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# Fractional Hamiltonian Systems with Vanishing Potentials

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**Abstract:** In this paper we deal with an existence result for a class of fractional Hamiltonian system in $\mathbb{R}$ with potentials vanishing at infinity and suitable nonlinearities which are super-linear at the origin and at infinity. We establish new compact embedding theorem in order to show the boundedness of a Cerami sequence.

**Keywords:** Fractional Hamiltonian systems, Liuoville-Weyl fractional operator, fractional Sobolev space, Cerami sequences, variational methods, vanishing potentials.

## 1 Introduction

Fractional derivatives and nonlocal operators are historically applied in the study of nonlocal or time-dependent processes. The first and well established application of fractional calculus in physics was in the framework of anomalous diffusion, which is related to features observed in many physical systems, for example; in dispersive transport in amorphous semiconductor, liquid crystals, polymers, proteins, etc. [1], [2], [3], [4].

In 1996-1997, by applying the fractional variational principle, Riewe [5] derived the following Euler-Lagrange equation

$$\sum_{i=1}^{N} {}_tD_b^{\alpha_i}[\partial_i F] + \sum_{i=1}^{\tilde{N}} {}_aD_t^{\beta_i}[\partial_{i+N}F] + \partial_{\tilde{N}+N+1}F = 0, \tag{1}$$

from the energy functional

$$\mathbf{I}(y) = \int_a^b F({}_aD_t^{\alpha_1}y(t),\cdots,{}_aD_t^{\alpha_N}y(t),{}_tD_b^{\beta_1}y(t),\cdots,{}_tD_b^{\beta_{\tilde{N}}}y(t),y(t),t)dt,$$

where $y:[a,b]\to\mathbb{R}^n$ and $\alpha_{\mathbf{I}},\beta_{\mathbf{I}}\in[0,1]$, $i=1\cdots N$, $\mathbf{I}=1,\cdots,\tilde{N}$ and $n,N,\tilde{N}\in\mathbb{N}$.

If the Lagrangian function $F$ has the form

$$F = \frac{1}{2}m\dot{y}^2 - B(y) + \frac{1}{2}\gamma i\left({}_aD_t^{\frac{1}{2}}[y]\right)^2, \tag{2}$$

Riewe obtained the Euler-Lagrange equation

$$m\ddot{y} = -\gamma i\left({}_tD_b^{\frac{1}{2}}\circ{}_aD_t^{\frac{1}{2}}\right)[y] - \frac{\partial B(y)}{\partial y}. \tag{3}$$

Recently, several different approaches have been developed to generalize the least action principle and the Euler-Lagrange equations to include fractional derivatives with singular and regular kernel, see for example [6], [7], [8].

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Nowadays critical point theory and variational methods have also turned out to be very effective tools in determining the existence of solutions for differential equations. The idea behind them is to try and find solutions of a given boundary value problem by looking for critical points of a suitable energy functional defined on an appropriate function space, for more details see [9], [10], [11], [12], and the references therein. Recently by using critical point theory and mountain pass theorem, Jiao and Zhou [13], showed the existence of nontrivial weak solutions for the following fractional boundary value problem

$$\begin{aligned} {}_xD_T^s({}_0D_x^s u(x)) &= \nabla F(x,u(x)), \;\; x\in[0,T], \\ u(0) &= u(T) = 0. \end{aligned} \tag{4}$$

In [14], Torres studied the fractional Hamiltonian systems

$$\begin{aligned} &{}_xD_\infty^s({}_{-\infty}D_x^s u(x)) + L(x)u(x) = \nabla W(x,u(x)), \;\; x\in\mathbb{R} \\ &u\in H^s(\mathbb{R},\mathbb{R}^N), \end{aligned} \tag{5}$$

where $s\in(1/2,1)$, $L:\mathbb{R}\to\mathbb{R}^{N\times N}$ is a continuous positive definite symmetric matrix and $W\in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$. Under the following hypothesis

(L) there exists an $l\in C(\mathbb{R},(0,\infty))$ with $l(x)\to\infty$ as $|x|\to\infty$ such that

$$(L(x)u,u)\geq l(x)|u|^2 \quad \text{for all } x\in\mathbb{R} \text{ and } u\in\mathbb{R}^N. \tag{6}$$

($FHS_1$) There is a constant $\theta>2$ such that

$$0<\theta W(x,u)\leq(\nabla W(x,u),u) \quad \text{for all } x\in\mathbb{R} \text{ and } u\in\mathbb{R}^N\backslash\{0\},$$

($FHS_2$) $|\nabla W(x,u)|=o(|u|)$ as $|u|\to 0$ uniformly with respect to $x\in\mathbb{R}$.

($FHS_3$) There exists $\overline{W}\in C(\mathbb{R}^N,\mathbb{R})$ such that

$$|W(x,u)|+|\nabla W(x,u)|\leq|\overline{W}(u)| \quad \text{for every } x\in\mathbb{R} \text{ and } u\in\mathbb{R}^N,$$

and by using the Mountain pass theorem, the author showed that (5) possesses at least one nontrivial weak solution. For more related works, the readers can see: [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32], [33], [34], [35], [36] and references therein. We point out that, these works considered the case $s\in(\frac{1}{2},1)$. The case $s\in(0,\frac{1}{2}]$, is still an open problem.

Motivated by these previous works and by the fact that after a bibliography review we did not find in the literature any paper dealing with the case $s\in(0,\frac{1}{2})$, in this work we deals with the existence of non zero weak solution for the following fractional system

$$\begin{cases} {}_xD_\infty^s({}_{-\infty}D_x^s u)+B(x)u=\Lambda_1(x)f(v), & x\in\mathbb{R} \\ {}_xD_\infty^s({}_{-\infty}D_x^s v)+B(x)v=\Lambda_2(x)g(u), & x\in\mathbb{R} \\ u,v\geq 0 \\ (u,v)\in\mathbb{I}_-^s(\mathbb{R})\times\mathbb{I}_-^s(\mathbb{R}), \end{cases} \tag{7}$$

where $s\in(0,\frac{1}{2})$, $f,\ g:\mathbb{R}\to\mathbb{R}$ are continuous nonnegative functions and $B,\ \Lambda_1,\ \Lambda_2:\mathbb{R}\to\mathbb{R}$ are nonnegative functions. Moreover, we suppose that $f,\ g$ satisfy

$$f(s),\ g(s)>0 \;\text{ for }\; s>0 \;\text{ and }\; f(s)=g(s)=0 \;\text{ for }\; s\leq 0,$$

and we assume the following conditions:

$(H_1)$ $f(s)=o(|s|)$ and $g(s)=o(|s|)$ as $|s|\to 0$;

$(H_2)$

$$\lim_{s\to+\infty}\frac{f(s)}{s}=+\infty, \quad \lim_{s\to+\infty}\frac{g(s)}{s}=+\infty;$$

$(H_3)$ There exist constants $\dfrac{1}{s+1}<\tau_1,\ \tau_2<1$, $c_0>0$ and $R_0>0$ such that

$$f(s)^{\tau_1}\leq c_0 s^{\tau_1}\tilde{F}(s), \quad g(s)^{\tau_2}\leq c_0 s^{\tau_2}\tilde{G}(s) \;\text{ for all } s\geq R_0$$

and $\tilde{F}(s),\ \tilde{G}(s)>0$ for $s\in(0,R_0)$, where

$$F(s)=\int_0^s f(t)dt, \;\; G(s)=\int_0^s g(t)dt, \;\; \tilde{F}(s):=\frac{1}{2}f(s)s-F(s) \quad \text{and} \quad \tilde{G}(s):=\frac{1}{2}g(s)s-G(s).$$

We notice that in the case $s \in (\frac{1}{2}, 1)$ the embedding of the fractional spaces $\mathbb{I}^s_-(\mathbb{R})$ into $C(\mathbb{R})$ play an important role, but in the case $s \in (0, \frac{1}{2}]$ this embedding is lost, so we need to show a new continuous embedding of $\mathbb{I}^s_-(\mathbb{R})$ into some $L^p(\mathbb{R})$ spaces (for more details see section 2). In order to deal with the lost of compactness, through all the paper we suppose that $(B, \Lambda) \in \mathscr{E}$ if the following conditions hold:

$(\mathscr{E}_{(i)})$ $B(x)$, $\Lambda(x) > 0$ $\forall x \in \mathbb{R}$ are measurable in $\mathbb{R}$ and $\Lambda \in L^\infty(\mathbb{R})$;

$(\mathscr{E}_{(ii)})$ For all $\delta \in (0,1]$, the function $\omega(x) := \dfrac{\Lambda(x)}{B^\delta(x)} > 0$, satisfies

$$\lim_{|x| \to \infty} \omega(x) = 0.$$

These conditions are crucial to introduce our new compact embedding theorem.

Now, we are in position to state our main result:

**Theorem 1.** *If $(B, \Lambda_1) \in \mathscr{E}$, $(B, \Lambda_2) \in \mathscr{E}$ and $f$, $g$ satisfy $(f_1)-(f_3)$, then, problem (7) possesses at least one pair $(u, B)$ of non negative weak solutions.*

When $s = 1$, problem (7) reduce to the following differential systems

$$\begin{cases} -u'' + B(x)u = \Lambda_1(x) f(v), & x \in \mathbb{R} \\ -v'' + B(x)v = \Lambda_2(x) g(u), & x \in \mathbb{R} \\ u, v \geq 0 \\ (u,v) \in H^1(\mathbb{R}) \times H^1(\mathbb{R}). \end{cases} \tag{8}$$

Recently several authors studied problem (8) in its general presentation like

$$\begin{cases} -\Delta u + B(x)u = F_v(x,u,v), & x \in \mathbb{R}^N \\ -\Delta v + B(x)v = F_u(x,u,v), & x \in \mathbb{R}^N, \end{cases} \tag{9}$$

where $N \geq 3$, $F \in C^1(\mathbb{R}^{N+2}, \mathbb{R})$, by using variational tools such as reduction methods, generalized mountain pass theorem, dual variational formulation, generalized fountain theorem and generalized linking theorems, see for instance [37], [38], [39], [40], [41], [42], [43], [44].

