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Zagreb and Wiener Indices of the Conjugacy Class Graph of the Quasi-Dihedral and Generalized Quaternion Groups

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Abstract: The numerical invariance of a chemical graph is referred to as a graph-theoretic or topological index. In this article, we determine some degree-based and distance-based topological indices (viz., the first and second Zagreb indices and the Wiener index) for the conjugacy class graph of the quasi-dihedral and generalized quaternion Groups.

Keywords: Wiener Index, generalized quaternion group, conjugacy class graph, quasi-dihedral group, zagreb index, maximum clique

1 Introduction

In this article, Γ denotes an undirected simple graph, $\mathbb{N}$ denotes the set of natural numbers.

Graph theory has applications in many areas of computing, social, and natural sciences and is also a jovial playground for the exploration of proof techniques in discrete mathematics. The numerical parameters of a graph which characterize its topology, depend only on the abstract structure and is independent of graph representations such as particular labelings or drawings of the graph are known as topological indices [1]. In this research, we determine the generalized first and second Zagreb indices and the Wiener index for the conjugacy class graph of the generalized quaternion and quasi-dihedral groups.

Let G be a group. Two elements p and b in a group G are *conjugate* to each other if for some $w \in G$, $p = w^{-1}bw$. The *conjugacy class* of the element p in a group G is denoted by p^G and is defined as $p^G = \{w^{-1}pw : w \in G\}$ [2]. $K(G)$ and $Z(G)$ represents the number of conjugacy classes and the center of the group G respectively [2,3] and $\mathbb{V}$ denotes the set of all non-central conjugacy classes in G. A graph $\Gamma = \Gamma(G)$ with vertex set $\mathbb{V}$ in which two distinct vertices viz., E and F are joined by an edge whenever $gcd(|E|, |F|) \neq 1$ is said to be a *conjugacy class graph* of the group G [4].

A graph $\Gamma = \Gamma(G)$ with p vertices is said to be a *complete* graph if each pair of vertices in Γ is connected by an edge and is denoted by K_p. A complete subgraph of a graph Γ is termed a *clique* of the graph Γ and a complete subgraph of the largest size is referred to as a *maximum clique* of the graph Γ [5]. We denote the maximum clique of a graph Γ by Γ_M^{cq}. The *valency* or *degree* of a graph vertex p of a graph Γ is the number of graph edges that touch p. The *distance* between two vertices a and p is the number of edges in the shortest path between vertex p and vertex a and is denoted by $d(p, a)$ [6].

In 1972, *N.Trinajstić* and *I.Gutman* introduced first and second Zagreb indices [7] and in 1947, Harold Wiener introduced the concept of Wiener index of the connected graph [8]. The first and second Zagreb indices of a graph Γ with p-vertices are denoted by $M_1(\Gamma)$ and $M_2(\Gamma)$ respectively and are defined as

$$M_1(\Gamma) = \sum_{w=1}^{p} deg(h_w)^2$$

where $deg(h_w)$ represents the degree of the vertex h_w; $w = 1,2,3...p$. and

$$M_2(\Gamma) = \sum_{h_w h_x \in E(\Gamma)} deg(h_w)deg(h_x)$$

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where h_w and h_x are the vertices connected by an edge in Γ.

The Wiener number or Wiener index of a graph Γ with p-vertices is denoted by $W(\Gamma)$ and is defined as the half of the sum of the lengths of the shortest paths between all pairs of vertices in the graph Γ i.e.,

$$W(\Gamma) = \frac{1}{2}\sum_{w=1}^{p}\sum_{x=1}^{p} d(h_w, h_x),$$

where $d(h_w, h_x)$ represents the shortest distance between the vertices h_w and h_x.

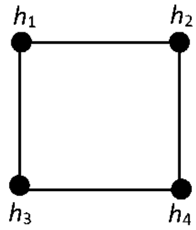

Fig. 1: Graph with four vertices

2 Preliminaries

In this section, we talk about some basic results and definitions on topological indices, graph theory, and group theory, which are required for proving the main results.

Definition 1.[7] ***The First Zagreb Index***
It is denoted by $M_1(\Gamma)$, where Γ is a connected graph with p-vertices and is defined as

$$M_1(\Gamma) = \sum_{w=1}^{p} deg(h_w)^2,$$

where $deg(h_w)$ represents degree of the vertex h_w; $w = 1,2,3...p$.

*Example 1.*The first Zagreb index of the graph is shown in Fig. (1) with four vertices h_1, h_2, h_3 and h_4 is given as follows

$$\begin{aligned} M_1(\Gamma) &= \sum_{w=1}^{4} deg(h_w)^2 \\ &= [deg(h_1)^2 + deg(h_2)^2 + deg(h_3)^2 + deg(h_4)^2] \\ &= [2^2 + 2^2 + 2^2 + 2^2] \\ &= 16. \end{aligned}$$

Definition 2.[7] ***The Second Zagreb Index***
It is denoted by $M_2(\Gamma)$, where Γ is a connected graph with p-vertices and is defined as

$$M_2(\Gamma) = \sum_{h_w h_x \in E(\Gamma)} deg(h_w)deg(h_x),$$

where h_w and h_x are the vertices that are connected by an edge in the graph Γ and $E(\Gamma)$ is the edge set of Γ.

