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Common Fixed Point Theorems for Families of Weakly Compatible Maps in 2-Metric Spaces

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In this article, a common fixed point theorem in 2-metric spaces is proved. Moreover, the main theorems of [2] and [15] in 2-metric spaces are investigated.

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1 Introduction

The concept of a 2-metric space is a natural generalization of a metric space. It has been introduced by Gähler [3–5] and extensively studied by some mathematicians such as Gähler [3–5], White [18], Iséki [6]. Moreover, a number of authors [1, 10, 13, 17] have studied the contractive, non-expansive and contraction type mapping in 2-metric spaces. On the other hand, Jungck [7] studied the common fixed points of commuting maps. Then Sessa [16] generalized the commuting maps by introducing the notion of weakly commuting and proved a common fixed point theorem for weakly commuting maps. Jungck [8] further made a generalization of weakly commuting maps by introducing the notion of compatible mappings. Moreover, Jungck and Rhoades [9] introduced the notion of coincidentally commuting or weakly compatible mappings. Several authors used these concepts to prove some common fixed point theorems on usual metric, as well as on different kinds of generalized metric spaces [1,2,11,15]). In this paper, the existence and approximation of a unique common fixed point of two families of weakly compatible self maps on a 2-metric space are proved. In order to study these theorems, we recall the definition of a 2-metric space which is given by Gähler as follows:

Definition 1.1 ([3])**.** A 2-metric space is a set with a real-valued function satisfying the following conditions:

(1) For distinct points $x, y \in X$, there exists a point $c \in X$ such that $d(x, y, c) \neq 0$;
(2) $d(x, y, c) = 0$ if at least two of x, y and c are equal;
(3) $d(x, y, c) = d(x, c, y) = d(c, y, x)$;
(4) $d(x, y, c) \leq d(x, y, z) + d(x, z, y) + d(z, y, c)$ for all $x, y, c, z \in X$.

The function d is called a 2-metric of the space X and pair (X, d) denotes a 2-metric space. It has shown by Gähler that a 2-metric space d is non-negative and although d is a continuous function of any one of its there arguments, it need not be continuous in two arguments. A 2-metric space d which is continuous in all of its arguments is said to be continuous.

Geometrically, a 2-metric $d(x, y, c)$ represents the area of a triangle with vertices x, y and c. Throughout in this paper, let (X, d) be 2-metric space unless mentioned otherwise and $B(X)$ is the set of all nonempty bounded subset of X.

Definition 1.2 ([14])**.** A sequence $\{x_n\}$ in (X, d) is said to be convergent to a point x in X, denoted by

$$\lim_{n\to\infty} x_n = x,$$

if

$$\lim_{n\to\infty} d(x_n, x, c) = 0$$

for all c in X. The point x is called the limit of the sequence $\{x_n\}$ in X.

Definition 1.3 ([14])**.** A sequence $\{x_n\}$ in (X, d) is said to be a Cauchy sequence if

$$\lim_{n\to\infty} d(x_n, x_m, c) = 0$$

for all c in X.

Definition 1.4 ([14])**.** The space (X, d) is said to be complete if every Cauchy sequence in converges to an element in X.

Remark 1.1. A convergent sequence is a Cauchy sequence in a metric space but in a 2-metric space a convergent sequence need not be a Cauchy sequence, but every convergent sequence is a Cauchy sequence when the 2-metric d is continuous on X (see [12]).

Define $\delta(A, B, C)$ for all A, B and C in $B(X)$, by

$$\delta(A, B, C) = \sup\{d(a, b, C) : a \in A, b \in B\},$$

where $d(a, b, C) = inf\{d(a, b, c) : c \in C\}$. If A consist of a single point a, we write $\delta(A, B, C) = \delta(a, B, C)$, if B and C consist of a single point b and c respectively, we write $\delta(A, B, C) = \delta(a, b, c)$. It follows immediately from the definition that

$$\delta(A, B, C) = \delta(A, C, B) = \delta(C, B, A) = \delta(C, A, B) = \delta(B, C, A) = \delta(B, A, C) \geq 0$$

and

$$\delta(A, B, C) \leq \delta(A, C, E) + \delta(A, E, C) + \delta(E, B, C),$$

for all A, B, C and E in $B(X)$. Moreover, $\delta(A, B, C) = 0$, if at least two of A, B and C consist of equal single points.

Definition 1.5 ([11]). Two single-valued mapping f and g of (X, d) into itself are compatible if

$$\lim_{n\to\infty} d(fgx_n, gfx_n, C) = 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = gx_n = t$$

for some t in X. It can be seen that two weakly commuting mapping are compatible but the converse is false.

Definition 1.6 ([1]). Self maps F and T of a 2-metric space (X, d) are weakly compatible if they commute at coincidence point, i.e. if $Fp = Tp$ for some point p in X, then $FTp = TFp$.

2 Main Results

In this section, the existence and approximation of a unique common fixed point theorem of two families of weakly compatible self maps on a complete 2-metric space is proved.

Define $\Phi = \{\varphi : \mathbb{R}^+ \to \mathbb{R}^+\}$, where $\mathbb{R}^+ = [0, +\infty)$ and each $\varphi \in \Phi$ satisfies the following conditions:

(a) φ is continuous on $\mathbb{R}^+$,
(b) φ is non-decreasing, and
(c) $\varphi(t) < t$ for each $t > 0$.