We note that in the papers cited above with respect to the problem (7) with $s \in (\frac{1}{2}, 1)$ and (9), the standard exercise is to show the boundedness of the Palais-Smale or Cerami sequences and the main difficulty is to prove the convergence of these sequences. Since we do not suppose the classical A-R condition, the most difficult part of our paper is to get the boundedness of the Cerami sequence (see Lemma 4 below).

We organize the paper in the following way: In section 2 we consider some preliminary results and present our variational framework. In section 3, by Linking theorem we show the existence and boundedness of a Cerami sequence for the associated functional to system (7). Finally, in section 4 we give a prove of Theorem 1.

## 2 Preliminary results

In this section, for the reader's convenience, firstly we introduce some basic definitions of fractional calculus, for more details the reader's can see [3], [45].

The Liouville-Weyl fractional derivatives of order $0 < s < 1$ are defined as

$$_{-\infty}D^s_x u(x) = \frac{d}{dx} {}_{-\infty}I^{1-s}_x u(x) \quad \text{and} \quad {}_xD^s_\infty u(x) = -\frac{d}{dx} {}_xI^{1-s}_\infty u(x), \tag{10}$$

where $_{-\infty}I^\alpha_x$ and $_xI^\alpha_\infty$ are the left and right Liouville-Weyl fractional integrals of order $0 < s < 1$ defined as

$$_{-\infty}I^s_x u(x) = \frac{1}{\Gamma(s)} \int_{-\infty}^x (x-\xi)^{s-1} u(\xi) d\xi \quad \text{and} \quad {}_xI^s_\infty u(x) = \frac{1}{\Gamma(s)} \int_x^\infty (\xi - x)^{s-1} u(\xi) d\xi.$$

Furthermore, for $u \in L^p(\mathbb{R})$, $p \geq 1$, we have

$$\mathscr{F}({}_{-\infty}I^s_x u(x)) = (i\omega)^{-s} \widehat{u}(\omega) \qquad \text{and} \qquad \mathscr{F}({}_xI^s_\infty u(x)) = (-i\omega)^{-s} \widehat{u}(\omega), \tag{11}$$

and for $u \in C_0^\infty(\mathbb{R})$, we have

$$\mathscr{F}({}_{-\infty}D^s_x u(x)) = (i\omega)^s \widehat{u}(\omega) \qquad \text{and} \qquad \mathscr{F}({}_xD^s_\infty u(x)) = (-i\omega)^s \widehat{u}(\omega). \tag{12}$$

**Theorem 2.** *1.Let $s > 0$ and $u \in C_0^\infty(\mathbb{R})$ then*

$$_{-\infty}I_x^s {}_{-\infty}D_x^s u = u \quad and \quad {}_xI_\infty^s {}_xD_\infty^s u = u$$

*2.Let $s > 0$ and $u \in C_0^\infty(\mathbb{R})$, then $_{-\infty}D_x^s u, {}_xD_\infty^s u \in L^p(\mathbb{R})$ for any $p \in [1,\infty)$.*

**Theorem 3.***Let $p \in (1,\infty]$ and $s \in (0,\frac{1}{p})$. Then the operators $_{-\infty}I_x^s, {}_xI_\infty^s : L^p(\mathbb{R}) \to L^{p_s^*}(\mathbb{R})$ are bounded, where $p_s^* = \frac{p}{1-ps}$ is called fractional critical exponent.*

**Proposition 1.***Let $s > 0$, $p > 1$ and $q > 1$ with*

$$\frac{1}{p} + \frac{1}{q} = 1 + s.$$

*Let $u \in L^p(\mathbb{R})$ and $B \in L^q(\mathbb{R})$, then*

$$\int_{\mathbb{R}} u(x) {}_{-\infty}I_x^s B(x) dx = \int_{\mathbb{R}} {}_xI_{+\infty}^s u(x) B(x) dx.$$

## *2.1 Fractional derivative spaces*

Motivated by the definition of Sobolev spaces [46], in this section we introduce the notion of weak fractional derivative and fractional space of Sobolev type, more precisely we have:

**Definition 1.***Let $s > 0$ and $u, w \in L_{loc}^1(\mathbb{R})$. The function $w$ is called weak left fractional derivative of $u$, which is denoted by $_{-\infty}\mathscr{D}_x^s u = w$, if and only if*

$$\int_{\mathbb{R}} u(x) {}_xD_{+\infty}^s \varphi(x) dx = \int_{\mathbb{R}} w(x)\varphi(x) dx, \ \ \forall \varphi \in C_0^\infty(\mathbb{R}). \tag{13}$$

*In a similar fashion, we say that $w$ is called weak right fractional derivative of $u$, denoted by $_x\mathscr{D}_\infty^s u = w$ if and only if*

$$\int_{\mathbb{R}} u(x) {}_{-\infty}D_x^s \varphi(x) dx = \int_{\mathbb{R}} w(x)\varphi(x) dx, \ \ \forall \varphi \in C_0^\infty(\mathbb{R}). \tag{14}$$

**Lemma 1.***[47] Let $u \in L_{loc}^1(\mathbb{R})$. If $u$ has a weak left (or right) fractional derivative, then it is unique up to a set of zero measure.*

**Lemma 2.***Let $u \in L_{loc}^1(\mathbb{R})$. The weak left (right) fractional derivative of $u$ is linear.*

*Proof.* Let $u, w \in L_{loc}^1(\mathbb{R})$ and $k \in \mathbb{R}$, then, for fixed $\varphi \in C_0^\infty(\mathbb{R})$ we have

$$\begin{aligned}\int_{\mathbb{R}} (ku+w)(x) {}_xD_\infty^s \varphi(x) dx &= k\int_{\mathbb{R}} u(x) {}_xD_\infty^s \varphi(x) dx + \int_{\mathbb{R}} w(x) {}_xD_\infty^s \varphi(x) dx \\ &= \int_{\mathbb{R}} [k {}_{-\infty}\mathscr{D}_x^s u(x) + {}_{-\infty}\mathscr{D}_x^s w(x)]\varphi(x) dx.\end{aligned}$$

Therefore, by definition of weak letf fractional derivative we get

$$_{-\infty}\mathscr{D}_x^s (ku+w) = k {}_{-\infty}\mathscr{D}_x^s u + {}_{-\infty}\mathscr{D}_x^s w.$$

In the same way we can show for $_x\mathscr{D}_\infty^s$. □

**Proposition 2.***If $u \in C_0^\infty(\mathbb{R})$, then*

$$_{-\infty}\mathscr{D}_x^s u = {}_{-\infty}D_x^s u \quad and \quad {}_x\mathscr{D}_\infty^s u = {}_xD_\infty^s u.$$

*Proof.* Since $u \in C_0^\infty(\mathbb{R})$, then

$$_{-\infty}D_x^s u(x) = \frac{d}{dx} {}_{-\infty}I_x^{1-s} u(x) = {}_{-\infty}I_x^{1-s} u'(x).$$

So, for any $\varphi \in C_0^\infty(\mathbb{R})$, by Proposition 1 we have

$$\begin{aligned}\int_{\mathbb{R}} u(x)_x D_\infty^s \varphi(x)dx &= -\int_{\mathbb{R}} u(x)_x I_\infty^{1-s}\varphi'(x)dx = -\int_{\mathbb{R}} {}_{-\infty}I_x^{1-s}u(x)\varphi'(x)dx \\ &= -\left[{}_{-\infty}I_x^{1-s}u(x)\varphi(x)\Big|_{-\infty}^{\infty} - \int_{\mathbb{R}}\frac{d}{dx}{}_{-\infty}I_x^{1-s}u(x)\varphi(x)dx\right] \\ &= \int_{\mathbb{R}} {}_{-\infty}D_x^s u(x)\varphi(x)dx,\end{aligned}$$

from which we get the desired result. □

**Definition 2.** *For $s \in (0,1)$ we define the fractional space ${}_{-\infty}\mathbb{I}_x^s(\mathbb{R})$ as*

$${}_{-\infty}\mathbb{I}_x^s(\mathbb{R}) = \{u \in L^2(\mathbb{R}) : \ {}_{-\infty}\mathscr{D}_x^s u \in L^2(\mathbb{R})\}$$

*endowed with the norm*

$$\|u\|_l = \left(\int_{\mathbb{R}} |u(x)|^2dx + \int_{\mathbb{R}} |{}_{-\infty}\mathscr{D}_x^s u(x)|^2dx\right)^{1/2}.$$

*In the same way, we define*

$${}_x\mathbb{I}_\infty^\alpha(\mathbb{R}) = \{u \in L^2(\mathbb{R}) : \ {}_x\mathscr{D}_\infty^s u \in L^2(\mathbb{R})\}.$$

*endowed with the norm*

$$\|u\|_r = \left(\int_{\mathbb{R}} |u(x)|^2dx + \int_{\mathbb{R}} |{}_x\mathscr{D}_\infty^s u(x)|^2dx\right)^{1/2}.$$

**Theorem 4.** *[47] Given $s \in (0,1)$, the fractional spaces ${}_{-\infty}\mathbb{I}_x^s(\mathbb{R}), {}_x\mathbb{I}_\infty^s(\mathbb{R})$ and $H^s(\mathbb{R})$ are identical spaces with equal norms, where $H^s(\mathbb{R})$ is the classical fractional Sobolev spaces.*

*Remark.* Since $C_0^\infty(\mathbb{R})$ is dense in $H^s(\mathbb{R})$, then by Theorem 4, we have that $C_0^\infty(\mathbb{R})$ is dense in ${}_{-\infty}\mathbb{I}_x^s(\mathbb{R}), {}_x\mathbb{I}_x^s(\mathbb{R})$.

