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Mixture of Lindley and Weibull Distributions: Properties and Estimation

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Abstract: The present paper addresses a mixture of one-parameter Lindley and Weibull distributions from both practical and theoretical points of view. On the other hand, the aim of this paper is to set the record straight about this mixture. First, we introduce the mixture of one-parameter Lindley and Weibull distributions. Consequently, we study the main statistical properties of the proposed mixture model with some graphs of both density and hazard rate functions. Next, we estimate the unknown parameters of the mixture of one-parameter Lindley and Weibull distributions via the generalized method of moments and the maximum likelihood method. However, the bias, mean squared error and the relative efficiency of the estimated parameters are calculated using a Monte Carlo simulation study. The coverage probability and average width of the estimated intervals are also computed to examine the quality of the estimation methods. Finally, we evaluate the performance of our results with some simulation experiments and real data applications.

Keywords: Two component mixture, Properties, Reliability and hazard rate functions, GMMEs, MLEs

1 Introduction

In recent decades, mixture models have been recognized as being appropriate for describing different types of data in many applications such as biology, genetics, medicine, economics, engineering, marketing, reliability studies and life testing problems, social sciences and many other fields. Generally, a mixture distribution is made by combining two or more distributions using mixing parameters (weights) which are non-negative and the sum of these weights must be one. The distributions that are mixed are called the components of the mixture. Therefore, a mixture distribution of two sub-populations could be a suitable model for characterizing the overall population. Indeed, mixture models have been investigated by many authors; for example, Everitt and Hand [1], Titterington, Smith and Makov [2], McLachlan and Basford [3], Lindsay [4] and McLaclachlan and Peel [5]. In this paper, we propose a mixture of one-parameter Lindley and Weibull distributions (MLWD) as an example of mixture models that can be used in many real-life applications. It is well known that Lindley distribution is important for modelling various sets of life time data and reliability. Similarly, Weibull distribution is one of the most popular distributions used in reliability, engineering, hydrology, energy, ecology and the environment. Hence, the MLWD is proposed. Lindley distribution was proposed by Lindley [6] in the context of fiducial and Bayesian statistics to illustrate the difference between fiducial and posterior distributions. On the other hand, the statistical properties of Lindley distributions were discussed by Ghitany, Atieh and Nadarajah [7]. They showed that Lindley distribution is a better model for some applications than those based on exponential distribution. Mazucheli and Achcar [8] showed that Lindley distribution can be used effectively for modelling strength data and these distributions have been suggested as a possible alternative to exponential or Weibull distributions. Regarding Weibull distribution, Waloddi Weibull was the first to support the usefulness of these distributions for modeling datasets of widely differing characters. The first study of Weibull distribution was performed by Weibull [9]. Then, a subsequent study was done by Weibull [10] where he has modeled data sets from many different disciplines and applications. A similar model had

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been proposed earlier by Rosen and Rammler [11], and the earliest known publication dealing with Weibull distribution was conducted by Fisher and Tippet [12]. However, modified Weibull distribution was proposed by Lai, Murthy and Xie [13]. Our paper is organized as follows: in Section 1, we introduce one-parameter Lindley and Weibull distributions and address previous studies related to these distributions. We present MLWD formula in Section 2. Section 3 comprises the main statistical properties of the proposed mixture model with some graphs of the density and hazard rate functions. In Section 4, we discuss the estimation methods of the unknown parameters of the MLWD with the generalized method of moments (GMM) and the maximum likelihood method (ML). We investigate a simulation experiment in Section 5 to examine the performance of the estimation methods for estimating the unknown parameters of the underlying mixture model. In Section 6 we apply the MLWD to two real datasets. We present conclusion in Section 7.

2 Proposed Mixture Model

In this section, we show the formula of the mixture of one-parameter Lindley and Weibull distributions and illustrate its components. The first component is a Lindley distribution with one parameter. A one-parameter Lindley distribution is a mixture of two components, namely, an exponential distribution (with scale parameter θ) and a Gamma distribution (with shape parameter 2 and scale parameter θ) according to a mixing proportion that is equal to $\frac{\theta}{\theta+1}$. Then, the probability density function (pdf) of the one parameter Lindley distribution takes the form

$$f(x;\theta) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x},\ x>0,\ \theta>0. \tag{1}$$

The second component of the MLWD is a basic Weibull distribution with two parameters and its pdf is defined mathematically as

$$f(x;\alpha,\beta) = \frac{\beta}{\alpha^\beta}\, x^{(\beta-1)}\, e^{-(\frac{x}{\alpha})^\beta},\ x\geq 0,\ \alpha,\beta>0. \tag{2}$$

where β and α are the shape and scale parameters, respectively, of the Weibull distribution. Thus, the pdf of the MLWD takes the following form

$$\begin{aligned} f(x;p,\theta,\alpha,\beta) &= p\left(\frac{\theta^2}{\theta+1}\ (1+x)\,e^{-\theta x}\right) + (1-p)\left(\frac{\beta}{\alpha^\beta}\, x^{(\beta-1)}\, e^{-(\frac{x}{\alpha})^\beta}\right), \\ &0<p<1; x>0; \theta,\alpha,\beta>0, \end{aligned} \tag{3}$$

where p is the mixing parameter which is a non-negative proportion. On the other hand, the cumulative distribution function (cdf) of the proposed mixture model is given by

$$\begin{aligned} F(x;p,\theta,\alpha,\beta) &= p\,F_1(x;\theta) + (1-p)\,F_2(x;\alpha,\beta); \\ &0<p<1; x>0; \theta,\alpha,\beta>0, \end{aligned} \tag{4}$$

where $F_1(x;\theta)$ is the cdf of the Lindley distribution with one parameter θ, which is given as

$$F_1(x;\theta) = 1-\frac{(\theta+1+\theta x)e^{-\theta x}}{\theta+1};\ x>0,\ \theta>0, \tag{5}$$

and $F_2(x;\alpha,\beta)$ is the cdf of Weibull distribution with two parameters (α,β) which is given by

$$F_2(x;\alpha,\beta) = 1-e^{-(\frac{x}{\alpha})^\beta};\ x\geq 0,\ \alpha,\beta>0. \tag{6}$$

3 Properties of the MLWD

Ghitany et al. [7] demonstrated a one-parameter Lindley distribution and its applications. In addition, Shanker, Fesshaye and Selvaraj [14] illustrated the main statistical properties of a Lindley distribution for both one parameter and two parameters. Regarding Weibull distribution, Lai, Murthy and Xie [15] briefly presented the main statistical properties of a basic Weibull distribution and listed the various extensions. In this section, we explain some of the properties of the

Table 1: The mode(s) and median of the MLWD.