Theorem 2.1. *Let $P_1, P_2, \ldots, P_{2n}, Q_0$ and Q_1 be mapping from 2-metric space (X, d) into itself, satisfying conditions:*

(I) $Q_0(X) \subseteq P_1P_3\cdots P_{2n-1}(X),\ Q_1(X) \subseteq P_2P_4\cdots P_{2n}(X)$*;*

(II)

$$
\begin{aligned}
P_2(P_4\cdots P_{2n}) &= (P_4\cdots P_{2n})P_2,\\
P_2P_4(P_6\cdots P_{2n}) &= (P_6\cdots P_{2n})P_2P_4,\\
&\vdots\\
P_2\cdots P_{2n-2}(P_{2n}) &= (P_{2n})P_2\cdots P_{2n-2},\\
Q_0(P_4\cdots P_{2n}) &= (P_4\cdots P_{2n})Q_0,\\
Q_0(P_6\cdots P_{2n}) &= (P_6\cdots P_{2n})Q_0,\\
&\vdots\\
Q_0P_{2n} &= P_{2n}Q_0,
\end{aligned}
$$

$$
\begin{aligned}
P_1(P_3\cdots P_{2n-1}) &= (P_3\cdots P_{2n-1})P_1,\\
P_1P_3(P_5\cdots P_{2n-1}) &= (P_5\cdots P_{2n-1})P_1P_3,\\
&\vdots\\
P_1\cdots P_{2n-3}(P_{2n-1}) &= (P_{2n-1})P_1\cdots P_{2n-3},\\
Q_1(P_3\cdots P_{2n-1}) &= (P_3\cdots P_{2n-1})Q_1,\\
Q_1(P_5\cdots P_{2n-1}) &= (P_5\cdots P_{2n-1})Q_1,\\
&\vdots\\
Q_1P_{2n-1} &= P_{2n-1}Q_1;
\end{aligned}
$$

(III) $P_2\cdots P_{2n}$ *or* Q_0 *is continuous;*
(IV) *The pair* $(Q_0, P_2\cdots P_{2n})$ *is compatible and the pair* $(Q_1, P_1\cdots P_{2n-1})$ *is weakly compatible;*
(V) *There exists* $\varphi\in\Phi$ *such that*

$$
\begin{aligned}
d(Q_0u, Q_1v, C) \le \mathrm{Max}\Big\{&\varphi[d(P_2P_4\cdots P_{2n}u, Q_0u, C)], \varphi[d(P_1P_3\cdots P_{2n-1}v, Q_1v, C)],\\
&\varphi[d(P_2P_4\cdots P_{2n}u, P_1P_3\cdots P_{2n-1}v, C)],\\
&\varphi\Big[\frac{1}{2}\big[d(P_1P_3\cdots P_{2n-1}v, Q_0u, C) + d(P_2P_4\cdots P_{2n}u, Q_1v, C)\big]\Big]\Big\}
\end{aligned}
$$

for all $u, v, C\in X$.

Then $P_1, P_2, \ldots, P_{2n}$, Q_0 *and* Q_1 *have a unique common fixed point in* X.

Proof. Let $x_0\in X$. From condition (I) there exists $x_1, x_2\in X$ such that $Q_0x_0 = P_1P_3\cdots P_{2n-1}x_1 = y_0$ and $Q_1x_1 = P_2P_4\cdots P_{2n}x_2 = y_1$. Inductively, we can construct sequence $\{x_n\}$ and $\{y_n\}$ in X

$$Q_0x_{2k} = P_1P_3\cdots P_{2n-1}x_{2k+1} = y_{2k}$$

and

$$Q_1x_{2k+1} = P_2P_4\cdots P_{2n}x_{2k+2} = y_{2k+1},$$

for $k \in \mathbb{N}$.

Putting $u = x_p = x_{2k}, v = x_{q+1} = x_{2m+1}, P'_1 = P_2P_4 \cdots P_{2n}$ and $P'_2 = P_1P_3 \cdots P_{2n-1}$ in condition (V), we have

$$\begin{aligned} d(Q_0x_{2k}, Q_1x_{2m+1}, C) \leq \mathrm{Max}\Big\{ & \varphi[d(P'_1x_{2k}, Q_0x_{2k}, C)], \varphi[d(P'_2x_{2m+1}, Q_1x_{2m+1}, C)], \\ & \varphi[d(P'_1x_{2k}, P'_2x_{2m+1}, C)], \\ & \varphi\Big[\frac{1}{2}\big[d(P'_2x_{2m+1}, Q_0x_{2k}, C) + d(P'_1x_{2k}, Q_1x_{2m+1}, C)\big]\Big]\Big\}, \end{aligned}$$

i.e.,

$$\begin{aligned} d(y_{2k}, y_{2m+1}, C) \leq \mathrm{Max}\Big\{ & \varphi[d(y_{2k-1}, y_{2k}, C)], \varphi[d(y_{2m}, y_{2m+1}, C)], \varphi[d(y_{2k-1}, y_{2m}, C)], \\ & \varphi\Big[\frac{1}{2}\big[d(y_{2m}, y_{2k}, C) + d(y_{2k-1}, y_{2m+1}, C)\big]\Big]\Big\}. \end{aligned}$$