**Theorem 5.** *If $u \in \mathbb{I}_-^s(\mathbb{R})$, then*

$$\mathscr{F}({}_{-\infty}\mathscr{D}_x^s u(x))(\xi) = (i\xi)^s \hat{u}(\xi).$$

*Proof.* Since $u \in \mathbb{I}_-^s(\mathbb{R})$, there exists $(\varphi_n)_{n\in\mathbb{N}} \subset C_0^\infty(\mathbb{R})$ such that

$$\|u - \varphi_n\|_l \to 0 \ \text{ as } \ n \to +\infty.$$

Then

$$\|u - \varphi_n\|_{L^2(\mathbb{R})} \to 0 \quad \text{and} \quad \|{}_{-\infty}\mathscr{D}_x^s u - {}_{-\infty}\mathscr{D}_x^s \varphi_n\|_{L^2(\mathbb{R})} \to 0 \ \text{ as } \ n \to +\infty. \tag{15}$$

Moreover, by (12) we have

$$\mathscr{F}({}_{-\infty}\mathscr{D}_x^s \varphi_n)(\xi) = (i\xi)^s \hat{\varphi}_n(\xi) \ \ \forall n.$$

Hence, by Plancherel theorem and (15) we get

$$\begin{aligned}&\|(i\xi)^s\hat{u} - \widehat{{}_{-\infty}\mathscr{D}_x^s u}\|_{L^2(\mathbb{R})} \le \|(i\xi)^s[\hat{u} - \hat{\varphi}_n]\|_{L^2(\mathbb{R})} + \|(i\xi)^s\hat{\varphi}_n - \widehat{{}_{-\infty}\mathscr{D}_x^s\varphi_n}\|_{L^2(\mathbb{R})} + \|\widehat{{}_{-\infty}\mathscr{D}_x^s(u-\varphi_n)}\|_{L^2(\mathbb{R})} \\ &\le \|u-\varphi_n\|_{L^2(\mathbb{R})} + \|{}_{-\infty}\mathscr{D}_x^s(u-\varphi_n)\|_{L^2(\mathbb{R})} \to 0 \ \text{ as } \ n \to +\infty\end{aligned}$$

Therefore

$$\mathscr{F}({}_{-\infty}\mathscr{D}_x^s u)(\xi) = (i\xi)^s\hat{u}(\xi) \ \text{ a.e. } \ \xi \in \mathbb{R}. \quad \square$$

As a consequence of this theorem we have the following result

**Corollary 1.** *If $u \in \mathbb{I}_-^s(\mathbb{R})$, then*

$$u = {}_{-\infty}I_x^s({}_{-\infty}\mathscr{D}_\infty^s u) \ \ a.e. \ \ x \in \mathbb{R}.$$

*Proof.* Since $u \in \mathbb{I}^s_-(\mathbb{R})$, then ${}_{-\infty}\mathscr{D}^s_x u \in L^2(\mathbb{R})$ and by Theorem 5 we have

$$\mathscr{F}({}_{-\infty}\mathscr{D}^s_x u) = (i\xi)^s \hat{u}.$$

Hence, by (11) and (12) we get

$$\begin{aligned}\mathscr{F}({}_{-\infty}I^s_x({}_{-\infty}\mathscr{D}^s_x u(x)))(\xi) &= (i\xi)^{-s}\widehat{{}_{-\infty}\mathscr{D}^s_x u(x)}(\xi)\\ &= (i\xi)^{-s}(i\xi)^s\hat{u}(\xi) = \hat{u}(\xi) \ \text{ a.e. } \ \xi \in \mathbb{R}.\end{aligned}$$

By the inversion Fourier theorem we get the desired result. □

*Remark.* Note that, if $u \in \mathbb{I}^s_-(\mathbb{R})$, then ${}_{-\infty}\mathscr{D}^s_x u \in L^2(\mathbb{R})$, moreover by Corollary 1 we get

$${}_{-\infty}I^{1-s}_x u(x) = {}_{-\infty}I^{1-s}_x\left({}_{-\infty}I^s_x {}_{-\infty}\mathscr{D}^s_x u(x)\right) = \int_{-\infty}^x {}_{-\infty}\mathscr{D}^s_\sigma u(\sigma)d\sigma. \quad (16)$$

Hence, integrating by parts

$$\begin{aligned}\int_{-\infty}^{\infty} {}_{-\infty}I^{1-s}_x u(x)\varphi'(x)dx &= \int_{-\infty}^{\infty}\frac{d}{dx}\left(\int_{-\infty}^x {}_{-\infty}\mathscr{D}^s_\sigma u(\sigma)d\sigma\right)\varphi(x)dx\\ &= \int_{-\infty}^{\infty} {}_{-\infty}\mathscr{D}^s_x u(x)\varphi(x)dx, \ \ \forall \varphi \in C^\infty_0(\mathbb{R}).\end{aligned}$$

So, by definition of weak derivative we get

$$\frac{d}{dx}{}_{-\infty}I^{1-s}_x u = {}_{-\infty}\mathscr{D}^s_x u,$$

where $\frac{d}{dx}$ is understood in the weak sense. Moreover, as ${}_{-\infty}\mathscr{D}^s_x u \in L^2(\mathbb{R})$, then ${}_{-\infty}I^{1-s}_x u \in H^1(\mathbb{R})$. Therefore, we can characterize the fractional space $\mathbb{I}^s_-(\mathbb{R})$ as:

$$\begin{aligned}\mathbb{I}^s_-(\mathbb{R}) &= \left\{u \in L^2(\mathbb{R}) : \ {}_{-\infty}\mathscr{D}^s_x u \in L^2(\mathbb{R})\right\}\\ &= \left\{u \in L^2(\mathbb{R}) : \ {}_{-\infty}I^{1-s}_x u \in H^1(\mathbb{R}) \text{ and } {}_{-\infty}\mathscr{D}^s_x u = \frac{d}{dx}{}_{-\infty}I^{1-s}_x u \text{ in the weak sense}\right\}.\end{aligned}$$

## *2.2 Embedding results*

Now we consider some embedding properties.

**Theorem 6.** *If $s \in (0, \frac{1}{2})$, then the embedding $\mathbb{I}^s_-(\mathbb{R}) \hookrightarrow L^{2^*_s}(\mathbb{R})$ is continuous and there is a positive constant $\mathscr{S}$ such that*

$$\|u\|_{L^{2^*_s}(\mathbb{R})} \le \mathscr{S}\|{}_{-\infty}\mathscr{D}^s_x u\|_{L^2(\mathbb{R})} \ \ \forall u \in \mathbb{I}^s_-(\mathbb{R}),$$

*where $2^*_s = \frac{2}{1-2s}$ is called the fractional Sobolev exponent.*

*Proof.* As $u \in \mathbb{I}^s_-(\mathbb{R})$, then ${}_{-\infty}\mathscr{D}^s_x u \in L^2(\mathbb{R})$. So by Corollary 1 and Theorem 3 we get

$$\begin{aligned}\|u\|_{L^{2^*_s}(\mathbb{R})} &= \|{}_{-\infty}I^s_x({}_{-\infty}\mathscr{D}^s_x)u\|_{L^{2^*_s}(\mathbb{R})}\\ &\le \mathscr{S}\|{}_{-\infty}\mathscr{D}^s_x u\|_{L^2(\mathbb{R})}. \quad \square\end{aligned}$$

As a consequence of this theorem, we have:

**Corollary 2.** *If $s \in (0, \frac{1}{2})$ and $\frac{1}{2^*_s} = \frac{1}{2} - s$, then*

$$\mathbb{I}^s_-(\mathbb{R}) \subset L^p(\mathbb{R})$$

*with continuous embedding for every $2 \le p \le 2^*_s$. That is, there is a constant $C > 0$ such that*

$$\|u\|_{L^p(\mathbb{R})} \le C\|u\|_I, \ \ \forall u \in \mathbb{I}^s_-(\mathbb{R}).$$

*Proof.* Let $u \in \mathbb{I}^s_-(\mathbb{R})$, then $u \in L^2(\mathbb{R})$ and by Theorem 6 $u \in L^{2^*_s}(\mathbb{R})$, thus $u \in L^2(\mathbb{R}) \cap L^{2^*_s}(\mathbb{R})$. Let $\theta \in (0,1)$ such that

$$\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{2^*_s}.$$

By Theorem 6, interpolation inequality and Young inequality we obtain

$$\begin{aligned}\|u\|_{L^p(\mathbb{R})} &\le \|u\|^{\theta}_{L^2(\mathbb{R})}\|u\|^{1-\theta}_{L^{2^*_s}(\mathbb{R})}\\ &\le \theta\|u\|_{L^2(\mathbb{R})} + (1-\theta)\|u\|_{L^{2^*_s}(\mathbb{R})}\\ &\le \|u\|_{L^2(\mathbb{R})} + \mathscr{S}\|{}_{-\infty}\mathscr{D}^s_x u\|_{L^2(\mathbb{R})} \le C\|u\|_{I},\end{aligned}$$

where $C = 1+\mathscr{S}$. □

**Theorem 7.** *If $\alpha = \frac{1}{2}$. Then*

$$\mathbb{I}^{1/2}_-(\mathbb{R}) \subset L^p(\mathbb{R}),$$

*with continuous embedding for all $p \in [2,+\infty)$.*

*Proof.* Let $\varepsilon > 0$ and $\beta = \frac{1}{2} - \varepsilon < \frac{1}{2}$. Hence, by Theorem 6 with $s$ replaced by $\beta$, we get

$$\|u\|_{L^{2^*_\beta}(\mathbb{R})} \le \mathscr{S}\|{}_{-\infty}\mathscr{D}^{\beta}_x u\|_{L^2(\mathbb{R})}.$$

Moreover, since $2^*_\beta = \frac{2}{1-(\frac{1}{2}-\varepsilon)}$ can be arbitrarily large and

$$\|{}_{-\infty}\mathscr{D}^{\frac{1}{2}-\varepsilon}_x u\|_{L^2(\mathbb{R})} \to \|{}_{-\infty}\mathscr{D}^{\frac{1}{2}}_x u\|_{L^2(\mathbb{R})}$$

as $\varepsilon \to 0$, we conclude that $\mathbb{I}^{\frac{1}{2}}_-(\mathbb{R})$ is embedded into $L^p(\mathbb{R})$ for every $p \in [2,\infty)$. □