Modality	$(p,\ \theta,\ \alpha,\ \beta)$	Mode(s)	Median
Unimodal case	0.50,0.75,2,1.25	0.4899	1.55688
	0.50,0.25,2,1.25	0.6266	2.77031
	0.50,0.25,2,1.50	1.0279	2.67277
Bimodal case	0.95,0.25,0.5,2.0	1.1741 , 3.0000	5.55671
	0.75,0.25,0.5,2.0	1.4101 , 3.0000	3.95131
	0.55,0.25,0.5,2.0	1.5066 , 3.0000	1.32436

MLWD with some graphs of the pdf and hazard rate functions. It is noticeable that the discussed properties will include mean, variance, mode, median, skewness and kurtosis, reliability and hazard rate functions.

1.Mean and variance
The expected value of the MLWD is given as

$$E(X) = p\left(\frac{\theta+2}{\theta(\theta+1)}\right) + (1-p)\alpha\Gamma(1+\frac{1}{\beta}),$$
$$0 < p < 1; x > 0;\ \ \theta,\ \alpha,\ \beta > 0, \tag{7}$$

while the variance is obtained by

$$\begin{aligned}V(X) = p&\left[\frac{2(\theta+3)}{\theta^2(\theta+1)} - p(\frac{\theta+2}{\theta(\theta+1)})^2\right] \\ &+ (1-p)\left[\alpha^2\Gamma(1+\frac{2}{\beta}) - (1-p)\alpha^2(\Gamma(1+\frac{1}{\beta}))^2\right] \\ &- 2\,p\,(1-p)(\frac{\theta+2}{\theta(\theta+1)})\alpha\,\Gamma(1+\frac{1}{\beta}), \\ &0 < p < 1; x > 0; \theta,\ \alpha,\ \beta > 0.\end{aligned} \tag{8}$$

2.Mode and median
The mode (modes) of the proposed MLWD is (are) obtained by solving the following nonlinear equation with respect to x

$$\frac{p\,\theta^2\,e^{-\theta\,x}}{\theta+1}\left[1-\theta\,(1+x)\right] + \frac{(1-p)\,\beta\,e^{-(\frac{x}{\alpha})^\beta}}{\alpha^\beta}\left[(\beta-1)\,x^{(\beta-2)} - \frac{\beta\,x^{2(\beta-1)}}{\alpha^\beta}\right] = 0. \tag{9}$$

Therefore, based on the cdf of the underlying mixture model, the median of the MLWD is obtained by solving the next equation with respect to x

$$F(x;p,\theta,\alpha,\beta) = p\,F_1(x;\theta) + (1-p)\,F_2(x;\alpha,\beta) = 0.5, \tag{10}$$

where the first component $F_1(x;\theta)$ is the cdf of the Lindley distribution with parameter θ, see (5), and the second component $F_2(x;\alpha,\beta)$ is the cdf of the Weibull distribution with parameters α and β, as given in (6). Figure 1 shows the pdf plot of the MLWD in the unimodal and bimodal cases. In Table 1, we present the mode and median of the MLWD through different choices of parameter according to the unimodal and bimodal cases. Table 1 shows that in unimodal case we have 3 different collection of the parameters (θ,α,β) with fixed $p = 0.50$ to show the effect of changing the parameter values of θ and β on the mode and median. Thus, we found that mode and median values are increase when the shape parameter θ of Lindley distribution decreases with fixed values of (α,β). On the other hand, when the shape parameter β of Weibel distribution decrease, mode increases and median decreases. According to bimodal case we noticed that modes and median are affected significantly by growing in the mixing proportion p in which mode values decrease while median values increase. The parameter values are chosen based on unimodal and bimodal cases of the density function of the MLWD.

Table 2: Skewness and kurtosis for MLWD.

Modality	$(p,\ \theta,\ \alpha,\beta)$	Sk	Kur
Unimodal case	0.50,0.75,2,1.25	1.5626	6.6229
	0.50,0.25,2,1.25	2.1057	8.9590
	0.50,0.25,2,1.50	2.1596	9.1635
Bimodal case	0.95,0.25,0.5,2.0	1.4343	6.0218
	0.75,0.25,0.5,2.0	1.5472	6.1968
	0.55,0.25,0.5,2.0	1.5472	6.1968

3.Measures of skewness and kurtosis

The skewness and kurtosis coefficients of the MLWD can be obtained according to the rth moment about the origin of the proposed mixture model as

$$\mu_r^{'} = p = (\frac{r!(\theta+r+1)}{\theta^r(\theta+1)}) + (1-p)\ \alpha^r \Gamma(1+\frac{r}{\beta}),\ r = 1,2,...,n. \tag{11}$$

The skewness (Sk) and kurtosis (Kur) of the MLWD can be derived, respectively, as

$$Sk = \frac{\mu_3}{\sigma^3}, Kur = \frac{\mu_4}{\sigma^4}, \tag{12}$$

where $\mu_3 = 2\mu^3 - 3\mu_2^{'}\mu + \mu_3^{'}$, $\mu_4 = -3\mu^4 + 6\mu_2^{'}\mu^2 - 4\mu_3^{'}\mu + \mu_4^{'}$.

Thus, $\mu = E(X)$ is given in (7), while $\sigma^3 = (Var(X))^{\frac{3}{2}}$,$\sigma^4 = (Var(X))^2$ and $V(X)$ can be calculated as given in (8). Table 2 shows some of the skewness and kurtosis values based on the unimodal and bimodal cases of the MLWD. Table 2 illustrates that the proposed mixture model MLWD is right or positively skewed since the values of the skewness measure are positive. According to the unimodal case, it is noticeable that the skewness and kurtosis values are affected by the increase in p since they are increasing. However, the skewness and kurtosis decrease when shape parameter β increases according to the bimodal case of the MLWD.