Thus,

$$\begin{aligned} d(y_p, y_{q+1}, C) \leq \mathrm{Max}\Big\{ & \varphi[d(y_{p-1}, y_p, C)], \varphi[d(y_q, y_{q+1}, C)], \varphi[d(y_{p-1}, y_q, C)] \\ & \varphi\Big[\frac{1}{2}\big[d(y_q, y_p, C) + d(y_{p-1}, y_{q+1}, C)\big]\Big]\Big\}. \end{aligned} \tag{2.1}$$

If $q = p$, then

$$\begin{aligned} \frac{1}{2}\Big[d(y_p, y_p, C) + d(y_{p-1}, y_{p+1}, C)\Big] &\leq \frac{1}{2}\Big[d(y_{p-1}, y_p, C) + d(y_p, y_{p+1}, C)\Big] \\ &\leq \mathrm{Max}\Big\{d(y_{p-1}, y_p, C), d(y_p, y_{p+1}, C)\Big\}. \end{aligned}$$

Thus, from (2.1) and the property (b) of φ,

$$d(y_p, y_{p+1}, C) \leq \varphi\big[\mathrm{Max}\big\{d(y_{p-1}, y_p, C), d(y_p, y_{p+1}, C)\big\}\big].$$

Since by the property (c) of φ, $d(y_p, y_{p+1}, C) \leq \varphi[d(y_p, y_{p+1}, C)]$ is impossible for $d(y_p, y_{p+1}, C) > 0$, we have

$$d(y_p, y_{p+1}, C) \leq \varphi\big[d(y_{p-1}, y_p, C)\big].$$

This means that

$$d(y_{2k}, y_{2k+1}, C) \leq \varphi\big[d(y_{2k-1}, y_{2k}, C)\big].$$

Similarly,

$$d(y_{2k+1}, y_{2k+2}, C) \leq \varphi\big[d(y_{2k}, y_{2k+1}, C)\big].$$

Therefore, for all n, even or odd, we have

$$d(y_n, y_{n+1}, C) \leq \varphi\big[d(y_{n-1}, y_n, C)\big]. \tag{2.2}$$

Hence $\{d(y_n, y_{n+1}, C)\}$ is non-decreasing and, therefore, $d(y_n, y_{n+1}, C) \to \alpha \geq 0$ as $n \to \infty$. Taking the limit in (2.2) we get $\alpha \leq \varphi(\alpha)$, and from (c), $\alpha = 0$. Thus

$$\lim_{n\to\infty} d(y_n, y_{n+1}, C) = 0.$$

Now, we show that $\{y_n\}$ is a Cauchy sequence in X.

Let $\epsilon > 0$ be arbitrary. We need to show that there exists an integer $N \geq 2$ such that

$$d(y_n, y_m, C) < \epsilon \quad \text{for all} \;\; m \geq n \geq N. \tag{2.3}$$

Since by the property (c) of φ, $\epsilon - \varphi(\epsilon) > 0$, and as φ is continuous, there exists a $\delta > 0$ such that

$$\epsilon < t < \epsilon + 5\delta \Rightarrow \varphi(t) < \varphi(\epsilon) + \frac{\epsilon - \varphi(\epsilon)}{3}. \tag{2.4}$$

Without loss of generality, we may assume that $\delta < [\epsilon - \varphi(\epsilon)]/3$. Since

$$d(y_n, y_{n+1}, C) \to 0$$

there exists an integer $N \geq 1$ such that

$$d(y_{n-2}, y_{n-1}, C) < \delta \;\; \text{for all} \;\; n \geq N. \tag{2.5}$$

By induction we shall show that

$$d(y_n, y_m, C) < \varphi(\epsilon) + \frac{\epsilon - \varphi(\epsilon)}{3} + 2\delta \quad \text{for all} \;\; m \geq n \geq N. \tag{2.6}$$

Let $n \geq N$ be fixed. Obviously, for $m = n + 1$, (2.6) holds from (2.5). Assuming that (2.6) holds for an integer $m \geq n + 1$, we shall prove that (2.6) holds for $m + 1$. We have to consider the following cases:

(i) $n = 2k$ and $m = 2q$. Then $d(y_n, y_m, C) = d(y_{2k}, y_{2q}, C)$ and

$$d(y_n, y_{m+1}, C) = d(y_{2k}, y_{2q+1}, C).$$

(ii) $n = 2k$ and $m = 2q + 1$. Then

$$d(y_n, y_{m+1}, C) \leq d(y_{2k}, y_{2q+1}, C) + d(y_m, y_{m+1}, C).$$

(iii) $n = 2k + 1$ and $m = 2q$. Then

$$d(y_n, y_{m+1}, C) \leq d(y_{2k}, y_{2q+1}, C) + d(y_{n-1}, y_n, C).$$

(iv) $n = 2k + 1$ and $m = 2q + 1$. Then

$$d(y_n, y_{m+1}, C) \leq d(y_{2k}, y_{2q+1}, C) + d(y_{n-1}, y_n, C) + d(y_m, y_{m+1}, C). \tag{2.7}$$

Consider the most complex case (iv). Since

$$d(y_{2k}, y_{2q+1}, C) = d(Q_0x_{2k}, Q_1x_{2q+1}, C),$$

then by (2.7) and (2.5),

$$d(y_n, y_{m+1}, C) \le d(Q_0x_{2k}, Q_1x_{2q+1}, C) + 2\delta. \tag{2.8}$$

Now we show that

$$d(Q_0x_{2k}, Q_1x_{2q+1}, C) \le \varphi(\varepsilon) + \frac{\epsilon\ - \varphi(\epsilon)}{3}. \tag{2.9}$$