For $s \in (0,1)$ we introduce a new fractional space

$$H^s_B(\mathbb{R}) := \left\{u \in \mathbb{I}^s_-(\mathbb{R}) : \int_{\mathbb{R}} B(x)|u(x)|^2 dx < \infty\right\}$$

equipped with the inner product

$$\langle u, w\rangle = \int_{\mathbb{R}} {}_{-\infty}\mathscr{D}^s_x u(x)\,{}_{-\infty}\mathscr{D}^s_x w(x)dx + \int_{\mathbb{R}^n} B(x)u(x)w(x)dx$$

and the norm

$$\|u\|^2 = \int_{\mathbb{R}} |{}_{-\infty}\mathscr{D}^s_x u(x)|^2 dx + \int_{\mathbb{R}^n} B(x)|u(x)|^2 dx.$$

**Proposition 3.** *Let $s \in (0,1)$. Suppose that $(B,\Lambda_1),(B,\Lambda_2) \in \mathscr{E}$ hold. Then the embedding*

$$H^s_B(\mathbb{R}^N) \hookrightarrow L^p_{\Lambda_1}(\mathbb{R}^N) \;\; and \;\; H^s_B(\mathbb{R}^N) \hookrightarrow L^p_{\Lambda_2}(\mathbb{R}^N)$$

*are continuous and compact for $p \in [2, 2^*_s)$.*

*Proof.* Since $p \in [2,2^*_s)$, then $\delta = \frac{2^*_s - p}{2^*_s - 2} \le 1$. Then, for some $R > 0$ we have

$$\begin{aligned}\int_{\mathbb{R}} \Lambda(x)|u|^p dx &= \int_{\mathbb{R}} B^{\delta}(x)\omega(x)|u|^p dx\\ &= \int_{-R}^{R} B^{\delta}(x)\omega(x)|u|^p dx + \int_{(-R,R)^c} B^{\delta}(x)\omega(x)|u|^p dx.\end{aligned}$$

For any $p' > p$, we have by Hölder inequality

$$\int_{-R}^{R} \Lambda(x)|u|^p dx \le \left(\int_{-R}^{R} |u|^{p'} dx\right)^{\frac{p}{p'}} \left(\int_{-R}^{R} \Lambda^r(x)dx\right)^{\frac{1}{r}} \tag{17}$$

where $r$ is such that

$$r \geq \frac{2_s^*}{2_s^* - p} \quad \text{and} \quad \frac{1}{r} + \frac{p}{p'} = 1. \tag{18}$$

Note that (18) is satisfies if and only if $p' \leq 2_s^*$. So for such $p'$ the embedding $\mathbb{I}_-^s(-R,R) \hookrightarrow L^{p'}(-R,R)$ holds. Moreover, if $B_R = \inf_{(-R,R)} B(x) > 0$, then

$$\|u\|^2_{\mathbb{I}_-^s(-R,R)} \leq \frac{B_R + 1}{B_R} \|u\|^2_{H_B^s(-R,R)} \tag{19}$$

Since $\Lambda \in L^\infty(-R,R)$, then $\Lambda \in L^\infty(-R,R)$ and

$$\int_{-R}^{R} \Lambda^r(x)dx \leq 2R\|\Lambda\|^r_{L^\infty(-R,R)}. \tag{20}$$

Therefore, by (17)-(20) we get

$$\begin{aligned} \int_{-R}^{R} \Lambda(x)|u|^p dx &\leq \left(\int_{-R}^{R} |u|^{p'} dx\right)^{\frac{p}{p'}} \left(\int_{-R}^{R} \Lambda^r(x)dx\right)^{\frac{1}{r}} \\ &\leq 2R\|\Lambda\|^r_{L^\infty((-R,R)} C^p_{p'} \|u\|^p_{\mathbb{I}_-^s(-R,R)} \\ &\leq 2R\|\Lambda\|^r_{L^\infty((-R,R))} C^p_{p'} \|u\|^p \end{aligned} \tag{21}$$

On the other hand, since $\omega(x) \to 0$ as $|x| \to \infty$, then there exists $M > 0$ such that

$$\omega(x) \leq M.$$

So by Hölder inequality and Theorem 6 we have

$$\begin{aligned} \int_{(-R,R)^c} B^\delta(x)\omega(x)|u|^p dx &\leq M \int_{(-R,R)^c} B^\delta(x)|u|^{\frac{2(2_s^*-p)}{2_s^*-2}} |u|^{\frac{2_s^*(p-2)}{2_s^*-2}} dx \\ &\leq M\left(\int_{\mathbb{R}} B(x)|u|^2 dx\right)^\delta \left(\int_{\mathbb{R}} |u|^{2_s^*} dx\right)^{\frac{p-2}{2_s^*-2}} \\ &\leq MC^{\frac{2_s^*(p-2)}{2(2_s^*-2)}} \|u\|^{2\delta} \|u\|^{\frac{2_s^*(p-2)}{2_s^*-2}} = \tilde{C}\|u\|^p \end{aligned} \tag{22}$$

Finally by (20) and (21)-(22), there exists a positive constant $\tilde{C}$ such that

$$\int_{\mathbb{R}} \Lambda(x)|u|^p dx \leq \tilde{C}\|u\|^p,$$

which implies that the embedding $H_B^s(\mathbb{R}) \hookrightarrow L^p_\Lambda(\mathbb{R})$ is continuous for all $p \in [2, 2_s^*)$.

We now turn to compactness. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of functions in $H_B^s(\mathbb{R})$ with

$$u_n \rightharpoonup 0 \text{ as } n \to \infty \quad \text{and} \quad \|u_n\| \text{ uniformly bounded.}$$

Since $\omega(x) \to 0$ as $|x| \to \infty$, given $\varepsilon > 0$, there is $R > 0$ such that

$$\omega(x) < \varepsilon \text{ for all } |x| > R.$$

Then as in (22) we get

$$\int_{(-R,R)^c} B^\delta \omega(x)|u_n|^p dx \leq \varepsilon\tilde{C}\|u_n\|^p. \tag{23}$$

On the other hand, by (19) and Sobolev theorem, the embedding $H_B^\alpha(-R,R) \hookrightarrow L^{p'}(-R,R)$ is compact, where $p'$ is given by (18). Then by (18) we get

$$\int_{-R}^{R} \Lambda(x)|u_n|^p dx \leq \varepsilon\tilde{C}. \tag{24}$$

Using (23) and (24) we reach the desired conclusion. □

Let $E = H^s_B(\mathbb{R}) \times H^s_B(\mathbb{R})$ with the inner product on $E$ given by

$$\langle (u,v),(\phi,\psi)\rangle_E = \int_{\mathbb{R}} {}_{-\infty}\mathscr{D}^s_x u(x) {}_{-\infty}\mathscr{D}^s_x \varphi(x)dx \int_{\mathbb{R}^N} B(x)u\phi dx + \int_{\mathbb{R}} {}_{-\infty}\mathscr{D}^s_x v(x) {}_{-\infty}\mathscr{D}^s_x \psi(x)dx + B(x)v\psi dx,$$

and corresponding norm

$$\|z\|^2_E = \int_{\mathbb{R}} |{}_{-\infty}\mathscr{D}^s_x u(x)|^2 dx + \int_{\mathbb{R}} B(x)u^2 dx + \int_{\mathbb{R}} |{}_{-\infty}\mathscr{D}^s_x v(x)|^2 dx + \int_{\mathbb{R}} B(x)v^2 dx,$$

with $z=(u,v)$. The fractional space $(E,\langle\cdot,\cdot\rangle)$ is a Hilbert space with dual $E^*$. We notice that we can decompose $E = E^+ \oplus E^-$, where $E^+, E^-$ are infinite dimensional and are defined as

$$E^+ := \{(u,u)\ ;\ u \in H^s_B(\mathbb{R})\},\quad E^- := \{(u,-u)\ ;\ u \in H^s_B(\mathbb{R})\}.$$

Moreover, for each $z=(u,v)\in E$ can be written as $z = z^+ + z^-$ where

$$z^+ = \left(\frac{u+v}{2}, \frac{u+v}{2}\right),\quad z^- = \left(\frac{u-v}{2}, -\frac{u-v}{2}\right).$$

Hence

$$\int_{\mathbb{R}} {}_{-\infty}\mathscr{D}^s_x u(x) {}_{-\infty}\mathscr{D}^s_x v(x)dx + \int_{\mathbb{R}} B(x)uvdx = \frac{1}{2}(\|z^+\|^2_E - \|z^-\|^2_E). \tag{25}$$

The energy functional $\mathbf{I} : E \to \mathbb{R}$ associated to problem (1) is defined as

$$\mathbf{I}(z) = \mathbf{I}(u,v) = \int_{\mathbb{R}} {}_{-\infty}\mathscr{D}^s_x u(x) {}_{-\infty}\mathscr{D}^s_x v(x)dx + \int_{\mathbb{R}} B(x)uvdx - \int_{\mathbb{R}} \Lambda_1(x)F(v)dx - \int_{\mathbb{R}} \Lambda_2(x)G(u)dx, \tag{26}$$

where $z=(u,v)$. Moreover $\mathbf{I}\in C^1(E,\mathbb{R})$ and its Fréchet derivative is given by

$$\begin{aligned}\mathbf{I}'(u,v)(\phi,\psi) = &\int_{\mathbb{R}} {}_{-\infty}\mathscr{D}^s_x u(x) {}_{-\infty}\mathscr{D}^s_x \phi(x)dx + \int_{\mathbb{R}} B(x)u\phi dx + \int_{\mathbb{R}} {}_{-\infty}\mathscr{D}^s_x v(x) {}_{-\infty}\mathscr{D}^s_x \psi(x)dx + \int_{\mathbb{R}} B(x)v\psi dx \\ &- \int_{\mathbb{R}^N} \Lambda_1(x)f(v)\psi dx - \int_{\mathbb{R}^N} \Lambda_2(x)g(u)\phi dx\end{aligned} \tag{27}$$

for every $(u,v),\ (\phi,\psi)\in E$. Thus, a pairs of weak solutions of (1) correspond to a critical points of the energy functional $\mathbf{I}$. Furthermore, equality (25), means that the quadratic part of $\mathbf{I}$ is given by $\frac{1}{2}(\|z^+\|E^2 - \|z^-\|^2_E)$ and is strongly indefinite since both $E^+$ and $E^-$ are infinite dimensional.