4.Reliability and hazard rate functions

This part discusses the reliability and hazard rate functions of the MLWD. We study the behavior of the hazard rate function (HRF) as x tends to zero or infinity with some plots of the HRF according to the unimodal and bimodal cases. The reliability and hazard rate functions of the MLWD are obtained, respectively, as follows

$$\begin{aligned} R(x,p,\theta,\alpha,\beta) &= p\ R_1(x,\theta) + (1-p)\ R_2(x,\alpha,\beta), \\ &= p\left[\frac{(\theta+1+\theta x)\ e^{-\theta x}}{\theta+1}\right] + (1-p)\left(e^{-(\frac{x}{\alpha})^\beta}\right) \\ & 0 < p < 1;\ \ x > 0;\ \ \theta,\alpha,\beta > 0, \end{aligned} \tag{13}$$

and

$$\begin{aligned} r(x,p,\theta,\alpha,\beta) &= \frac{p\left(\frac{\theta^2}{\theta+1}\ \ (1+x)\ e^{-\theta x}\right) + (1-p)\left(\frac{\beta}{\alpha^\beta}x^{(\beta-1)}e^{-(\frac{x}{\alpha})^\beta}\right)}{p\left(\frac{(\theta+1+\theta x)\ \ e^{-\theta x}}{\theta+1}\right) + (1-p)\ e^{-(\frac{x}{\alpha})^\beta}}, \\ & 0 < p < 1;\ \ x > 0; \theta,\alpha,\beta > 0. \end{aligned} \tag{14}$$

Further, the HRF of the MLWD satisfies the following limits that are proven in Lemma below.

Lemma

$$\lim_{x\to 0} r(x;p,\theta,\alpha,\beta) = \frac{p\ \theta^2}{\theta+1}, \tag{15}$$

and

$$\lim_{x\to\infty} r(x;p,\theta,\alpha,\beta)=\theta. \tag{16}$$

Proof From (15), we solve and prove the limit as follows

$$\begin{aligned}
\lim_{x\to 0} r(x,p,\theta,\alpha,\beta) &= \lim_{x\to 0}\frac{p\left(\frac{\theta^2}{\theta+1}\ (1+x)\ e^{-\theta x}\right)+(1-p)\left(\frac{\beta}{\alpha^\beta}\ x^{(\beta-1)}\ e^{-(\frac{x}{\alpha})^\beta}\right)}{p\left(\frac{(\theta+1+\theta x)\ \ e^{-\theta x}}{\theta+1}\right)+(1-p)\ e^{-(\frac{x}{\alpha})^\beta}}\\
&=\frac{\frac{p\theta^2}{\theta+1}}{p+(1-p)}\\
&=\frac{p\theta^2}{\theta+1}.
\end{aligned} \tag{17}$$

To prove (16) we divide $r(x,p,\theta,\alpha,\beta)$ by $e^{-\theta x}$ and use the following definition with some properties of the limit as x tends to infinity with the following definition.
Definition Let $f(x),g(x)$ be functions defined on a reduced neighborhood of a, where a is a real number ∞ or $-\infty$. Then we say that $f\cong g$ at a if $\lim\limits_{x\to a}\frac{f(x)}{g(x)}=1$.

Now, suppose that $A=\frac{p\theta^2}{\theta+1}(1+x), B=\frac{(1-p)\beta}{\alpha^\beta}x^{(\beta-1)}e^{-(\frac{x}{\alpha})^\beta+\theta x}, C=p\left(\frac{\theta+1+\theta x}{\theta+1}\right)$ and $D=(1-p)e^{-(\frac{x}{\alpha})^\beta+\theta x}$. Then, we have

$$\begin{aligned}
\lim_{x\to\infty} r(x;p,\theta,\alpha,\beta) &= \lim_{x\to\infty}\frac{A+B}{C+D}\\
&=\lim_{x\to\infty}\frac{A}{C+D}+\lim_{x\to\infty}\frac{B}{C+D}.
\end{aligned} \tag{18}$$

We use some of the limit properties to solve the required limits as given below

$$\lim_{x\to\infty}\frac{A}{C+D}=\frac{\lim\limits_{x\to\infty}\frac{p\theta^2(1+x)}{\theta+1}}{\lim\limits_{x\to\infty}p\left(\frac{\theta+1+\theta x}{\theta+1}\right)+\lim\limits_{x\to\infty}(1-p)e^{-(\frac{x}{\alpha})^\beta+\theta x}} \tag{19}$$

where $\lim\limits_{x\to\infty}D=0$ since $-(\frac{x}{\alpha})^\beta+\theta x<0$. Therefore, we solve $\lim\limits_{x\to\infty}\frac{A}{C}$ after dividing the quantity $\frac{A}{C}$ by $(1+x)$ as follows

$$\lim_{x\to\infty}\frac{A}{C}=\lim_{x\to\infty}\frac{\theta^2}{\frac{1}{(1+x)}+\theta}=\theta. \tag{20}$$

Next,

$$\lim_{x\to\infty}\frac{B}{C+D}=\frac{\lim\limits_{x\to\infty}\frac{(1-p)\beta x^{(\beta-1)}\ e^{-(\frac{x}{\alpha})^\beta+\theta x}}{\alpha^\beta}}{\lim\limits_{x\to\infty}\frac{p(\theta+1+\theta x)}{\theta+1}+\lim\limits_{x\to\infty}(1-p)e^{-(\frac{x}{\alpha})^\beta+\theta x}}. \tag{21}$$

Since $\lim\limits_{x\to\infty}D=0$, then we have the limit as $\lim\limits_{x\to\infty}\frac{B}{C}$. For simplicity we represent B as given below

$$B=\frac{(1-p)\beta}{\alpha^\beta}x^{(\beta-1)}e^{-(\frac{x}{\alpha})^\beta+\theta x}=\frac{(1-p)\beta}{\alpha^\beta}\left(\frac{x^\beta}{x}\right)e^{-x^\beta\left[\frac{1}{\alpha^\beta}-\frac{\theta}{x^{(\beta-1)}}\right]}. \tag{22}$$

Now we multiply (22) by $\frac{-\alpha^\beta}{-\alpha^\beta}$ to obtain an equivalent function as illustrated in the previous definition. Thus, let $f(x)=\frac{1}{\alpha^\beta}-\frac{\theta}{x^{(\beta-1)}}$ and $g(x)=\frac{1}{\alpha^\beta}$, then

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\left(\frac{\frac{1}{\alpha^\beta}-\frac{\theta}{x^{(\beta-1)}}}{\frac{1}{\alpha^\beta}}\right)=\lim_{x\to\infty}\left(1-\frac{\theta\ \alpha^\beta}{x^{(\beta-1)}}\right)=1, \tag{23}$$

which implies that $f \cong g$ and we get

$$e^{-x^{\beta}\left(\frac{1}{\alpha^{\beta}}-\frac{\theta}{x^{(\beta-1)}}\right)} \cong e^{-x^{\beta}\left(\frac{1}{\alpha^{\beta}}\right)}. \tag{24}$$

Thus,

$$\lim_{x\to\infty} B = \frac{(1-p)\beta}{\alpha^{\beta}} \lim_{x\to\infty}\left(\frac{-\alpha^{\beta}}{x}\right) \lim_{x\to\infty}\left[-\left(\frac{x}{\alpha}\right)^{\beta} e^{-\left(\frac{x}{\alpha}\right)^{\beta}}\right] = 0. \tag{25}$$

Therefore, from (25) we get

$$\lim_{x\to\infty} \frac{B}{C+D} = 0. \tag{26}$$

Substituting the results (20) and (26) in (18), we prove the required limit as $x \longrightarrow \infty$ of the lemma. Figures 2 and 3 effectively show the validity of the lemma when $x \longrightarrow 0$ or ∞ since we have proven that the limit of the HRF when x tends to 0 is equal to $\frac{p\,\theta^2}{\theta+1}$ and as x tends to ∞, it is equal to θ [see Figure 2($p = 0.50$), Figure 3($p = 0.95$)]. Also, we illustrate four cases of the HRF graph for the MLWD to study the behavior of the HRF curve based on the unimodal and bimodal cases.

