y$ and thus $X \leq_{HR} Y, X \leq_{MRL} Y$ and $X \leq_{ST} Y$.

8 Renyi Entropy

The entropy of a random variable X is a measure of variation of the uncertainty. It is applied in many fields, such as engineering, statistical mechanics, finance, information theory, biomedical and economics. A popular entropy measure is the Renyi entropy [21] which is defined as

$$RE(\delta) = \frac{1}{1-\delta}log\int_0^{\infty}(f(x))^{\delta}dx; \delta > 0, \delta \neq 1. \tag{9}$$

Theorem 3. *The Renyi entropy of the Gharaibeh distribution random variable X is given by*

$$RE(\delta) = \frac{1}{1-\delta}log\left[\left(\beta^6+\beta^4+\beta^2+1\right)^{-\delta}\sum_{i=1}^{\delta}\sum_{j=1}^{i}\sum_{k=1}^{\delta-i}\binom{\delta}{i}\binom{i}{j}\binom{\delta-i}{k}20^{i-j}120^{-i}\beta^{7\delta-i-k}\frac{(3i+2j+k)!}{(\beta\delta)^{3i+2j+k+1}}\right]$$

*Proof.*Using the pdf of the Gharaibeh distribution in (1) and plugging in (9), we have

$$
\begin{aligned}
RE(\delta) &= \frac{1}{1-\delta} log \int_0^{\infty} \left[\frac{\beta^6}{120(\beta^6+\beta^4+\beta^2+1)} (x^5+20x^3+120x+120\beta) e^{-\beta x} \right]^{\delta} dx \\
&= \frac{1}{1-\delta} log \left[\left(\frac{\beta^6}{120(\beta^6+\beta^4+\beta^2+1)} \right)^{\delta} \int_0^{\infty} (x^5+20x^3+120x+120\beta)^{\delta} e^{-\beta\delta x} \right] dx \qquad (10)
\end{aligned}
$$

Using Binomial series, we have

$$
\begin{aligned}
(x^5+20x^3+120x+120\beta)^{\delta} &= (20x^3(\frac{x^2}{20}+1)+120\beta(\frac{x}{\beta}+1))^{\delta} \\
&= \sum_{i=1}^{\delta} \binom{\delta}{i} (20x^3(\frac{x^2}{20}+1))^i (120\beta(\frac{x}{\beta}+1))^{\delta-i} \\
&= \sum_{i=1}^{\delta} \binom{\delta}{i} (20x^3)^i (120\beta)^{\delta-i} \sum_{j=1}^{i} \binom{i}{j} (\frac{x^2}{20})^j \sum_{k=1}^{\delta-i} \binom{\delta-i}{k} (\frac{x}{\beta})^k \\
&= \sum_{i=1}^{\delta} \sum_{j=1}^{i} \sum_{k=1}^{\delta-i} \binom{\delta}{i} \binom{i}{j} \binom{\delta-i}{k} 20^{i-j} 120^{\delta-i} \beta^{\delta-i-k} x^{3i+2j+k} \qquad (11)
\end{aligned}
$$

Plugging (11) in (10), we have

$$
\begin{aligned}
RE(\delta) &= \frac{1}{1-\delta} log \Bigg[\left(\frac{\beta^6}{120(\beta^6+\beta^4+\beta^2+1)} \right)^{\delta} \sum_{i=1}^{\delta} \sum_{j=1}^{i} \sum_{k=1}^{\delta-i} \binom{\delta}{i} \binom{i}{j} \binom{\delta-i}{k} \\
&\qquad \times 20^{i-j} 120^{\delta-i} \beta^{\delta-i-k} \int_0^{\infty} x^{3i+2j+k} e^{-\beta\delta x} dx \Bigg] \\
&= \frac{1}{1-\delta} log \left[\left(\beta^6+\beta^4+\beta^2+1 \right)^{-\delta} \sum_{i=1}^{\delta} \sum_{j=1}^{i} \sum_{k=1}^{\delta-i} \binom{\delta}{i} \binom{i}{j} \binom{\delta-i}{k} 20^{i-j} 120^{-i} \beta^{7\delta-i-k} \frac{(3i+2j+k)!}{(\beta\delta)^{3i+2j+k+1}} \right].
\end{aligned}
$$