Before proceeding, we consider some previos results introduced in [48], [49]. Set $(X,\|\cdot\|)$ be a real Banach space, $I\in C^1(X,\mathbb{R})$ and $c\in\mathbb{R}$. We say that a sequence $\{x_n\}\subset X$ is a Cerami sequence at level $c$ denote by $(C)_c$, if

$$I(x_n)\to c \quad \text{and} \quad (1+\|x_n\|)\|I'(x_n)\|_* \to 0 \quad \text{as} \quad n\to\infty$$

where $\|\cdot\|_*)$ denote the norm of the dual space $X^*$. Moreover, we say that $I$ satisfies the Cerami condition if every $(C)_c$ sequence has a strongly convergent subsequence in $X$.

Let $H^-$ be a closed subspace of a separable Hilbert space $H$ with norm $\|\cdot\|_H$ and let $H^+ := (H^-)^\perp$. For $u\in H$ we shall write $u = u^+ + u^-$, where $u^\pm \in H^\pm$. On $H$ we define a new norm

$$\|u\|_\tau := \max\left\{\|u^+\|_H, \sum_{\Lambda=1}^{\infty} \frac{1}{2^\Lambda}|\langle u^-, e_\Lambda\rangle|\right\},$$

where $\{e_\Lambda\}$ is a total orthonormal sequence in $H^-$. The topology induced by $\|\cdot\|_\tau$ is called the $\tau$-topology. We recall from [48] that a homotopy $h = I - g : A\times[0,1]\to H$ is called admissible, with $A\subset H$, if

(i)$h$ is $\tau$-continuous, which means, $h(u_n,s_n)\to h(s,u)$ in $\tau$-topology as $n\to\infty$ whenever $u_n\to u$ in $\tau$-topology and $s_n\to s$ as $n\to\infty$;

(ii)$g$ is $\tau$-locally finite-dimensional, i.e., for each $(u,s) \in A \times [0,1]$ there is a neighborhood $U$ of $(u,s)$ in the product topology of $(H,\tau)$ and $[0,1]$ such that $g(U \cap (A \times [0,1]))$ is contained in a finite-dimensional subspace of $H$.

Notice that admissible homotopies are continuous in the strong topology. Also, if $\{u_m\}$ is a bounded sequence in $H$, then $u_m \to u$ in the $\tau$-topology if, and only if, $u_m \rightharpoonup u$ in $H^-$ and $u_m \to u$ in $H^+$.

To prove our main result we use the following proposition.

**Proposition 4.**[49, *Theorem 2.1] Let $H = H^+ \oplus H^-$ be a separable Hilbert space with $H^-$ orthogonal to $H^+$ and $\Phi \in C^1(H,\mathbb{R})$. Suppose*

*(i)$\Phi(z) = \frac{1}{2}(\|z^+\|^2 - \|z^-\|) - \Psi(z)$, where $\Psi \in C^1(H,\mathbb{R})$ is bounded below, weakly sequentially lower semicontinuous and $\Psi'$ is weakly sequentially continuous.*

*(ii)There exist $z_0 \in H^+ \setminus \{0\}$, $\alpha > 0$ and $R > r > 0$ such that $\Phi|_{N_r} \geq \alpha$ and $\Phi|_{\partial M_{R,z_0}} \leq 0$, where*

$$M_{R,z_0} = \{z = z^- + tz_0\ ;\ \|z\| \leq R,\ t \geq 0\},\ \ N_r = \{z \in H^+\ ;\ \|z\| = r\}.$$

*Then, there exists a $(C)_c$ sequence for $\Phi$, where*

$$c := \inf_{h \in \Gamma} \sup_{u \in M} \Phi(h(u,1)),$$

*and*

$$\Gamma := \{h \in C(M \times [0,1], H)\ ;\ h \text{ is admissible, } h(u,0) = u \text{ and } \Phi(h(u,s)) \leq \max\{\Phi(u), -1\} \text{ for all } s \in [0,1]\}.$$

*Moreover, $c \geq \alpha$.*

## 3 Cerami condition

In this section, we are going to show that the energy functional $\mathbf{I}$ satisfies the Cerami condition. Moreover, we show some convergence results which will be essential in the proof of Theorem 1. We start with the following remark:

*Remark.* Suppose that $(H_3)$ holds, then for $|s|$ large enough

$$|f(s)|^{\tau_1} \leq \frac{1}{2} c_0 |f(s)||s|^{\tau_1 + 1}.$$

Hence, combining $(H_1)$ with $(H_3)$, given $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$0 < f(s) \leq \varepsilon s + C_\varepsilon s^{p-1}, \ \text{ for } s > 0,$$

where $p \in (2, 2_s^*)$. The same holds for the function $g$, that is, under conditions $(H_1)$ and $(H_3)$, given $\varepsilon > 0$ there is $C_\varepsilon$ such that

$$0 < g(s) \leq \varepsilon s + C_\varepsilon s^{q-1}, \ \text{ for } s > 0,$$

where $q \in (2, 2_s^*)$.

We notice that, by (25) the energy functional $\mathbf{I}$ satisfies Proposition 4 part (i). The following lemma shows that Proposition 4 part (ii) also is satisfies for the functional $\mathbf{I}$.

**Lemma 3.** *Under the assumptions of Theorem 1 we obtain:*

*(i)$\mathbf{I}|_{N_r} \geq \alpha$ for some $r,\ \alpha > 0$;*

*(ii)For any $z_0 = (u_0, v_0) \in E^+ \setminus \{0\}$ with $\|z_0\| = 1$, there is $R > r$ such that $\mathbf{I}|_{\partial M_{R,z_0}} \leq 0$, where $r$ is given in (i).*

*Proof.* **(i)** For each $z \in N_r$, there exists $u \in H_B^s(\mathbb{R}^N)$ with $z = (u,u)$ and $\|z\| = r$. By Proposition 3 and the previous remark, we choose $r > 0$ small enough such that

$$\begin{aligned}\mathbf{I}(z) &= \int_{\mathbb{R}} |_{-\infty}\mathscr{D}_x^s u(x)|^2 dx + \int_{\mathbb{R}} B(x) u^2 dx - \int_{\mathbb{R}} \Lambda_1(x) F(u) dx - \int_{\mathbb{R}} \Lambda_2(x) G(u) dx \\ &\geq \|z\|_E^2 - \varepsilon \|z\|_E^2 - \frac{c_\varepsilon}{p}\|z\|_E^p - \frac{c_\varepsilon}{q}\|z\|_E^q = (1-\varepsilon)\|z\|_E^2 - \frac{c_\varepsilon}{p}\|z\|_E^p - \frac{c_\varepsilon}{q}\|z\|_E^q \geq \alpha > 0.\end{aligned}$$

So $(i)$ follows.
**(ii)** Let $z \in \partial M_{R,z_0}$, then $z = z^- + \rho z_0$ with $\|z\|_E = R$, $\rho > 0$ or $\|z\|_E < R$, $\rho = 0$. Suppose that $\rho = 0$, then $z \in E^-$, that is, $z = (u, -u)$ and $(\mathscr{E}_{(i)})$, Remark 3 yield that

$$\mathbf{I}(z) = \mathbf{I}(u,-u) = -\frac{1}{2}\|z^-\|_E^2 - \int_{\mathbb{R}} \Lambda_1(x)F(-u)dx - \int_{\mathbb{R}} \Lambda_2(x)G(u)dx \le 0.$$

Now, we assume that $\rho > 0$. By contradiction, suppose that there is a sequence

$$\{z_n\} \subset \partial M_{R_n,z_0},\ \ z_n = z_n^- + \rho_n z_0,\ \ \rho_n > 0,\ \ \|z_0\|_E = 1,\ \ \|z_n\|_E = R_n \to \infty,$$

such that

$$\mathbf{I}(z_n) = \mathbf{I}(u_n, v_n) = \frac{1}{2}(\rho_n^2\|z_0\|_E^2 - \|z_n^-\|_E^2) - \int_{\mathbb{R}} \Lambda_1(x)F(v_n)dx - \int_{\mathbb{R}} \Lambda_2(x)G(u_n)dx > 0.$$

Let $\delta_n = \dfrac{\rho_n}{\|z_n\|_E}$ and $w_n^- = \dfrac{z_n^-}{\|z_n\|_E}$. Hence,

$$\frac{\mathbf{I}(z_n)}{\|z_n\|_E^2} = \frac{\mathbf{I}(u_n,v_n)}{\|z_n\|_E^2} = \frac{1}{2}(\delta_n^2 - \|w_n^-\|_E^2) - \int_{\mathbb{R}} \Lambda_1(x)\frac{F(v_n)}{\|z_n\|_E^2}dx - \int_{\mathbb{R}} \Lambda_2(x)\frac{G(u_n)}{\|z_n\|_E^2} > 0. \tag{28}$$

Consequently

$$\delta_n \ge \|w_n^-\|_E. \tag{29}$$

Note that

$$\delta_n^2 + \|w_n^-\|_E^2 = \frac{\rho_n^2\|z_0\|_E^2}{\|z_n\|_E^2} + \frac{\|z_n^-\|_E^2}{\|z_n\|_E^2} = 1 \tag{30}$$

and then it follows from (29) and (30) that $\dfrac{1}{\sqrt{2}} \le \delta_n \le 1$ and $w_n^-$ is bounded. Hence, there is $\delta \ge 0$ such that up to a subsequence, $\delta_n \to \delta$ and $w_n^- \rightharpoonup w^- = (\phi, -\phi)$ in $E$ as $n \to \infty$.