(1) Figure 2(a) and (b) are graphed using $p = 0.50$ and $p = 0.95$, respectively which represent the unimodal case of MLWD. In Figure 2, the HRF curve increases gradually until it reaches the value of θ, then it is stable. In addition, we conclude that the variation in mixing proportion p affects the increasing rate of the curve. Therefore, when p increases from 0.50 to 0.95, then the HRF curve of the MLWD matches the HRF curve of the one-parameter Lindley distribution with parameter $\theta = 0.75$. Moreover, the HRF curve increases in $(0,x^*)$, as shown in Figure 2, which denotes an increasing hazard rate until it reaches point x^* and in constant in (x^*,∞), which represents a constant hazard rate.

(2) Figure 3(a) and (b) are graphed using $p = 0.45$ and $p = 0.95$, respectively that show the bimodal case of MLWD. In Figure 3, the curve of HRF also increases gradually in $(0,x_1^*)$ to reach the mode values at $x_1^* = 1.3757$ and $x_1^* = 0.1674$ [see Figure 3(a) and (b)]. Then, in (x_1^*,x_2^*) the HRF curve begins to decrease until it reaches the value of θ in (x_2^*,∞). Besides, the variation in the mixing proportion p affects the HRF curve. Therefore, when p decreases from 0.95 to 0.45, the peak of the HRF curve grows according to the kurtosis value, which increases approximately from 6.02 to 8.83.

4 Estimation

This section illustrates the estimation methods of the unknown parameters of the MLWD via the generalized method of moments and the maximum likelihood, denoted by the GMM and ML, respectively.

4.1 Generalized Method of Moments

In this part, we get the GMM estimates for the MLWD by defining the coast function $Q_n(\Theta), \Theta = (p,\theta,\alpha,\beta)$ based on the moment conditions [see Hall [16]]. Therefore, we can get the GMM estimates of the unknown parameters $\Theta = (p,\theta,\alpha,\beta)$ by finding the parameter values that minimize the coast function $Q_n(\Theta)$ which measures the deviation of the moment conditions. Consequently, we define the coast function as follows

$$Q_n(\Theta) = n^{-1}\sum_{i=1}^{n}\left[x_i - E(X)\;\; x_i^2 - E(X^2)\;\; x_i^3 - E(X^3)\;\; x_i^4 - E(X^4)\right]'$$
$$\times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} n^{-1}\sum_{i=1}^{n}\begin{bmatrix} x_i - E(X) \\ x_i^2 - E(X^2) \\ x_i^3 - E(X^3) \\ x_i^4 - E(X^4) \end{bmatrix}$$
$$= \left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^2 - \left(\frac{2\sum_{i=1}^{n} x_i}{n}\right)E(X) + (E(X))^2 + \left(\frac{\sum_{i=1}^{n} x_i^2}{n}\right)^2$$
$$- \left(\frac{2\sum_{i=1}^{n} x_i^2}{n}\right)E(X^2) + (E(X^2))^2 + \left(\frac{\sum_{i=1}^{n} x_i^3}{n}\right)^2$$
$$- \left(\frac{2\sum_{i=1}^{n} x_i^3}{n}\right)E(X^3) + (E(X^3))^2 + \left(\frac{\sum_{i=1}^{n} x_i^4}{n}\right)^2$$
$$- \left(\frac{2\sum_{i=1}^{n} x_i^4}{n}\right)E(X^4) + (E(X^4))^2, \tag{27}$$

where

$$E(X) = p\left(\frac{\theta+2}{\theta(\theta+1)}\right) + (1-p)\alpha\Gamma(1+\frac{1}{\beta}),$$

$$E(X^2) = p\left(\frac{2(\theta+3)}{\theta^2(\theta+1)}\right) + (1-p)\alpha^2\Gamma(1+\frac{2}{\beta}),$$

$$E(X^3) = p\left(\frac{6(\theta+4)}{\theta^3(\theta+1)}\right) + (1-p)\alpha^3\Gamma(1+\frac{3}{\beta}),$$

and

$$E(X^4) = p\left(\frac{24(\theta+5)}{\theta^4(\theta+1)}\right) + (1-p)\alpha^4\Gamma(1+\frac{4}{\beta}).$$

Now, to get the GMM estimates of the MLWD we differentiate (27) with respect to each component of $\Theta = (p,\theta,\alpha,\beta)$ and equate these derivative equations to zero. Thus, we have the following system of non-linear equations

$$\frac{\partial Q_n(\Theta)}{\partial p} = 2\left[\left(\frac{24(5+\theta)}{\theta^4(\theta+1)} - \alpha^4\Gamma(1+\frac{4}{\beta})\right)\left(\frac{24p(5+\theta)}{\theta^4(\theta+1)} + (1-p)\alpha^4\Gamma(1+\frac{4}{\beta})\right)\right] = 0, \tag{28}$$

$$\frac{\partial Q_n(\Theta)}{\partial \theta} = 48p\left[\frac{1}{\theta^4(\theta+1)} - \left(\frac{\theta^3(4+5\theta)(5+\theta)}{(\theta^4(\theta+1))^2}\right)\left(\frac{24p(5+\theta)}{\theta^4(\theta+1)} + (1-p)\alpha^4\Gamma(1+\frac{4}{\beta})\right)\right] = 0, \tag{29}$$

$$\frac{\partial Q_n(\Theta)}{\partial \alpha} = \left[8(1-p)\alpha^3\Gamma(1+\frac{4}{\beta})\right]\left[\frac{24p(5+\theta)}{\theta^4(\theta+1)} + (1-p)\alpha^4\Gamma(1+\frac{4}{\beta})\right] = 0, \tag{30}$$

$$\frac{\partial Q_n(\Theta)}{\partial \beta} = -8\left[(1-p)\alpha^4\left(\frac{24p(5+\theta)}{\theta^4(\theta+1)} + (1-p)\alpha^4\Gamma(1+\frac{4}{\beta})\right)\psi\left(1+\frac{4}{\beta}\right)\left(\frac{\Gamma(1+\frac{4}{\beta})}{\beta^2}\right)\right] = 0, \tag{31}$$

where ψ is the digamma function. Therefore, we solve the system of nonlinear equations (28-31) and get the GMM estimates of the underlying model parameters. We use a (gmm) R software package to solve the system and obtain the estimates. All the numerical results are shown later in Section 5.