From (V),

$$\begin{aligned} d(Q_0x_{2k}, Q_1x_{2q+1}, C) \le \mathrm{Max}\Big\{ & \varphi[d(y_{2k-1}, y_{2k}, C)], \varphi[\mathrm{d(y_{2q}, y_{2q+1}, C)}], \\ & \varphi\big[d(y_{2k-1}, y_{2q}), C)\big], \\ & \varphi\Big[\frac{1}{2}\big[d(y_{2q}, y_{2k}, C) + d(y_{2k-1}, y_{2q+1}, C)\big]\Big]\Big\}. \end{aligned}$$

If we denote

$$\begin{aligned} t_{n,m} = \mathrm{Max}\Big\{ & d(y_{2k-1}, y_{2k}, C), d(y_{2q}, y_{2q+1}, C), d(y_{2k-1}, y_{2q}, C), \\ & \frac{1}{2}\Big[d(y_{2q}, y_{2k}, C) + d(y_{2k-1}, y_{2q+1}, C)\Big]\Big\}, \end{aligned}$$

then by the property (b) of φ,

$$d(Q_0x_{2k}, Q_1x_{2m+1}, C) \le \varphi(t_{n,m}). \tag{2.10}$$

Next we estimate $t_{n,m}$. Since $n = 2k + 1$, $m = 2q + 1$, by the induction hypotheses we have

$$d(y_{2k+1}, y_{2q+1}, C) < \varphi(\epsilon) + \frac{\epsilon\ - \varphi(\epsilon)}{3}\ + 2\delta. \tag{2.11}$$

From (2.5), we find that

$$d(y_{2k-1}, y_{2k}, C) = d(y_{n-2}, y_{n-1}, C) < \delta,$$

$$d(y_{2q}, y_{2q+1}, C) = d(y_{m-1}, y_m, C) < \delta.$$

Further, by the triangle inequality, (2.11) and (2.5), we get

$$\begin{aligned} d(y_{2k-1}, y_{2q}, C) &\le d(y_{2k+1}, y_{2q+1}, C) + d(y_{n-2}, y_{n-1}, C) + d(y_{n-1}, y_n, C) \\ &\quad + d(y_{m-1}, y_m, C) \\ &\le \varphi(\varepsilon) + \frac{\epsilon\ - \varphi(\epsilon)}{3} + 2\delta + 3\delta < \varepsilon + 5\delta \end{aligned}$$

and, by (2.10), we have

$$\begin{aligned}\frac{1}{2}\Big[d(y_{2q}, y_{2k}, C) + d(y_{2k-1}, y_{2q+1}, C)\Big] &\le \frac{1}{2}\Big[d(y_{2k+1}, y_{2q+1}, C) + d(y_{n-1}, y_n, C)\\ &\quad + d(y_{m-1}, y_m, C) + d(y_{2k+1}, y_{2q+1}, C)\\ &\quad + d(y_{n-2}, y_{n-1}, C) + d(y_{n-1}, y_n, C)\Big]\\ &\le \varphi(\varepsilon) + \frac{\epsilon\ - \varphi(\epsilon)}{3} + 4\delta\\ &< \varepsilon + 4\delta.\end{aligned}$$

Thus $t_{n,m} < \varepsilon + 5\delta$. Then from (2.4), we find that

$$\varphi(t_{n,m}) < \varphi(\varepsilon) + \frac{\epsilon\ - \varphi(\epsilon)}{3}.$$

Now that (2.10) implies (2.9), and (2.8) and (2.9) imply (2.6). Since $\delta < [\epsilon\ - \varphi(\epsilon)]/3$, then (2.6) implies (2.3). Hence we conclude that $\{y_n\}$ is a Cauchy sequence in X.

Since X is complete, there exists some $z \in X$ such that $y_n \to z$. Also, for its subsequences we have

$$\begin{aligned} Q_1x_{2k+1} \to z \text{ and } \quad & P_1P_3 \cdots P_{2n-1}x_{2k+1} \to z,\\ Q_0x_{2k} \to z \text{ and } \quad & P_2P_4 \cdots P_{2n}x_{2k} \to z.\end{aligned}$$

Case 1. $P_2P_4 \cdots P_{2n}$ is continuous.

Denote $P'_1 = P_2P_4 \cdots P_{2n}$. Since P'_1 is continuous, $P'_1 \circ P'_1x_{2k} \to P'_1z$ and $P'_1Q_0x_{2k} \to P'_1z$. Also, as (Q_0, P'_1) is compatible, it implies that $Q_0P'_1x_{2k} \to P'_1z$.

a) Putting $u = P_2P_4 \cdots P_{2n}x_{2k} = P'_1x_{2k},\ v = x_{2k+1}$, and $P'_2 = P_1P_3 \cdots P_{2n-1}$ in condition (V), we have

$$\begin{aligned}&d(Q_0P'_1x_{2k}, Q_1x_{2k+1}, C)\\ &\le \text{Max}\Big\{\varphi[d(P'_1P'_1x_{2k}, Q_0P'_1x_{2k}, C)], \varphi[d(P'_2x_{2k+1}, Q_1x_{2k+1}, C)],\\ &\qquad \varphi[d(P'_1P'_1x_{2k}, P'_2x_{2k+1}, C)],\\ &\qquad \varphi\Big[\frac{1}{2}\big[d(P'_2x_{2k+1}, Q_0P'_1x_{2k}, C) + d(\ P'_1P'_1x_{2k}, Q_1x_{2k+1}, C)\big]\Big]\Big\}.\end{aligned}$$