Notice that if $\delta = 0$, then (28) yields that

$$\|w_n^-\|_E \to 0,\ \ \int_{\mathbb{R}} \Lambda_1(x)\frac{F(v_n)}{\|z_n\|_E^2} \to 0,\ \ \int_{\mathbb{R}} \Lambda_2(x)\frac{G(u_n)}{\|z_n\|_E^2} \to 0.$$

Thus,

$$1 = \delta_n^2 + \|w_n^-\|_E^2 \to 0$$

which is a contradiction.

Therefore, $\delta > 0$, i.e., $\dfrac{\rho_n^2}{\|z_n\|_E^2} \to \delta^2 > 0$ and as $\|z_n\|_E \to \infty$ as $n \to \infty$, it follows that $\rho_n \to \infty$. Notice that

$$\frac{u_n}{\|z_n\|_E} = \frac{\rho_n u_0 + \phi_n}{\|z_n\|_E} \rightharpoonup \delta u_0 + \phi$$

and

$$\frac{B_n}{\|z_n\|_E} = \frac{\rho_n u_0 - \phi_n}{\|z_n\|_E} \rightharpoonup \delta u_0 - \phi$$

in $E$. Hence, by Proposition 3, up to a subsequence we have

$$\frac{u_n(x)}{\|z_n\|_E} = \frac{\rho_n u_0(x) + \phi_n(x)}{\|z_n\|_E} \to \delta u_0(x) + \phi(x)$$

a.e. in $\mathbb{R}$ and

$$\frac{B_n(x)}{\|z_n\|_E} = \frac{\rho_n u_0(x) - \phi_n(x)}{\|z_n\|_E} \to \delta u_0(x) - \phi(x)$$

a.e. in $\mathbb{R}$ as $n \to \infty$.

Let us denote $A_1 = \{x \in \mathbb{R}\ ;\ \delta u_0(x) + \phi(x) \neq 0\}$. Hence, for a.e. $x \in A_1$

$$\lim_{n\to\infty} \frac{\rho_n u_0(x) + \phi_n(x)}{\|z_n\|_E} = \delta u_0(x) + \phi(x) \neq 0,$$

which means that for a.e. $x \in A_1$

$$u_n(x) = \rho_n u_0(x) + \phi_n(x) \to +\infty \ \text{ as } \ n \to \infty. \tag{31}$$

In the same way, if we denote $A_2 = \{x \in \mathbb{R}^N\ ;\ \delta u_0(x) - \phi(x) \neq 0\}$, then for a.e. $x \in A_2$ we have

$$v_n(x) = \rho_n u_0(x) - \phi_n(x) \to +\infty \ \text{ as } \ n \to \infty. \tag{32}$$

Fatou's lemma and (28), (31), (32), $(H_2)$ yield that

$$\begin{aligned}
0 &\leq \frac{1}{2}(\delta^2 - \|w^-\|_E^2) - \liminf_{n\to\infty} \int_{\mathbb{R}} \Lambda_1(x) \frac{F(v_n)v_n^2}{v_n^2\|z_n\|_E^2} - \liminf_{n\to\infty} \int_{\mathbb{R}} \Lambda_2(x) \frac{G(u_n)u_n^2}{u_n^2\|z_n\|_E^2} \\
&\leq \frac{1}{2}(\delta^2 - \|w^-\|_E^2) - \liminf_{n\to\infty} \int_{A_2} \Lambda_1(x) \frac{F(v_n)v_n^2}{v_n^2\|z_n\|_E^2} - \liminf_{n\to\infty} \int_{A_1} \Lambda_2(x) \frac{G(u_n)u_n^2}{u_n^2\|z_n\|_E^2} \\
&\leq \frac{1}{2}(\delta^2 - \|w^-\|_E^2) - \int_{A_2} \liminf_{n\to\infty} \Lambda_1(x) \frac{F(v_n)}{v_n^2}(\delta u_0 - \phi)^2 - \int_{A_1} \liminf_{n\to\infty} \Lambda_2(x) \frac{G(u_n)}{u_n^2}(\delta u_0 + \phi)^2 \\
&= -\infty
\end{aligned}$$

This is a contradiction and hence the Lemma 3 is proved. □

**Lemma 4.** *Let $\{z_n\} \subset E$ be a $(C)_c$-sequence of the energy functional* $\mathbf{I}$, *then $\{z_n\}$ is bounded in $E$*

*Proof.* Let $\{z_n\} \subset E$ such that

$$\mathbf{I}(z_n) \to c \ \text{ and } \ (1 + \|z_n\|_E)\|\mathbf{I}'(z_n)\|_{E^*} \to 0 \ \text{ as } \ n \to \infty. \tag{33}$$

Hence,

$$\mathbf{I}(z_n) \to c \quad \text{and} \quad \mathbf{I}'(z_n)z_n \to 0 \ \text{ as } \ n \to \infty. \tag{34}$$

Denote $z_n = (u_n, v_n)$. By (34), for $n$ large enough,

$$c + o(1) = \mathbf{I}(z_n) - \frac{1}{2}\mathbf{I}'(z_n)z_n = \frac{1}{2}\left( \int_{\mathbb{R}} \Lambda_1(x)\tilde{F}(v_n)dx + \int_{\mathbb{R}} \Lambda_2(x)\tilde{G}(u_n)dx \right). \tag{35}$$

By contradiction, suppose that $\|z_n\|_E \to \infty$. Set

$$w_n = \frac{z_n}{\|z_n\|_E} = \left( \frac{u_n}{\|z_n\|_E}, \frac{v_n}{\|z_n\|_E} \right) := (w_n^1, w_n^2).$$

So $\{w_n\}$ is bounded in $E$ with $\|w_n\|_E = 1$ and there is $w := (w^1, w^2) \in E$ such that up to a subsequence we have

$$w_n \rightharpoonup w := (w^1, w^2).$$

By Proposition 3

$$w_n(x) \to w(x) \ \text{ a.e. in } \mathbb{R}.$$

Since

$$\begin{aligned}
\mathbf{I}'(z_n)(z_n^+ - z_n^-) &= \int_{\mathbb{R}} |{}_{-\infty}\mathscr{D}_x^s u_n(x)|^2 dx + \int_{\mathbb{R}} B(x)u_n^2 dx + \int_{\mathbb{R}} |{}_{-\infty}\mathscr{D}_x^s v_n(x)|^2 dx + \int_{\mathbb{R}} B(x)v_n^2 dx \\
&\quad - \int_{\mathbb{R}} \Lambda_1(x)f(v_n)u_n dx - \int_{\mathbb{R}} \Lambda_2(x)g(u_n)v_n dx \\
&= \|z_n\|_E^2 - \int_{\mathbb{R}} \Lambda_1(x)f(v_n)u_n dx - \int_{\mathbb{R}} \Lambda_2(x)g(u_n)v_n dx.
\end{aligned}$$

Then

$$\frac{\mathbf{I}'(z_n)(z_n^+ - z_n^-)}{\|z_n\|_E^2} = 1 - \int_{\mathbb{R}} \Lambda_1(x)\frac{f(v_n)u_n}{\|z_n\|_E^2}dx - \int_{\mathbb{R}} \Lambda_2(x)\frac{g(u_n)v_n}{\|z_n\|_E^2}dx. \tag{36}$$

As $\|z_n\|_E = \|z_n^+ - z_n^-\|_E$, (34) and (36) yield that

$$\lim_{n\to\infty}\left(\int_{\mathbb{R}} \Lambda_1(x)\frac{f(v_n)u_n}{\|z_n\|_E^2}dx + \int_{\mathbb{R}} \Lambda_2(x)\frac{g(u_n)v_n}{\|z_n\|_E^2}dx\right) = 1. \tag{37}$$

Let $0 \le a < b \le +\infty$ and consider the following sets

$$A_n(a,b) = \{x \in \mathbb{R}\,;\ a \le v_n(x) < b\},$$
$$B_n(a,b) = \{x \in \mathbb{R}\,;\ a \le u_n(x) < b\}.$$

From (35), for $n$ sufficiently large we have

$$\begin{aligned} c + o(1) = &\int_{A_n(0,a)} \Lambda_1(x)\tilde{F}(v_n)dx + \int_{A_n(a,b)} \Lambda_1(x)\tilde{F}(v_n)dx + \int_{A_n(b,+\infty)} \Lambda_1(x)\tilde{F}(v_n)dx \\ &+ \int_{B_n(0,a)} \Lambda_2(x)\tilde{G}(u_n)dx + \int_{B_n(a,b)} \Lambda_2(x)\tilde{G}(u_n)dx + \int_{B_n(b,+\infty)} \Lambda_2(x)\tilde{G}(u_n)dx \end{aligned} \tag{38}$$

Set $C_3 > 0$ such that $\|w\|^2_{L^t_{\Lambda_1}(\mathbb{R})} \le C_3\|w\|^2$ for each $w \in H^s_B(\mathbb{R})$ and $t \in [2, 2^*]$. By $(f_1)$, there exists $a > 0$ such that

$$|f(s)| \le \frac{|s|}{12C_3}, \quad \text{for each } |s| \le a.$$

Hence, for all $n \in \mathbb{N}$, we have

$$\begin{aligned} \int_{A_n(0,a)} \Lambda_1(x)\frac{u_n f(v_n)}{\|z_n\|_E^2}dx &\le \int_{A_n(0,a)} \Lambda_1(x)\frac{u_n f(v_n)}{\|z_n\|_E^2}dx \\ &\le \frac{1}{12C_3}\int_{A_n(0,a)} \Lambda_1(x) w_n^1 w_n^2 dx \\ &\le \frac{1}{12C_3}\left(\int_{A_n(0,a)} \Lambda_1(x)|w_n^1|^2dx\right)^{\frac{1}{2}}\left(\int_{A_n(0,a)} \Lambda_1(x)|w_n^2|^2dx\right)^{\frac{1}{2}} \\ &\le \frac{1}{12}\|w_n^1\|\|w_n^2\| \le \frac{1}{12} \end{aligned}$$

In the same way, we get

$$\int_{B_n(0,a)} \Lambda_2(x)\frac{v_n g(u_n)}{\|z_n\|_E^2}dx \le \frac{1}{12}$$

for all $n \in \mathbb{N}$.