4.2 Maximum Likelihood Estimation

In this part, the maximum likelihood estimates for the unknown parameters of the MLWD are obtained. Thus, suppose $x_1, x_2, \ldots, x_n$ is a random sample from the MLWD, then the log-likelihood function is given as

$$L^* = \log L = \sum_{i=1}^{n} \log(p\frac{\theta^2}{\theta+1}(1+x_i)e^{-\theta x_i} + (1-p)\left(\frac{\beta}{\alpha^\beta}\, x_i^{(\beta-1)}\, e^{-(\frac{x_i}{\alpha})^\beta}\right),\ \ 0<p<1;\ \ x>0;\ \ \theta,\alpha,\beta>0 \tag{32}$$

Differentiating the log-likelihood function in (32) with respect to each parameter (p,θ,α,β), we get the first order derivatives of L^*. The maximum likelihood estimates of the four parameters of the MLWD can be obtained by equating these derivative equations to zero and solving this system of nonlinear equations as shown below

$$\frac{\partial L^*}{\partial p} = \sum_{i=1}^{n} \frac{f_1(x_i;\theta) - f_2(x_i;\alpha,\beta)}{pf_1(x_i;\theta)+(1-p)f_2(x_i;\alpha,\beta)} = 0, \tag{33}$$

$$\frac{\partial L^*}{\partial \theta} = \sum_{i=1}^{n} \frac{p(1+x_i)e^{-\theta x_i}\left(\frac{\theta(2+\theta)}{(\theta+1)^2} - \frac{x_i\theta^2}{(\theta+1)}\right)}{pf_1(x_i;\theta)+(1-p)f_2(x_i;\alpha,\beta)} = 0, \tag{34}$$

$$\frac{\partial L^*}{\partial \alpha} = \sum_{i=1}^{n} \frac{(1-p)\beta x_i^{(\beta-1)}e^{-(\frac{x_i}{\alpha})^\beta}\left[\beta x_i^\beta \alpha^{-(2\beta+1)} - \beta\alpha^{-(\beta+1)}\right]}{pf_1(x_i;\theta)+(1-p)f_2(x_i;\alpha,\beta)} = 0, \tag{35}$$

$$\frac{\partial L^*}{\partial \beta} = \sum_{i=1}^{n} \frac{(1-p)\left[\frac{\alpha^\beta x_i^{(\beta-1)}e^{-(\frac{x_i}{\alpha})^\beta}(1-\beta\log\alpha)}{\alpha^{2\beta}} + f_2(x_i;\alpha,\beta)(1+\log x_i) + (\frac{x_i}{\alpha})^\beta\log(\frac{\alpha}{x_i})\right]}{pf_1(x_i;\theta)+(1-p)f_2(x_i;\alpha,\beta)} = 0, \tag{36}$$

where $f_1(x_i;\theta)$ and $f_2(x_i;\alpha,\beta)$ are the pdf of the one-parameter Lindley distribution and Weibull distribution, as given in (1) and (2), respectively. The system of nonlinear equations (33-36) is solved using some R software packages to get the maximum likelihood estimates for the unknown parameters of the MLWD. The numerical results are presented in Section 5.

5 Simulation Study

This section contains the numerical results for the simulation experiment of the proposed mixture model. We perform some simulation studies for the MLWD, including solving the nonlinear equations of the GMM and ML, which were given in Section 4. Moreover, we use some statistical measures of the estimates to examine the performance of the estimation methods and the quality of the estimates. Thus, we calculate the bias, mean squared error (MSE) and the relative efficiency (RE) of the estimates according to the unimodal and bimodal cases. In addition, we estimate the confidence intervals for the MLWD parameters in the unimodal case and calculate the average width with the coverage probability of these estimated intervals for both the GMM and ML methods. The coverage probability is a statistical technique for calculating the proportion of the time that the confidence interval takes up or covering the true value of the population parameters. Consequently, to generate the random samples from the MLWD and to estimate the unknown parameters of the underlying mixture model, we apply the procedure of a Monte Carlo simulation experiment that is shown in the following steps:
(1) Generate random samples of sizes 50 and 100 from the MLWD model using different choices of mixing proportion p and the other model parameters (θ,α,β) according to the unimodal and bimodal cases. The random samples used in the simulation process are generated as illustrated below:

(i) Generate one variate u from the uniform distribution $U(0,1)$ by the uniform generator (runif) in the R software.
(ii) If $u \le p$, then we generate a random variate from the first component (one-parameter Lindley distribution) using the R function (rlindley). Otherwise, we generate a random variate from the second component (Weibull distribution) using the function (rweibull) in the R software.
(iii) Go to (i) until we get the required random sample of size n (50 and 100).

Table 3: Bias, MSE and RE of the MLWD estimated parameters.

$\Theta = (p, \theta, \alpha, \beta)$	n	GMM Bias & MSE				MLE Bias & MSE				RE			
		$\hat{p}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{p}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{p}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$
(0.50, 0.75, 2.00, 1.25)*	50	-0.3129	-0.0154	-0.2079	-0.0733	-0.2985	-0.0436	-0.1822	0.1238	6.8584	2.3925	1.9630	10.5231
		0.1264	0.0186	0.1298	0.0216	0.8669	0.0445	0.2548	0.2273				
	100	-0.3268	-0.0265	-0.2058	-0.0948	-0.3078	-0.0619	-0.0867	0.0821	2.5617	2.0291	1.8889	8.9572
		0.1216	0.0172	0.0864	0.0187	0.3115	0.0349	0.1632	0.1675				
(0.95, 0.25, 0.50, 2.00)**	50	0.2503	0.0241	1.6454	0.9162	-0.0415	0.0097	1.0341	0.0391	0.0349	0.2000	0.7636	0.4454
		0.3292	0.0045	3.2607	3.0592	0.0115	0.0009	2.4898	1.3626				
	100	0.4214	0.0460	1.3718	0.2974	0.0072	0.0043	0.4330	0.1899	0.0047	0.1667	0.2516	0.3955
		0.3001	0.0036	1.9875	0.9286	0.0014	0.0006	0.5000	0.3673				

*Unimodel case.