9 Mean Deviations about Mean and Median

Mean deviations about mean and median can be used to measure the dispersion and the spread in a population from the center. They are defined, respectively, as

$$
MD_{mean} = \int_0^{\infty} |x-\mu| f(x) dx = 2\mu F(\mu) - 2\int_0^{\mu} x f(x) dx, \qquad (12)
$$

$$
MD_{median} = \int_0^{\infty} |x-M| f(x) dx = \mu - 2\int_0^{M} x f(x) dx, \qquad (13)
$$

where $\mu = E(X)$ and M is the population median. Using the pdf of Gharaibeh distribution in (1), we have

$$
\int_0^{\mu} x f(x) dx = \mu - \frac{\begin{bmatrix} 120\mu\beta^7 + (\mu^6+20\mu^4+120\mu^2+120)\beta^6 \\ +(6\mu^5+80\mu^3+240\mu)\beta^5 + (30\mu^4+240\mu^2+240)\beta^4 \\ +(120\mu^3+480\mu)\beta^3 + (360\mu^2+480)\beta^2 + 720\mu\beta + 720 \end{bmatrix} e^{-\mu\beta}}{120\beta(\beta^6+\beta^4+\beta^2+1)} \qquad (14)
$$

and

$$
\int_0^{M} x f(x) dx = \mu - \frac{\begin{bmatrix} 120M\beta^7 + (M^6+20M^4+120M^2+120)\beta^6 \\ +(6M^5+80M^3+240M)\beta^5 + (30M^4+240M^2+240)\beta^4 \\ +(120M^3+480M)\beta^3 + (360M^2+480)\beta^2 + 720M\beta + 720 \end{bmatrix} e^{-M\beta}}{120\beta(\beta^6+\beta^4+\beta^2+1)} \qquad (15)
$$

Using (2), (14) and (15) in (12) and (13), the mean deviations about mean and median of Gharaibeh distribution can be , respectively, simplified as

$$MD_{mean} = \frac{e^{-\mu\beta}}{60\beta(\beta^6+\beta^4+\beta^2+1)}\begin{bmatrix}120\mu^2\beta^6+(\mu^5+20\mu^3+120\mu)\beta^5+(10\mu^4+120\mu^2+240)\beta^4\\+(60\mu^3+360\mu)\beta^3+(240\mu^2+480)\beta^2+600\mu\beta+720\end{bmatrix},$$

and

$$MD_{median} = \frac{e^{-M\beta}}{60\beta(\beta^6+\beta^4+\beta^2+1)}\begin{bmatrix}120M\beta^7+(M^6+20M^4+120M^2+120)\beta^6\\+(6M^5+80M^3+240M)\beta^5+(30M^4+240M^2+240)\beta^4\\+(120M^3+480M)\beta^3+(360M^2+480)\beta^2+720M\beta+720\end{bmatrix}-\mu.$$

10 Maximum Likelihood Estimate

Suppose $X_1, X_2, \ldots, X_n$ are a random sample of size n from the Gharaibeh distribution with a pdf $f(x)$ in (1) and parameter β . The likelihood function is defined as

$$\begin{aligned}L(\beta|x_1,x_2,\ldots,x_n) &= \prod_{i=1}^{n}\frac{\beta^6}{120(\beta^6+\beta^4+\beta^2+1)}(x_i^5+20x_i^3+120x_i+120\beta)e^{-\beta x_i}\\&= \left(\frac{\beta^6}{120(\beta^6+\beta^4+\beta^2+1)}\right)^n\prod_{i=1}^{n}\left(x_i^5+20x_i^3+120x_i+120\beta\right)e^{-\beta\sum_{i=1}^{n}x_i}.\end{aligned}$$

Thus, the log-likelihood function is given by

$$\begin{aligned}L^* &= lnL(\beta|x_1,x_2,\ldots,x_n)\\&= n ln\left(\frac{\beta^6}{120(\beta^6+\beta^4+\beta^2+1)}\right)+\sum_{i=1}^{n} ln\left(x_i^5+20x_i^3+120x_i+120\beta\right)-\beta\sum_{i=1}^{n}x_i.\end{aligned}$$

Therefore, the maximum likelihood estimate (MLE) of β is the solution of the following nonlinear equation

$$\frac{\partial L^*}{\partial\beta} = \frac{6n}{\beta}-\frac{n(6\beta^5+4\beta^3+2\beta)}{\beta^6+\beta^4+\beta^2+1}+\sum_{i=1}^{n}\frac{120}{x_i^5+20x_i^3+120x_i+120\beta}-\sum_{i=1}^{n}x_i = 0,$$

which can be solved using numerical methods.