Now we consider the sets $A_n(b,+\infty)$ and $B_n(b,+\infty)$. By (38) we obtain

$$c + o(1) \ge \int_{A_n(b,+\infty)} \Lambda_1(x)\tilde{F}(v_n)dx \ge \inf_{|v|\ge b} \tilde{F}(v)\int_{A_n(b,+\infty)} \Lambda_1(x)dx.$$

Since $\lim_{b\to+\infty} \inf_{|v|\ge b} \tilde{F}(v) = +\infty$, it follows that

$$\lim_{b\to+\infty}\int_{A_n(b,+\infty)} \Lambda_1(x)dx = 0. \tag{39}$$

Hence,

$$\begin{aligned} \int_{A_n(b,+\infty)} \Lambda_1(x)|w_n^2|^{s_1}dx &= \int_{A_n(b,+\infty)} \Lambda_1^{\frac{2_s^*-s_1}{2_s^*}}(x)\Lambda_1^{\frac{s_1}{2_s^*}}(x)|w_n^2|^{s_1}dx \\ &\le \left(\int_{A_n(b,+\infty)} \Lambda_1(x)dx\right)^{\frac{2_s^*-s_1}{2_s^*}}\left(\int_{A_n(b,+\infty)} \Lambda_1(x)|w_n^2|^{2_s^*}dx\right)^{\frac{s_1}{2_s^*}} \\ &\le C\left(\int_{A_n(b,+\infty)} \Lambda_1(x)dx\right)^{\frac{2_s^*-s_1}{2_s^*}} \to 0 \end{aligned} \tag{40}$$

as $b \to \infty$ uniformly in $n$, where

$$s_1 = \frac{1}{\frac{3}{2} - \frac{1}{\tau_1}},$$

$\tau_1$ is given in hypothesis $(f_3)$ and the last inequality follows from the facts that $\|w_n^1\| \leq 1$ and $\Lambda_1 \in L^\infty(\mathbb{R})$. In the same way we can show that

$$\lim_{b\to+\infty} \int_{B_n(b,+\infty)} \Lambda_2(x)|w_n^1|^{s_2}dx = 0 \tag{41}$$

uniformly in $n$, where

$$s_2 = \frac{1}{\frac{3}{2} - \frac{1}{\tau_2}}$$

and $\tau_2$ is given in hypothesis $(f_3)$. Thus, Hölder's inequality, $(f_3)$, (40), (38) and (40) yield that

$$\begin{aligned}
&\int_{A_n(b,+\infty)} \Lambda_1(x)\frac{f(v_n)u_n}{\|_E z_n\|_E^2}dx = \int_{A_n(b,+\infty)} \Lambda_1^{\frac{1}{\tau_1}}(x)\Lambda_1^{\frac{1}{s_1}}(x)\Lambda_1^{\frac{1}{2_s^*}}(x)\frac{f(v_n)u_n}{\|z_n\|_E^2}dx\\
&= \int_{A_n(b,+\infty)} \Lambda_1^{\frac{1}{\tau_1}}(x)\Lambda_1^{\frac{1}{s_1}}(x)\Lambda_1^{\frac{1}{2_s^*}}(x)\frac{f(v_n)w_n^1}{\|z_n\|_E}dx\\
&= \int_{A_n(b,+\infty)} \Lambda_1^{\frac{1}{\tau_1}}(x)\Lambda_1^{\frac{1}{s_1}}(x)\Lambda_1^{\frac{1}{2_s^*}}(x)\frac{f(v_n)}{v_n}w_n^1 w_n^2 dx\\
&\leq \left(\int_{A_n(b,+\infty)} \Lambda_1(x)\left(\frac{|f(v_n)|}{|v_n|}\right)^{\tau_1}dx\right)^{\frac{1}{\tau_1}}\left(\int_{A_n(b,+\infty)} \Lambda_1(x)|w_n^1|^{s_1}dx\right)^{\frac{1}{s_1}}\left(\int_{A_n(b,+\infty)} \Lambda_1(x)|w_n^2|^{2_s^*}dx\right)^{\frac{1}{2_s^*}}\\
&\leq \left(c_0\int_{A_n(b,+\infty)} \Lambda_1(x)\tilde{F}(v_n)dx\right)^{\frac{1}{\tau_1}}\left(\int_{A_n(b,+\infty)} \Lambda_1(x)|w_n^1|^{s_1}dx\right)^{\frac{1}{s_1}}\left(\int_{A_n(b,+\infty)} \Lambda_1(x)|w_n^2|^{2_s^*}dx\right)^{\frac{1}{2_s^*}}\\
&\leq C\left(\int_{A_n(b,+\infty)} \Lambda_1(x)|w_n^1|^{s_1}dx\right)^{\frac{1}{s_1}} < \frac{1}{12}
\end{aligned}$$

for $b$ large enough uniformly in $n$, where we can take $b$ large independent of $\frac{1}{12}$. Analogously, using generalized Hölder inequality and (41), one can show that

$$\int_{B_n(b,+\infty)} \Lambda_2(x)\frac{g(u_n)v_n}{\|z_n\|_E^2}dx < \frac{1}{12}.$$

Finally, we deal with the sets $A_n(a,b)$ and $B_n(a,b)$, where $a,b$ were chosen when we dealt with the sets $A_n(0,a)$ and $B_n(0,a)$ and the sets $A_n(b,+\infty)$ and $B_n(b,+\infty)$ respectively.

Firstly, notice that by Remark 3 there exist constants, which we denote by the same letter $C$ independent of $n$ which depends on $a$ and $b$, such that

$$|f(v_n)| \leq C|v_n| \tag{42}$$

for all $x \in A_n(a,b)$ and

$$|g(u_n)| \leq C|u_n| \tag{43}$$

for all $x \in B_n(a,b)$. Thus, from $\mathscr{E}_{(ii)}$, (42) and (43), for every $\eta \in \left(\frac{1}{48}, \frac{1}{24}\right)$, there is a $N_0 \in \mathbb{N}$ large enough such that

$$\frac{f(s)}{s} \leq \eta \frac{B(x)}{\Lambda_1(x)} \tag{44}$$

and

$$\frac{g(s)}{s} \leq \eta \frac{B(x)}{\Lambda_2(x)}, \tag{45}$$

for $s \in (a,b)$ and $|x| > N_0$.

So, we have

$$\begin{aligned}\int_{\substack{A_n(a,b)\\|x|>N_0}}\Lambda_1(x)\frac{f(v_n)u_n}{\|z_n\|_E^2}dx &= \int_{\substack{A_n(a,b)\\|x|>N_0}}\Lambda_1(x)\frac{f(v_n)w_n^1}{\|z_n\|_E}dx\\ &= \int_{\substack{A_n(a,b)\\|x|>N_0}}\Lambda_1(x)\frac{f(v_n)}{v_n}\frac{v_n}{\|z_n\|_E}w_n^1dx\\ &= \int_{\substack{A_n(a,b)\\|x|>N_0}}\Lambda_1(x)\frac{f(v_n)}{v_n}w_n^2w_n^1dx\\ &\le \int_{\substack{A_n(a,b)\\|x|>N_0}}\eta B(x)w_n^2w_n^1dx \le \int_{\mathbb{R}}\eta B(x)w_n^2w_n^1dx\\ &\le \eta\left(\int_{\mathbb{R}^N}B(x)|w_n^1|^2dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}B(x)|w_n^2|^2dx\right)^{\frac{1}{2}}\\ &\le \eta\|w_n^1\|\|w_n^2\| \le \eta < \frac{1}{24}.\end{aligned}$$

Analogously, we have

$$\int_{\substack{B_n(a,b)\\|x|>N_0}}\Lambda_2(x)\frac{g(u_n)v_n}{\|z_n\|_E^2}dx < \frac{1}{24}.$$

Now, since $\Lambda_1 \in L^\infty(\mathbb{R})$ then $\Lambda_1 \in L^1(A_n(a,b)\cap B_{N_0}(0))$ and since $\|_E z_n\|_E \to \infty$ as $n\to\infty$, we get

$$\int_{\substack{A_n(a,b)\\|x|\le N_0}}\Lambda_1(x)|w_n^2|^2dx = \frac{1}{\|z_n\|_E^2}\int_{\substack{A_n(a,b)\\|x|\le N_0}}\Lambda_1(x)v_n^2dx \le \frac{b^2}{\|z_n\|_E^2}\int_{\substack{A_n(a,b)\\|x|\le N_0}}\Lambda_1(x)dx \to 0 \tag{46}$$

as $n\to\infty$. In the same way we have

$$\int_{\substack{B_n(a,b)\\|x|\le N_0}}\Lambda_2(x)|w_n^1|^2dx \to 0 \tag{47}$$

as $n\to\infty$.