Letting $k \to \infty$, we get

$$\begin{aligned}d(P'_1z, z, C) \le \text{Max}\Big\{&\varphi[d(P'_1z, P'_1z, C)], \varphi[d(z, z, C)],\\ &\varphi[d(P'_1z, z, C)], \varphi\Big[\frac{1}{2}\big[d(z, P'_1z, C) + d(P'_1z, z, C)\big]\Big]\Big\}.\end{aligned}$$

Hence $d(P'_1z, z, C) \le \varphi[d(P'_1z, z, C)]$. If we suppose that $d(P'_1z, z, C) > 0$, then we have

$$d(P'_1z, z, C) \le \varphi[d(P'_1z, z, C)] < d(P'_1z, z, C),$$

which is a contradiction. Thus $P'_1 z = z$, i. e., $P_2P_4\cdots P_{2n}z = z$.

b) Putting $u = z$, $v = x_{2k+1}$, $P'_1 = P_2P_4\cdots P_{2n}$ and $P'_2 = P_1P_3\cdots P_{2n-1}$ in condition (V), we have

$$\begin{aligned} d(Q_0z, Q_1x_{2k+1}, C) \le \mathrm{Max}\Big\{&\varphi[d(P'_1z, Q_0z, C)], \varphi[d(P'_2x_{2k+1}, Q_1x_{2k+1}, C)],\\ &\varphi[d(P'_1z, P'_2x_{2k+1}, C)],\\ &\varphi\Big[\frac{1}{2}\big[d(P'_2x_{2k+1}, Q_0z, C) + d(P'_1z, Q_1x_{2k+1}, C)\big]\Big]\Big\}. \end{aligned}$$

Letting $k \to \infty$, we get

$$\begin{aligned} d(Q_0z, z, C) &\le \mathrm{Max}\Big\{\varphi[d(z, Q_0z, C)], \varphi[d(z, z, C)], \varphi[d(z, z, C)],\\ &\qquad \varphi\Big[\frac{1}{2}\big[d(z, Q_0z, C) + d(z, z, C)\big]\Big]\Big\}\\ &= \varphi[d(z, Q_0z, C)]. \end{aligned}$$

So, $d(Q_0z, z, C) \le \varphi[d(Q_0z, z, C)]$. Hence $d(Q_0z, z, C) = 0$, and $Q_0z = P_2P_4\cdots P_{2n}z = z$.

c) Putting $u = P_4\cdots P_{2n}z$, $v = x_{2k+1}$, $P'_1 = P_2P_4\cdots P_{2n}$ and $P'_2 = P_1P_3\cdots P_{2n-1}$ in condition (V), and using the condition $P_2(P_4\cdots P_{2n}) = (P_4\cdots P_{2n})P_2$ and $Q_0(P_4\cdots P_{2n}) = (P_4\cdots P_{2n})Q_0$ in condition (II), we get

$$\begin{aligned} &d(Q_0P_4\cdots P_{2n}z, Q_1x_{2k+1}, C)\\ &\le \mathrm{Max}\Big\{\varphi[d(P'_1P_4\cdots P_{2n}z, Q_0P_4\cdots P_{2n}z, C)], \varphi[d(P'_1P_4\cdots P_{2n}z, P'_2x_{2k+1}, C)],\\ &\qquad \varphi[d(P'_2x_{2k+1}, Q_1x_{2k+1}, C)],\\ &\qquad \varphi\Big[\frac{1}{2}\big[d(P'_2x_{2k+1}, Q_0P_4\cdots P_{2n}z, C) + d(P'_1P_4\cdots P_{2n}z, Q_1x_{2k+1}, C)\big]\Big]\Big\}. \end{aligned}$$

Letting $k \to \infty$, we get

$$\begin{aligned} d(P_4\cdots P_{2n}z, z, C) &\le \mathrm{Max}\Big\{\varphi[d(P_4\cdots P_{2n}z, P_4\cdots P_{2n}z, C)],\\ &\qquad \varphi[d(z, z, C)], \varphi[d(P_4\cdots P_{2n}z, z, C)],\\ &\qquad \varphi\Big[\frac{1}{2}\big[d(z, P_4\cdots P_{2n}z, C) + d(P_4\cdots P_{2n}z, z, C)\big]\Big]\Big\}\\ &= \varphi[d(P_4\cdots P_{2n}z, z, C)]. \end{aligned}$$

Hence it follows that $P_4\cdots P_{2n}z = z$. Then $P_2(P_4\cdots P_{2n}z) = P_2z$ and so $P_2z = P_2P_4\cdots P_{2n}z = z$.