11 Real Data Applications

In this section, the goodness of fit of the Gharaibeh distribution is investigated and compared to other existing distributions using three real data sets. The first data set is reported by [22] and given in Table 2, which corresponds to the survival times (in years) of a group of patients given chemotherapy treatment alone. The second data set in Table 3 is the strength data of glass of the aircraft window reported by [23]. The third data set given in Table 4 and used by [24] represents vinyl chloride data obtained from clean upgradient monitoring wells in mg/l.

Table 2: Data 1

0.047	0.115	0.121	0.132	0.164	0.197	0.203	0.260	0.282	0.296
0.334	0.395	0.458	0.466	0.501	0.507	0.529	0.534	0.540	0.570
0.641	0.644	0.696	0.841	0.863	1.099	1.219	1.271	1.326	1.447
1.485	1.553	1.581	1.589	2.178	2.343	2.416	2.444	2.825	2.830
3.578	3.658	3.743	3.978	4.003	4.033				

For each data set, the goodness of fit of the Gharaibeh distribution is compared with the following distributions:

–Exponential distribution: $f(x) = \theta e^{-x\theta}$;$x > 0, \theta > 0$.

–Lindley distribution: $f(x) = \frac{\theta^2}{\theta+1}(1+x)e^{-x\theta}$;$x > 0, \theta > 0$.

Table 3: Data 2

18.83	20.8	21.657	23.03	23.23	24.05	24.321	25.5	25.52	25.8	26.69
26.77	26.78	27.05	27.67	29.9	31.11	33.2	33.73	33.76	33.89	34.76
35.75	35.91	36.98	37.08	37.09	39.58	44.045	45.29	45.381		

Table 4: Data 3

5.1	1.2	1.3	0.6	0.5	2.4	0.5	1.1	8	0.8	0.4	0.6	0.9	0.4	2	0.5	5.3
3.2	2.7	2.9	2.5	2.3	1	0.2	0.1	0.1	1.8	0.9	2	4	6.8	1.2	0.4	0.2

–Rama distribution: $f(x) = \frac{\theta^4}{\theta^3+6}(1+x^3)e^{-x\theta} \quad ;x > 0, \theta > 0.$

–Aradhana distribution: $f(x) = \frac{\theta^3}{\theta^2+2\theta+2}(1+x)^2 e^{-x\theta}; x > 0, \theta > 0.$

–Ishita distribution: $f(x) = \frac{\theta^3}{\theta^3+2}(\theta+x^2)e^{-x\theta}; x > 0, \theta > 0.$

–Akshaya distribution: $f(x) = \frac{\theta^4}{\theta^3+3\theta^2+6\theta+6}(1+x)^3 e^{-x\theta}; x > 0, \theta > 0.$

based on the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), -2 log likelihood ($-2logL$), Kolmogorov Smirnov (KS) statistic and its p-value and the results are summarized in Table 5.

For the three real data sets, it is noticeable that the Gharaibeh distribution provides the best fit as it shows the lowest values of the $-2logL$, BIC, AIC, KS statistic and highest p-value among all fitted distributions.

Table 5: $-2logL$, AIC, BIC, KS Statistic and its p-value of the fitted distributions.