Consequently, by (42) and (46) we have

$$\begin{aligned}\int_{\substack{A_n(a,b)\\|x|\le N_0}}\Lambda_1(x)\frac{f(v_n)u_n}{\|z_n\|_E^2}dx &= \int_{\substack{A_n(a,b)\\|x|\le N_0}}\Lambda_1(x)\frac{f(v_n)w_n^1}{\|z_n\|_E}dx \le C\int_{\substack{A_n(a,b)\\|x|\le N_0}}\Lambda_1(x)\frac{v_nw_n^1}{\|z_n\|_E}dx\\ &= C\int_{\substack{A_n(a,b)\\|x|\le N_0}}\Lambda_1(x)w_n^1w_n^2dx \le C\|w_n^1\|_{H_B^1(\mathbb{R}^N)}\left(\int_{\substack{A_n(a,b)\\|x|\le N_0}}\Lambda_1(x)|w_n^2|^2dx\right)^{1/2}\\ &< \frac{1}{24},\end{aligned}$$

Analogously we see that

$$\int_{\substack{B_n(a,b)\\|x|\le N_0}}\Lambda_2(x)\frac{g(u_n)v_n}{\|z_n\|_E^2}dx < \frac{1}{24}.$$

Therefore, we obtain

$$\int_{A_n(a,b)}\Lambda_1(x)\frac{f(v_n)u_n}{\|z_n\|_E^2}dx < \frac{1}{12}$$

and

$$\int_{B_n(a,b)}\Lambda_2(x)\frac{g(u_n)v_n}{\|z_n\|_E^2}dx < \frac{1}{12}.$$

Gathering all these informations, we obtain

$$\int_{\mathbb{R}^N}\Lambda_1(x)\frac{f(v_n)u_n}{\|z_n\|_E^2}dx + \int_{\mathbb{R}^N}\Lambda_2(x)\frac{g(u_n)v_n}{\|z_n\|_E^2}dx < \frac{1}{2} < 1,$$

which is a contradiction with (37). Therefore, $\{z_n\}$ is bounded in $E$. □

**Lemma 5.** *Let $(B,\Lambda_1)\in\mathscr{E}$, $(B,\Lambda_2)\in\mathscr{E}$, $f$, $g$ satisfy $(H_1)-(H_3)$ and $\{z_n\}\subset E$ be a bounded sequence. If $z_n=(u_n,v_n)$ is such that $z_n\rightharpoonup z=(u,v)$ in $E$, then*

$$\int_{\mathbb{R}}\Lambda_1(x)f(v_n)u_n dx\to\int_{\mathbb{R}}\Lambda_1(x)f(v)udx$$

*and*

$$\int_{\mathbb{R}}\Lambda_2(x)g(u_n)v_n dx\to\int_{\mathbb{R}}\Lambda_2(x)g(u)vdx$$

*as $n\to\infty$.*

*Proof.* We only show the second part of the lemma. By Remark 3, for all $\varepsilon>0$, there is $c_\varepsilon>0$ large enough such that

$$g(t)\le\varepsilon|t|+c_\varepsilon|t|^{q-1}. \tag{48}$$

Since $(u_n,v_n)=z_n\rightharpoonup z=(u,v)$ in $E$, Proposition 3 implies that $u_n(x)\to u(x)$ a.e. in $\mathbb{R}$ and $u_n\to u$ in $L^t_{\Lambda_2}(\mathbb{R})$ for all $t\in[2,2^*)$. Thus,

$$\int_{\mathbb{R}}\Lambda_2(x)g(u_n)|v_n-v|dx\le\varepsilon\|u_n\|_{L^2_{\Lambda_2}(\mathbb{R})}\|v_n-v\|_{L^2_{\Lambda_2}(\mathbb{R})}+c_\varepsilon\|u_n\|^{q-1}_{L^q_{\Lambda_2}(\mathbb{R})}\|v_n-v\|_{L^q_{\Lambda_2}(\mathbb{R})}\to 0\ \text{ as } n\to\infty. \tag{49}$$

Moreover, since $u_n\to u$ in $L^t_{\Lambda_2}(\mathbb{R})$ for all $t\in[2,2^*)$, we have

$$\int_{\mathbb{R}}|\Lambda_2^{\frac{1}{t}}(x)u_n-\Lambda_2^{\frac{1}{t}}(x)u|^t dx\to 0\ \text{ as } n\to\infty.$$

So, by [46, Theorem 4.9], there is a subsequence of $\Lambda_2^{\frac{1}{t}}(x)u_n$ which we still denote by $\Lambda_2^{\frac{1}{t}}(x)u_n$ and a function $h\in L^t(\mathbb{R})$ such that $\Lambda_2^{\frac{1}{t}}(x)u_n(x)\to\Lambda_2^{\frac{1}{t}}(x)u(x)$ a.e. on $\mathbb{R}$ and $|\Lambda_2^{\frac{1}{t}}(x)u_n(x)|\le h(x)$, for all $n$, a.e. on $\mathbb{R}$. This facts combines with (48) yield that

$$\begin{aligned}|\Lambda_2(x)g(u_n)v|&\le\varepsilon\Lambda_2(x)|u_n|v+c_\varepsilon\Lambda_2(x)|u_n|^{q-1}v\\&=\varepsilon\Lambda_2^{\frac{1}{2}}(x)|u_n|\Lambda_2^{\frac{1}{2}}(x)v+c_\varepsilon(\Lambda_2(x)|u_n|^q)^{\frac{q-1}{q}}\Lambda_2^{\frac{1}{q}}(x)v\\&\le\left(\varepsilon h_1(x)\Lambda_2^{\frac{1}{2}}(x)v+c_\varepsilon h_2^{q-1}(x)\Lambda_2^{\frac{1}{q}}(x)v\right)\in L^1(\mathbb{R}),\end{aligned}$$

where $h_1\in L^2(\mathbb{R})$ and $h_2\in L^q(\mathbb{R})$ were obtained by [46, Theorem 4.9].

Therefore, by the Lebesgue dominated convergence theorem we obtain

$$\int_{\mathbb{R}}\Lambda_2(x)g(u_n)vdx\to\int_{\mathbb{R}}\Lambda_2(x)g(u)vdx \tag{50}$$

as $n\to\infty$.

Combining (49) and (50) we obtain

$$\int_{\mathbb{R}}\Lambda_2(x)g(u_n)v_n dx\to\int_{\mathbb{R}}\Lambda_2(x)g(u)vdx$$

as $n\to\infty$. In the same way, we can show that

$$\int_{\mathbb{R}}\Lambda_1(x)f(v_n)u_n dx\to\int_{\mathbb{R}}\Lambda_1(x)f(v)udx$$

as $n\to\infty$. The Lemma is proved. □

## 4 Proof of theorem 1

In this section we are going to prove Theorem 1 . Notice that $E=E^+\oplus E^-$, where

$$E^+=\{(u,u)\ ;\ u\in H^s_B(\mathbb{R})\}\quad\text{and}\quad E^-=\{(u,-u)\ ;\ u\in H^s_B(\mathbb{R})\}.$$

Moreover, for $z=(u,v)\in E$, we have

$$\mathbf{I}(z)=\frac{1}{2}(\|z^+\|_E-\|z^-\|_E)-\int_{\mathbb{R}}\Lambda_1(x)F(v)dx-\int_{\mathbb{R}}\Lambda_2(x)G(u)dx.$$

Note that the functional

$$\Phi(z)=\Phi(u,v)=\int_{\mathbb{R}}\Lambda_1(x)F(v)dx+\int_{\mathbb{R}}\Lambda_2(x)G(u)dx$$

is of class $C^1(E,\mathbb{R})$. Moreover, by Proposition 3 and Fatou's lemma we have that $\Phi(z)\geq 0$ is weakly lower semicontinuous and $\Phi'$ is weakly sequentially continuous in $E^*$. By Lemma 3, there exist $r>0$, $\alpha>0$ such that $\mathbf{I}|_{N_r}\geq\alpha$, where $N_r=\{z\in E^+\ ;\ \|z\|_E=r\}$ and for such $r$, there exist $R>r$ and a $z_0\in E^+\setminus\{0\}$ with $\|z_0\|_E=1$ such that $\mathbf{I}|_{\partial M_R}\leq 0$, with $M_R=\{z=z^-+\rho z_0\ ;\ z^-\in E^-,\ \|z\|_E\leq R,\ \rho\geq 0\}$.

Therefore, by Proposition 4 there is a $(C)_c$-sequence $\{z_n\}\subset E$ for $\mathbf{I}$ which is bounded in $E$ by Lemma 4. Then, up to a subsequence, we may assume that $z_n\rightharpoonup z$ in $E$.

Since $\|\mathbf{I}'(u_n,v_n)\|_{E^*}\to 0$, we have

$$\|z_n\|_E^2-\int_{\mathbb{R}}\Lambda_1(x)f(v_n)u_n-\int_{\mathbb{R}}\Lambda_2(x)g(u_n)v_n=\mathbf{I}'(u_n,v_n)(v_n,u_n)=o_n(1)$$

and hence

$$\lim_{n\to\infty}\|z_n\|_E^2=\lim_{n\to\infty}\left(\int_{\mathbb{R}}\Lambda_1(x)f(v_n)u_ndx+\int_{\mathbb{R}}\Lambda_2(x)g(u_n)v_ndx\right).$$

By Lemma 5,

$$\lim_{n\to\infty}\int_{\mathbb{R}}\Lambda_1(x)f(v_n)u_ndx=\int_{\mathbb{R}}\Lambda_1(x)f(v)udx$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}}\Lambda_2(x)g(u_n)v_ndx=\int_{\mathbb{R}}\Lambda_2(x)g(u)vdx,$$

then,

$$\lim_{n\to\infty}\|z_n\|_E^2=\int_{\mathbb{R}}\Lambda_1(x)f(v)udx+\int_{\mathbb{R}}\Lambda_2(x)g(u)vdx. \tag{51}$$

Also, since $\mathbf{I}'(u_n,v_n)(v,u)=o_n(1)$, we obtain

$$\|z\|_E^2=\int_{\mathbb{R}}\Big[\Lambda_1(x)f(v)u+\Lambda_2(x)g(u)v\Big]dx. \tag{52}$$

Hence, by (51) and (52),

$$\lim_{n\to\infty}\|z_n\|_E^2=\|z\|_E^2$$

which shows that $z_n\to z$ in $E$.
So, $z=(u,v)$ is a weak solution pair of problem (1) such that $\mathbf{I}(u,v)=c\geq\alpha>0$. □

## 5 Conclusion

The aim of this paper was to study the existence of weak solution for a class of fractional Hamiltonian system in $\mathbb{R}$ with potentials vanishing and order $s\in(0,\frac{1}{2})$. By introducing new compact embedding result, we are able to get our result by using generalized linking theorem.

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