Continuing this procedure, we obtain

$$Q_0z = P_2z = P_4z = \cdots = P_{2n}z = z.$$

d) As $Q_0(X) \subseteq P_1P_3\cdots P_{2n-1}(X)$, there exists $v \in X$ such that $z = Q_0z = P_1P_3\cdots P_{2n-1}v$. Putting $u = x_{2k}, P'_1 = P_2P_4\cdots P_{2n}$ and $P'_2 = P_1P_3\cdots P_{2n-1}$ in condition (V), we have that

$$\begin{aligned} d(Q_0x_{2k}, Q_1v, C) \leq \mathrm{Max}\Big\{ & \varphi[d(P'_1x_{2k}, Q_0x_{2k}, C)], \varphi[d(P'_2v, Q_1v, C)], \\ & \varphi[d(P'_1x_{2k}, P'_2v, C)], \\ & \varphi\Big[\frac{1}{2}\big[d(P'_2v, Q_0x_{2k}, C) + d(P'_1x_{2k}, Q_1v, C)\big]\Big]\Big\}. \end{aligned}$$

Letting $k \to \infty$, we get

$$\begin{aligned} d(z, Q_1v, C) \leq \mathrm{Max}\Big\{ & \varphi[d(z, z, C)], \varphi[d(z, Q_1v, C)], \varphi[d(z, z, C)], \\ & \varphi\Big[\frac{1}{2}\big[d(z, z, C) + d(z, Q_1v, C)\big]\Big]\Big\}. \end{aligned}$$

So, $d(z, Q_1v, C) \leq \varphi[d(z, Q_1v, C)]$. Therefore $Q_1v = z$. Hence $P_1P_3\cdots P_{2n-1}v = Q_1v = z$. As $(Q_1, P_1\cdots P_{2n-1})$ is weakly compatible, we have

$$P_1P_3\cdots P_{2n-1}Q_1v = Q_1P_1P_3\cdots P_{2n-1}v.$$

Thus $P_1P_3\cdots P_{2n-1}z = Q_1z$.

e) Putting $u = x_{2k},\ v = z,\ P'_1 = P_2P_4\cdots P_{2n}$ and $P'_2 = P_1P_3\cdots P_{2n-1}$ in condition (V), we have

$$\begin{aligned} d(Q_0x_{2k}, Q_1z, C) \leq \mathrm{Max}\Big\{ & \varphi[d(P'_1x_{2k}, Q_0x_{2k}, C)], \varphi[d(P'_2z, Q_1z, C)], \\ & \varphi[d(P'_1x_{2k}, P'_2z, C)], \\ & \varphi\Big[\frac{1}{2}\big[d(P'_2z, Q_0x_{2k}, C) + d(P'_1x_{2k}, Q_1z, C)\big]\Big]\Big\}. \end{aligned}$$

Letting $k \to \infty$, we get

$$\begin{aligned} d(z, Q_1z, C) \leq \mathrm{Max}\Big\{ & \varphi[d(z, z, C)], \varphi[d(Q_1z, Q_1z, C)], \varphi[d(z, Q_1z, C)], \\ & \varphi\Big[\frac{1}{2}\big[d(Q_1z, z, C) + d(z, Q_1z, C)\big]\Big]\Big\} \\ = \varphi[d(z, Q_1z, C)]. & \end{aligned}$$

So, $d(z, Q_1z, C) \leq \varphi[d(z, Q_1z, C)]$. Therefore $Q_1z = z$. Hence, $P_1P_3\cdots P_{2n-1}z = Q_1z = z$.

f) Putting $u = x_{2k},\ v = P_3\cdots P_{2n-1}z,\ P'_1 = P_2P_4\cdots P_{2n}$ and $P'_2 = P_1P_3\cdots P_{2n-1}$ in condition (V), we have

$$d(Q_0x_{2k}, Q_1P_3\cdots P_{2n-1}z, C)$$

$$\leq \mathrm{Max}\Big\{\varphi[d(\ {P'}_1x_{2k}, Q_0x_{2k}, C)], \varphi[d({P'}_2P_3\cdots P_{2n-1}z, Q_1P_3\cdots P_{2n-1}z, C)],$$
$$\varphi[d({P'}_1x_{2k}, {P'}_2P_3\cdots P_{2n-1}z, C)],$$
$$\varphi\Big[\frac{1}{2}\big[d({P'}_2P_3\cdots P_{2n-1}z, Q_0x_{2k}, C) + d({P'}_1x_{2k}, Q_1P_3\cdots P_{2n-1}z, C)\big]\Big]\Big\}.$$

Letting $k \to \infty$, we get

$$d(z, P_3\cdots P_{2n-1}z, C) \leq \mathrm{Max}\Big\{\varphi[d(z, z, C)], \varphi[d(P_3\cdots P_{2n-1}z, P_3\cdots P_{2n-1}z, C)],$$
$$\varphi[d(z, P_3\cdots P_{2n-1}z, C)],$$
$$\varphi\Big[\frac{1}{2}\big[d(P_3\cdots P_{2n-1}z, z) + d(z, P_3\cdots P_{2n-1}z, C)\big]\Big]\Big\}$$
$$= \varphi[d(z, P_3\cdots P_{2n-1}z, C)].$$

So $d(z, P_3\cdots P_{2n-1}z, C) \leq \varphi[d(z, P_3\cdots P_{2n-1}z, C)]$. Therefore $P_3\cdots P_{2n-1}z = z$. Hence $P_1z = z$. Continuing this procedure, we have

$$Q_1z = P_1z = P_3z = \cdots = P_{2n-1}z.$$

Thus we haveproved

$$Q_0z = Q_1z = P_1z = P_2z = \cdots = P_{2n-1}z = P_{2n}z = z.$$

Case 2. Q_0 is continuous.