	Distribution	$-2logL$	*AIC*	*BIC*	*KS Statistic*	*p-value*
Data 1	***Exponential***	117.8673	119.8673	121.6959	0.0932	0.7842
	Lindley	118.4908	120.4908	122.3195	0.1325	0.3626
	Rama	118.4683	120.4683	122.297	0.1064	0.6361
	Aradhana	120.5853	122.5853	124.414	0.1653	0.1443
	Ishita	117.7228	119.7229	121.5516	0.1073	0.6262
	Akshaya	123.8164	125.8164	127.6451	0.1935	0.05533
	Gharaibeh	**117.3092**	**119.3094**	**121.138**	**0.0769**	**0.9292**
Data 2	***Exponential***	274.5288	276.5289	277.9629	0.4587	0.0000
	Lindley	253.9884	255.9884	257.4224	0.3655	0.0003
	Rama	232.7924	234.7924	236.2263	0.2538	0.0301
	Aradhana	242.229	244.2289	245.6629	0.3062	0.0044
	Ishita	240.487	242.4871	243.921	0.2979	0.0061
	Akshaya	234.4386	236.4386	237.8726	0.2628	0.0223
	Gharaibeh	**223.9228**	**225.9227**	**227.3567**	**0.2017**	**0.1396**
Data 3	***Exponential***	110.9052	112.9052	114.4316	0.089	0.9507
	Lindley	112.6072	114.6073	116.1336	0.1326	0.5881
	Rama	118.6832	120.6832	122.2096	0.1768	0.2381
	Aradhana	116.0632	118.0633	119.5897	0.1695	0.2826
	Ishita	114.6058	116.6058	118.1322	0.1405	0.5136
	Akshaya	120.8176	122.8177	124.344	0.2004	0.1305
	Gharaibeh	**110.7768**	**112.7768**	**114.3032**	**0.086**	**0.9632**

Furthermore, the maximum likelihood estimates (MLEs) with their corresponding standard errors and confidence intervals of the parameters for all fitted distributions are obtained and given in Table 6.

Table 6: The MLEs of the parameters of the fitted distributions and their confidence intervals.

	Distribution	*MLE*	*Standard Error*	95% *Confidence Interval* Lower Limit	95% *Confidence Interval* Upper Limit
	Exponential	0.7549	0.1113	0.5367	0.9730
	Lindley	1.1122	0.1249	0.8673	1.3570
	Rama	1.8535	0.126	1.6065	2.1004
	Aradhana	1.4874	0.1371	1.2186	1.7561
	Ishita	1.4015	0.1043	1.1970	1.6059
	Akshaya	1.8753	0.1485	1.5842	2.1663
Data 1	***Gharaibeh***	1.4519	0.0954	1.2649	1.6388
	Exponential	0.0324	0.0058	0.0210	0.0437
	Lindley	0.0629	0.008	0.0472	0.0785
	Rama	0.1297	0.0116	0.1069	0.1524
	Aradhana	0.0943	0.0097	0.0752	0.1133
	Ishita	0.0973	0.0101	0.0775	0.1170
	Akshaya	0.1257	0.0112	0.1037	0.1476
Data 2	***Gharaibeh***	0.1922	0.0139	0.1649	0.2194
	Exponential	0.532	0.0912	0.3532	0.7107
	Lindley	0.8238	0.1054	0.6172	1.0303
	Rama	1.5309	0.1175	1.3006	1.7612
	Aradhana	1.1328	0.1181	0.9013	1.3642
	Ishita	1.157	0.0962	0.9684	1.3455
	Akshaya	1.4534	0.1299	1.1987	1.7080
Data 3	***Gharaibeh***	1.2575	0.0851	1.0907	1.4242

12 Conclusion

In this paper, a new continuous one parameter distribution (i.e. Gharaibeh distribution) was proposed and investigated. Some important mathematical and statistical features, such as moments, mean, variance, skewness, kurtosis, coefficient of variation, moment generating function, mean deviations about the mean and median, Bonferroni and Lorenz curves, order statistics, and stochastic ordering were investigated. Also, the reliability, hazard rate, cumulative hazard, reversed hazard, odds and mean residual life functions were obtained. Moreover, Renyi entropy and maximum likelihood estimate of the distribution parameter were derived. Applications to three real data sets were presented to demonstrate the importance and usefulness of the Gharaibeh distribution. Gharaibeh distribution outperformed other competing distributions, so it is appropriate for modeling lifetime data.

Acknowledgement

The author is grateful to the anonymous referees for the careful checking of the details and the constructive comments that improved this paper.

Conflict of interest There is no conflict of interest in this work.

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