**Bimodel case.

relative efficiency (RE)=$\frac{MSE_{ML}(\hat{\Theta})}{MSE_{GMM}(\hat{\Theta})}$

Table 4: Average width of the estimated CIs for the MLWD.

$\Theta = (p, \theta, \alpha, \beta)$	n	GMM Confidence Level 90%				GMM 95%				MLE Confidence Level 90%				MLE 95%			
		$\hat{p}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{p}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{p}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{p}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$
(0.50, 0.75, 2.00, 1.25)	50	0.0785	0.0630	0.1369	0.0592	0.0936	0.0751	0.1631	0.0705	0.4103	0.0960	0.2190	0.2142	0.4889	0.1144	0.2610	0.2553
	100	0.0400	0.0423	0.0691	0.0325	0.0477	0.0504	0.0823	0.0387	0.1532	0.0580	0.1298	0.1319	0.1825	0.0691	0.1547	0.1572

Table 5: Coverage probability of the estimated CIs for the MLWD.

$\Theta = (p, \theta, \alpha, \beta)$	n	GMM Confidence Level 90%				GMM 95%				MLE Confidence Level 90%				MLE 95%			
		$\hat{p}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{p}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{p}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{p}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$
(0.50, 0.75, 2.00, 1.25)	50	99.9	98.6	99.6	100	100	98.7	99.8	100	90	88.3	88.2	86.8	90.4	91.9	92	91.5
	100	99.4	99	99.6	99.7	99.8	99	99.7	99.8	93.9	87.7	86.9	88.6	94.5	92.3	91.2	93.8

(2) The GMM and ML estimates are obtained by solving the system of nonlinear equations (28-31) and (33-36), respectively, which were investigated in Section 4 using the R packages (gmm) and (nleqslv), respectively.

(3) Repeat steps (1-2) based on 1000 replications.

Therefore, we calculate the bias, MSE and RE for the point estimation of the model parameters and the average width with the coverage probability for the estimated confidence intervals. All the numerical results for the simulation process of the MLWD are shown and discussed in Tables 3, 4 and 5. Table 3 indicates that the estimated bias of the model parameters is over-estimated in the bimodal case except for p at $n = 50$ for the ML method. According to the GMM and ML estimation methods, the estimated bias is under-estimated in the unimodal case except for β. However, it can be seen that the MSE decreases as n increases for both the GMM and ML, which shows that all the estimates of the MLWD parameters satisfy the consistency. In addition, when estimating the unknown parameters of the MLWD in the unimodal case, it can be seen that the RE value is greater than one, which shows that the GMM is better than the ML for estimating the unknown parameters of the MLWD, and vice versa, in the bimodal case where the ML is better than the GMM for estimating the unknown parameters of the MLWD. Thus, it is noticeable that there is good competition between the GMM and ML for estimating the MLWD parameters.

Table 4 shows the average width of the estimated confidence intervals in the unimodal case for the MLWD parameters based on 1000 replications to ensure that we obtain a perfect estimated interval. Table 4 shows that the GMM has smaller values for the average width than those of the ML. Thus, we can conclude that two factors affect the average width of the estimated intervals, namely, the sample size and the confidence level. As the sample size increases, the value of the

Table 6: MLEs, AIC, K-S and p-value for Application (1).

Model	MLE	AIC	K-S	P-value
MLWD	$\hat{p}=0.0181$ $\hat{\theta}=2.0402$ $\hat{\alpha}=3.0973$ $\hat{\beta}=3.6370$	179.135	0.0783	0.8136
Mixture of two one parameter Lindley distributions	$\hat{p}=0.1909$ $\hat{\theta}_1=0.5903$ $\hat{\theta}_2=0.5904$	250.7681	0.2978	1.649e-05
Mixture of two Exponential distributions	$\hat{p}=0.6170$ $\hat{\theta}_1=0.3624$ $\hat{\theta}_2=0.3624$	271.9887	0.9320	$\leq$ 2.2e-16
Mixture of two Weibull distributions	$\hat{p}=0.0768$ $\hat{\alpha}_1=2.1432$ $\hat{\beta}_1=1.6259$ $\hat{\alpha}_2=3.1285$ $\hat{\beta}_2=3.8142$	180.8973	0.0800	0.7919

average width decreases as the estimated intervals become very narrow. Furthermore, the average width increases when the confidence level increases and the confidence interval becomes wider.

Table 5 shows the coverage probability of the estimated confidence intervals for the MLWD parameters. Besides, if the interval length of the model parameter is wide enough to contain the original parameter, then the coverage probability will increase to reach the confidence level or more. Table 5 illustrates that there is a higher coverage probability in the GMM than the ML. Thus, the estimation method and the sample size have a significant effect on the estimated confidence intervals. Also, the coverage probability increases when the confidence level increases and it becomes close to the confidence level or more.

6 Data Analysis

In this section, we apply a real dataset to the proposed MLWD model. Two different applications will be considered in order to illustrate the usefulness of the MLWD and to show that it is a better model for fitting data than other mixture models, such as a Weibull mixture, a one-parameter Lindley mixture or an exponential mixture. In each application, the model parameters are estimated by the maximum likelihood, as described in Section 4, using R software.

Application (1)

The dataset considered in this application consists of an uncensored dataset from Nichols and Padgett [17] on the breaking stress of carbon fibers (in Gba). Table 6 shows the maximum likelihood estimates of the unknown parameters for the MLWD and some suggested mixture models such as a mixture of two exponential distributions and a mixture of two Weibull distributions. In addition, the Akaike information criterion (AIC) and the statistic value of the Kolmogorov Smirnov (K-S) test with a p-value are calculated using the (fitdist) and (ks.test) functions from the R package software. All the results are shown below in the following table.