Since Q_0 is continuous, $Q_0^2x_{2k} \to Q_0z$. As $(Q_0, P_2P_4\cdots P_{2n})$ is compatible, we have $(P_2P_4\cdots P_{2n})Q_0x_{2k} \to Q_0z$.

g) Putting $u = Q_0x_{2k},\ v = x_{2k+1},\ {P'}_1 = P_2P_4\cdots P_{2n}$ and ${P'}_2 = P_1P_3\cdots P_{2n-1}$ in condition (V), we have

$$d(Q_0Q_0x_{2k}, Q_1x_{2k+1}, C)$$
$$\leq \mathrm{Max}\Big\{\varphi[d({P'}_1Q_0x_{2k}, Q_0Q_0x_{2k}, C)], \varphi[d({P'}_2x_{2k+1}, Q_1x_{2k+1}, C)],$$
$$\varphi[d({P'}_1Q_0x_{2k}, {P'}_2x_{2k+1}, C)],$$
$$\varphi\Big[\frac{1}{2}\big[d({P'}_2x_{2k+1}, Q_0Q_0x_{2k}, C) + d({P'}_1Q_0x_{2k}, Q_1x_{2k+1}, C)\big]\Big]\Big\}.$$

Letting $k \to \infty$, we get

$$d(Q_0z, z, C) \leq \mathrm{Max}\{\varphi[d(Q_0z, Q_0z, C)], \varphi[d(z, z, C)], \varphi[d(Q_0z, z, C)],$$
$$\varphi\Big[\frac{1}{2}\big[d(z, Q_0z, C) + d(Q_0z, z, C)\big]\Big]\Big\}$$
$$= \varphi[d(Q_0z, z, C)].$$

So $d(Q_0z, z, C) \leq \varphi[d(Q_0z, z, C)]$. Therefore $Q_0z = z$. Now, using steps d), e) and f), and continuing step f) give us

$$Q_1z = P_1z = P_3z = \cdots = P_{2n-1}z = z.$$

h) As $Q_1(X) \subseteq P_2 \cdots P_{2n}(X)$ there exists $w \in X$ such that $z = Q_1z = P_2 \cdots P_{2n}w$. Putting $u = w,\ v = x_{2k+1},\ P'_1 = P_2P_4 \cdots P_{2n}$ and $P'_2 = P_1P_3 \cdots P_{2n-1}$ in condition (V), we have

$$\begin{aligned} d(Q_0w, Q_1x_{2k+1}, C) \leq \mathrm{Max}\Big\{&\varphi[d(P'_1w, Q_0w, C)],\\ &\varphi[d(P'_2x_{2k+1}, Q_1x_{2k+1}, C)], \varphi[d(P'_1w, P'_2x_{2k+1}, C)],\\ &\varphi\Big[\frac{1}{2}\big[d(P'_2x_{2k+1}, Q_0w, C) + d(P'_1w, Q_1x_{2k+1}, C)\big]\Big]\Big\}. \end{aligned}$$

Letting $k \to \infty$, we get

$$\begin{aligned} d(Q_0w, z, C) &\leq \mathrm{Max}\Big\{\varphi[d(z, Q_0w, C)], \varphi[d(z, z, C)], \varphi[d(z, z, C)],\\ &\qquad \varphi\Big[\frac{1}{2}\big[d(z, Q_0w, C) + d(z, z, C)\big]\Big]\Big\}\\ &= \varphi[d(z, Q_0w, C)]. \end{aligned}$$

So $d(Q_0w, z, C) \leq \varphi[d(Q_0w, z, C)]$. Therefore $Q_0w = z = P_2 \cdots P_{2n}w$. As $(Q_0, P_2 \cdots P_{2n})$ is weakly compatible, we have

$$Q_0z = P_2P_4 \cdots P_{2n}z = z.$$

Similarly as in the step c) it can be shown that $P_2z = P_4z = \cdots = P_{2n}z = Q_0z = z$. Thus we proved that

$$Q_0z = Q_1z = P_1z = P_2z = P_3z = \cdots = P_{2n}z = z.$$

Proof of uniqueness. Let z' be another common fixed point of mentioned maps, then $Q_0z' = Q_1z' = P_1z' = P_2z' = \cdots = P_{2n}z' = z'$. Putting $u = z, v = z', P'_1 = P_2P_4 \cdots P_{2n}$ and $P'_2 = P_1P_3 \cdots P_{2n-1}$ in condition (V), we have

$$\begin{aligned} d(Q_0z, Q_1z', C) &\leq \mathrm{Max}\Big\{\varphi[d(P'_1z, Q_0z, C)], \varphi[d(P'_2z', Q_1z', C)], \varphi[d(P'_1z, P'_2z', C)]\\ &\qquad \varphi\Big[\frac{1}{2}\big[d(P'_2z', Q_0z, C) + d(P'_1z, Q_1z', C)\big]\Big]\Big\}\\ &= \varphi[d(z, z', C)]. \end{aligned}$$

It means that

$$d(z, z', C) \leq \varphi[d(z, z', C)].$$

Thus $z = z'$ and this shows that z is a unique common fixed point of the maps. □

Now we shall prove a common fixed point theorem, which is a slight generalization of Theorem 2.1.