Table 6 indicates that MLWD fits the data better than other mixture models in terms of AIC criteria and K-S test. Therefore, we can conclude that the MLWD is a perfect model for fitting the data since it has the smallest AIC value and the largest p-value compared to other mixture models, such as a Lindley mixture, an exponential mixture and a Weibull mixture. Figure 4 shows the plotted histogram and the empirical cumulative distribution function (ECDF) for the data of Application (1). To obtain approximately $100(1-\delta)\%$ confidence intervals for the model parameters, we use the components of the matrix $I^{-1}(\hat{\Theta})$, which is the inverse of Fisher information matrix. On the other hand, Fisher information matrix is a symmetric matrix that is calculated to obtain the asymptotic variances and covariances of the maximum likelihood estimates of the model parameters. We use the components of matrix $I^{-1}(\hat{\Theta})$ to obtain approximately $100(1-\delta)\%$ confidence intervals for the model parameters as $\hat{\Theta} \pm Z_{\delta/2}\sqrt{V(\hat{\Theta})}$, where $V(\hat{\Theta})$ are the variances of the parameters obtained from the diagonal elements of $I^{-1}(\hat{\Theta})$, and $Z_{\delta/2}$ is the upper $(\delta/2)$ percentile of the standard normal distribution. Consequently, the variance-covariance matrix of the MLWD parameters $\hat{\Theta}=(\hat{p},\hat{\theta},\hat{\alpha},\hat{\beta})$ for Application (1)

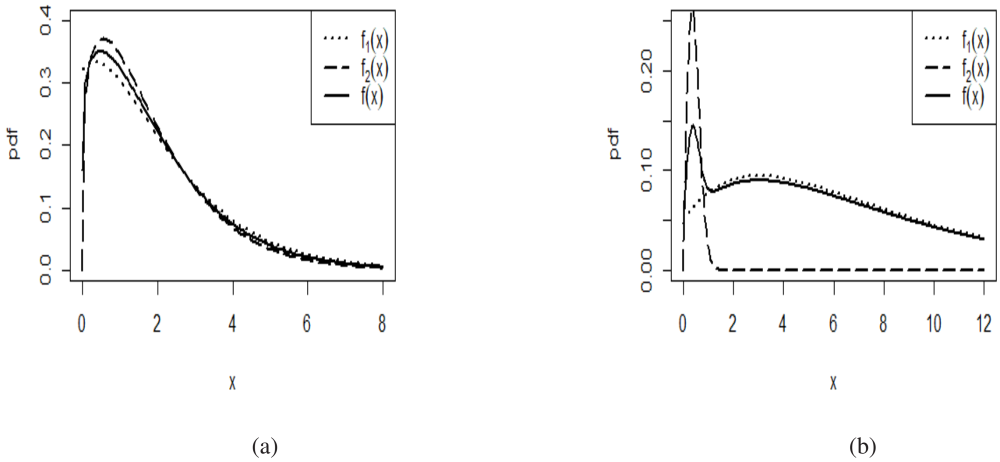


Fig. 1: pdf plot of MLWD with unimodal and bimodal cases with parameters $(p = 0.5, \theta = 0.75, \alpha = 2, \beta = 1)\&(p = 0.95, \theta = 0.25, \alpha = 0.5, \beta = 2)$

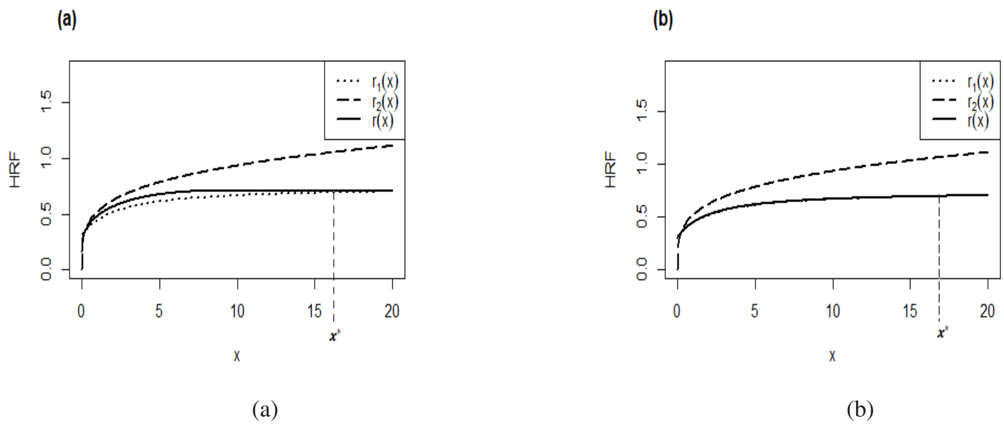


Fig. 2: HRF plot of MLWD with $\Theta = (p = (0.5, 0.95), \theta = 0.75, \alpha = 2, \beta = 1.25)$

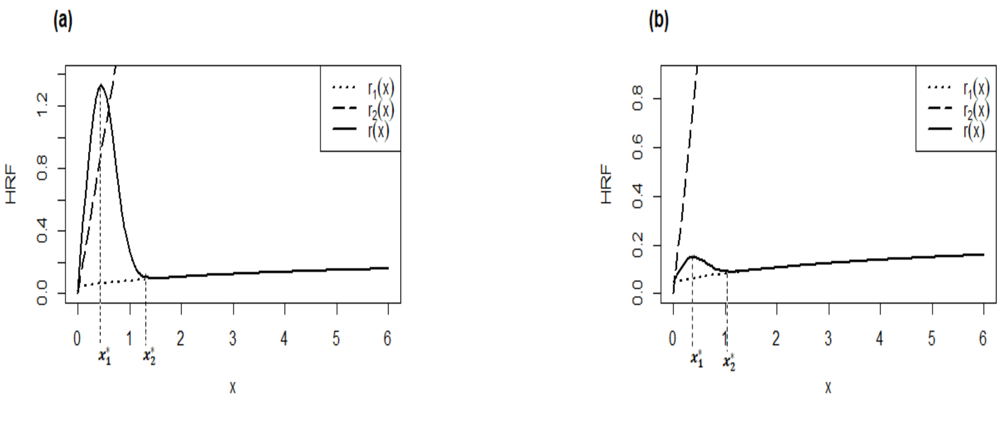
can be obtained by inverting Fisher information matrix, as given below:

$$I^{-1}(\hat{\Theta}) = \begin{bmatrix} 27.4047 & -1.0216 & -0.2349 & -0.1951 \\ -1.0216 & 0.1884 & 0.0262 & 0.0101 \\ -0.2349 & 0.0262 & 0.0154 & 0.0022 \\ -0.1951 & 0.0101 & 0.0022 & 0.0020 \end{bmatrix}$$

Therefore, the 90% CIs of the estimated parameters based on the variance-covariance matrix of $\hat{\Theta} = (\hat{p}, \hat{\theta}, \hat{\alpha}, \hat{\beta})$ are as follows:

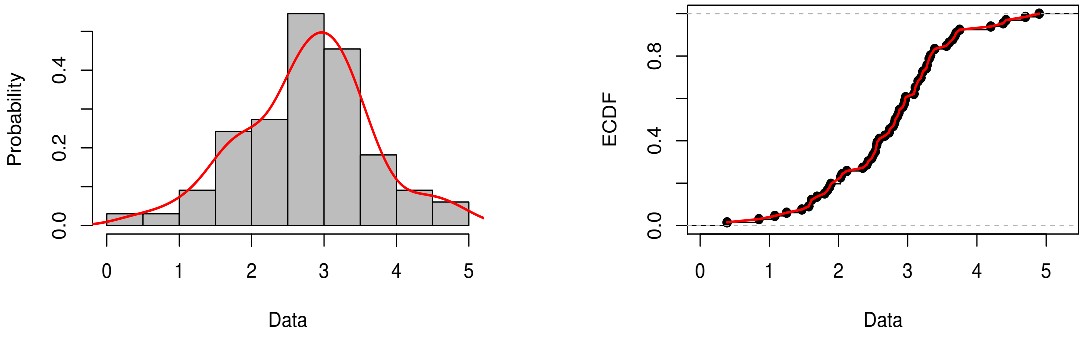
Parameter	$\hat{p}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$
90% CIs	0.0110 , 0.0251	1.2144 , 2.8660	3.0777 , 3.1169	3.5686 , 3.7055

Application (2)
In this application, the data was taken from Shanker, Fesshaye and Selvaraj [14], which represented an uncensored dataset



(a) (b)

Fig. 3: HRF plot of MLWD with $\Theta = (p = (0.45, 0.95), \theta = 0.25, \alpha = 0.5, \beta = 2.0)$



(a) (b)

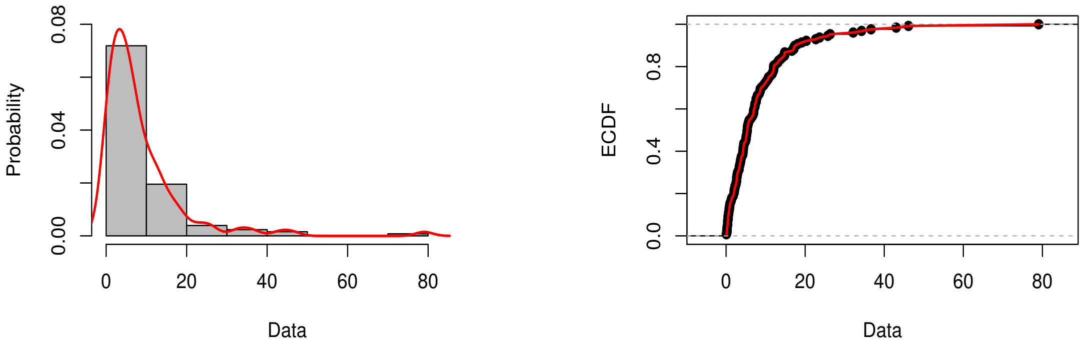
Fig. 4: Histogram and ECDF plots for the breaking stress of carbon fibers (in Gba).

corresponding to the remission times (in months) of a random sample of 128 bladder cancer patients as reported in Lee and Wang [18]. Table 7 shows the results for this data analysis including the maximum likelihood estimate of the unknown parameters, the AIC and the statistic value of the K-S test with a p-value.

From Table 7, we can conclude that the MLWD is a perfect model for fitting data compared to other mixture models such as a mixture of two Weibull distributions and a mixture of two one-parameter Lindley distributions based on the results of the K-S test and the AIC criteria. Figure 5 shows the plotted histogram and ECDF for the data of Application (2). The variance-covariance matrix of $\hat{\Theta} = (\hat{p}, \hat{\theta}, \hat{\alpha}, \hat{\beta})$ can be obtained as follows:

$$I^{-1}(\hat{\Theta}) = \begin{bmatrix} 0.00089 & 0.00113 & -0.02156 & 0.00217 \\ 0.00113 & 0.01104 & -0.04481 & 0.00485 \\ -0.02156 & -0.04481 & 1.27044 & -0.07893 \\ 0.00217 & 0.00485 & -0.07893 & 0.00808 \end{bmatrix}$$

Thus, we can construct the 90% CIs of the estimated model parameters from the MLWD and get the following intervals for each estimated parameter.



(a) (b)

Fig. 5: Histogram and ECDF plots for the remission times of 128 bladder cancer patients.

Table 7: MLEs, AIC, K-S and p-value for Application (2).

Model	MLE	AIC	K-S	P-value
MLWD	$\hat{p} = 0.0885$ $\hat{\theta} = 0.0675$ $\hat{\alpha} = 6.7648$ $\hat{\beta} = 1.0592$	808.6982	0.0406	0.9841
Mixture of two one parameter Lindley distributions	$\hat{p} = 0.2021$ $\hat{\theta}_1 = 0.0912$ $\hat{\theta}_2 = 0.3240$	810.9522	0.0704	0.5500
Mixture of two Weibull distributions	$\hat{p} = 0.9552$ $\hat{\alpha}_1 = 8.8496$ $\hat{\beta}_1 = 0.9765$ $\hat{\alpha}_2 = 0.4733$ $\hat{\beta}_2 = 2.5654$	812.5417	0.0583	0.7766

Parameter	$\hat{p}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$
90% CIs	0.0784 , 0.0987	0.0641 , 0.0709	6.6371 , 6.8925	1.0473, 1.0711

7 Conclusion

In this paper, we discussed a mixture of one-parameter Lindley and Weibull distributions. The statistical properties of the proposed mixture model, such as mean, variance, mode, median, measures of skewness and kurtosis. The behavior of the HRF of the MLWD was investigated with some graphs of the density and hazard rate functions. The GMM and ML methods were used to estimate the unknown parameters of the underlying mixture model. Moreover, a simulation experiment was performed in order to explore the performance of the estimation techniques and to compare between them based on 1000 replications. Furthermore, we calculated some measures via a simulation study such as the estimated bias, the MSE and the RE. Besides, we estimated the confidence intervals for the model parameters in the unimodal case to calculate the average width and the coverage probability of the estimated intervals at the confidence levels of 90% and 95%. Thus, in the unimodal case of the MLWD we found that the GMM has better results for estimating unknown parameters than the ML according to the value of the RE. On the other hand, the ML is better than the GMM in bimodal case of the MLWD since it has smaller MSE values. Also, we illustrated two applications of the MLWD through a real datasets used by other researchers. Hence, the MLWD was adequate and useful for modeling various sets of life time data and reliability. Moreover, the confidence intervals of the model parameters were calculated for the two applications based on the components of Fisher information matrix. Finally, we concluded that the MLWD is a perfect model for fitting data compared to other mixture models, such as a Weibull mixture, an exponential mixture or a Lindley mixture.

Conflict of Interest

The authors declare that they have no conflict of interest.

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