Theorem 2.2. *Let (X,d) be a complete 2-metric space and let $\{T_\alpha\}_{\alpha\in J}$ and $\{P_i\}_{i=1}^{2n}$ be two families of self-mappings of X. Suppose that there exists a fixed $\beta\in J$ such that*

(I) *$T_\alpha(X)\subseteq P_2P_4\cdots P_{2n}(X)$ for each $\alpha\in J$ and $T_\beta(X)\subseteq P_1P_3\cdots P_{2n-1}(X)$ for some $\beta\in J$;*

(II)

$$\begin{aligned} P_2(P_4\cdots P_{2n}) &= (P_4\cdots P_{2n})P_2,\\ P_2P_4(P_6\cdots P_{2n}) &= (P_6\cdots P_{2n})P_2P_4,\\ &\vdots\\ P_2\cdots P_{2n-2}(P_{2n}) &= (P_{2n})P_2\cdots P_{2n-2},\\ T_\beta(P_4\cdots P_{2n}) &= (P_4\cdots P_{2n})T_\beta,\\ T_\beta(P_6\cdots P_{2n}) &= (P_6\cdots P_{2n})T_\beta,\\ &\vdots\\ T_\beta P_{2n} &= P_{2n}T_\beta, \end{aligned}$$

$$\begin{aligned} P_1(P_3\cdots P_{2n-1}) &= (P_3\cdots P_{2n-1})P_1,\\ P_1P_3(P_5\cdots P_{2n-1}) &= (P_5\cdots P_{2n-1})P_1P_3,\\ &\vdots\\ P_1\cdots P_{2n-3}(P_{2n-1}) &= (P_{2n-1})P_1\cdots P_{2n-3},\\ T_\alpha(P_3\cdots P_{2n-1}) &= (P_3\cdots P_{2n-1})T_\alpha,\\ T_\alpha(P_5\cdots P_{2n-1}) &= (P_5\cdots P_{2n-1})T_\alpha,\\ &\vdots\\ T_\alpha P_{2n-1} &= P_{2n-1}Q_1; \end{aligned}$$

(III) *$P_2\cdots P_{2n}$ or T_β is continuous;*

(IV) *The pair $(T_\beta, P_2\cdots P_{2n})$ is compatible and the pairs $(T_\alpha, P_1\cdots P_{2n-1})$ are weakly compatible;*

(V) *There exists $\varphi=\varphi(\alpha)\in\Phi$ such that for all $u,v\in X$*

$$\begin{aligned} d(T_\beta u, T_\alpha v, C) \le \mathrm{Max}\Big\{&\varphi[d(P_2P_4\cdots P_{2n}u, T_\beta u, C)], \varphi[d(P_1P_3\cdots P_{2n-1}v, T_\alpha v, C)],\\ &\varphi[d(P_2P_4\cdots P_{2n}u, P_1P_3\cdots P_{2n-1}v, C)],\\ &\varphi\Big[\frac{1}{2}\big[d(P_1P_3\cdots P_{2n-1}v, T_\beta u, C) + d(P_2P_4\cdots P_{2n}u, T_\alpha v, C)\big]\Big]\Big\} \end{aligned}$$

Then all P_i and T_α have a unique common fixed point in X.

Proof. Let T_{α_0} be a fixed element in $\{T_\alpha\}_{\alpha\in J}$. By Theorem 2.1 with $Q_0 = T_\beta$ and $Q_1 = T_{\alpha_0}$ it follows that there exists some $z \in X$ such that

$$T_\beta z = T_{\alpha_0} z = P_2P_4\cdots P_{2n}z = P_1P_3\cdots P_{2n-1}z = z.$$

Let $\alpha \in J$ be arbitrary. Then from (V)

$$\begin{aligned} d(T_\beta z, T_\alpha z, C) \le \mathrm{Max}\Big\{ &\varphi[d(P_2P_4\cdots P_{2n}z, T_\beta z, C)], \varphi[d(P_1P_3\cdots P_{2n-1}z, T_\alpha z, C)],\\ &\varphi[d(P_2P_4\cdots P_{2n}z, P_1P_3\cdots P_{2n-1}z, C)],\\ &\varphi\Big[\frac{1}{2}\big[d(P_1P_3\cdots P_{2n-1}v, T_\beta z, C) + d(P_2P_4\cdots P_{2n}u, T_\alpha z, C)\big]\Big]\Big\} \end{aligned}$$

and hence

$$\begin{aligned} d(z, T_\alpha z, C) &\le \mathrm{Max}\Big\{\varphi(d(z,z,C)), \varphi[d(z, T_\alpha z, C)], \varphi[d(z,z,C)],\\ &\qquad \varphi\Big[\frac{1}{2}\big[d(z,z,C) + d(z, T_\alpha z, C)\big]\Big]\Big\}\\ &\le \varphi[d(z, T_\alpha z, C)]. \end{aligned}$$

If we suppose that $d(z, T_\alpha z, C) > 0$, then property (c) of φ shows that

$$d(z, T_\alpha z, C) \le \varphi[d(z, T_\alpha z, C)] < d(z, T_\alpha z, C),$$

which is a contradiction. Thus $T_\alpha z = z$ for each $\alpha \in J$. Since (V) implies the uniqueness of common fixed point and Theorem 2.2 is proved. □

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