*Example 2.*The second Zagreb index of the graph is shown in Fig. (1) with four vertices h_1, h_2, h_3, and h_4 is given as follows

$$\begin{aligned} M_2(\Gamma) &= \sum_{h_w h_x \in E(\Gamma)} deg(h_w)deg(h_x) \\ &= [deg(h_1)deg(h_2) + deg(h_1)deg(h_3) \\ &\quad + deg(h_2)deg(h_4) + deg(h_3)deg(h_4)] \\ &= [2.2 + 2.2 + 2.2 + 2.2] \\ &= 16. \end{aligned}$$

Definition 3.[8] ***The Wiener Index***
It is denoted by $W(\Gamma)$, where Γ is a connected graph with p-vertices and is defined as

$$W(\Gamma) = \frac{1}{2}\sum_{w=1}^{p}\sum_{x=1}^{p} d(h_w, h_x),$$

where $d(h_w, h_x)$ is the shortest path between the vertices h_w and h_x.

*Example 3.*The Wiener index of the graph shown in Fig. 1 with four vertices h_1, h_2, h_3 and h_4 is given as follows

$$\begin{aligned} W(\Gamma) &= \frac{1}{2}\sum_{w=1}^{4}\sum_{x=1}^{4} d(h_w, h_x) \\ &= \frac{1}{2}\left[\sum_{w=1}^{4} d(h_w,h_1) + d(h_w,h_2) + d(h_w,h_3) + d(h_w,h_4)\right] \\ &= \frac{1}{2}[d(h_1,h_1) + ... + d(h_1,h_4) + d(h_2,h_1) + ... \\ &\quad + d(h_2,h_4) + d(h_3,h_1) + ... + d(h_3,h_4) + d(h_4,h_1) \\ &\quad + ... + d(h_4,h_4)] \\ &= \frac{1}{2}(0+1+1+2+0+1+1+2+0+1+ \\ &\quad 1+2+0+1+1+2) \\ &= 8. \end{aligned}$$

Definition 4.[5] ***Maximum Clique***
Let Γ represents a graph. Then the largest complete subgraph of the graph Γ is said to be the maximum clique of the graph Γ.

Proposition 1.[9] *Let $G = Q_{4z} = < c, d \mid c^{2z} = 1,\ c^z = d^2,\ d^{-1}cd = c^{-1} >$ be the generalized quaternion group of order $4z$, where z is a positive integer; $z \geq 2$. Then the graph $\Gamma(Q_{4z})$ of the group Q_{4z} is*

$$\Gamma(Q_{4z}) = \begin{cases} K_{z+1} & \text{if } z \text{ is even}, \\ K_{z-1} \bigcup K_2 & \text{if } z \text{ is odd}. \end{cases}$$

Proposition 2.[9] *Let $G = QD_{2^z} = < c, d \mid c^{2^{(z-1)}} = d^2 = 1,\ dcd^{-1} = c^{2^{(z-2)}-1} >$ be the quasi-dihedral group of order 2^z, where z is a* *positive integer; $z \geq 4$. Then the graph $\Gamma(QD_{2^z})$ of the group QD_{2^z} is*

$$\Gamma(QD_{2^z}) = K_{2^{(z-2)}+1}.$$

*Remark.*1. Let $G = Q_{4z} = < c,d \mid c^{2z} = 1,\ c^z = d^2,\ d^{-1}cd = c^{-1} >$ be the generalized quaternion group of order $4z$, where z is a positive integer; $z \geq 2$. Then the maximum clique of the graph $\Gamma(Q_{4z})$ of the group (Q_{4z}) is

$$\Gamma_M^{cq} = \begin{cases} K_{z+1} & \text{if } z \text{ is even,} \\ K_{z-1} & \text{if } z \text{ is odd.} \end{cases}$$

3 Zagreb and Wiener indices of the Conjugacy Class Graph of Quasidihedral and Generalized Quaternion Groups

The main results of this article are discussed in this section. The distance and degree-based topological indices are calculated for connected graphs only. For generalized quaternion group Q_{4z}, the conjugacy class graph is connected when z is even and is disconnected for odd z (by Proposition 1). Therefore, there is no problem in obtaining the Wiener and the first and second Zagreb indices for Q_{4z} with even z. So, for the case when z is odd, it is not possible. Therefore, for this case, we find the Wiener and the Zagreb indices for the maximum clique of the graph $\Gamma(Q_{4z})$ of the generalized quaternion group Q_{4z}. Now, for calculating the Wiener and the first and second Zagreb indices of the conjugacy class graph of Q_{4z}, we consider two cases, i.e., when z is even and when z is even. For even z, we have the following theorems:

Theorem 1.*Let $G = Q_{4z} = < c,d \mid c^{2z} = 1,\ c^z = d^2,\ d^{-1}cd = c^{-1} >$ be the generalized quaternion group of order $4z$, where z is a positive even integer; $z \geq 2$. Then the Wiener index of the graph $\Gamma(Q_{4z})$ is*

$$W(\Gamma(Q_{4z})) = \frac{z(z+1)}{2}.$$

*Proof.*From proposition 1, we find that the graph $\Gamma(Q_{4z})$ of the group Q_{4z} is K_{z+1} with $z+1$ vertices and is a complete graph. Then the totality of vertices of the graph $\Gamma(Q_{4z})$ is

$$|V(\Gamma(Q_{4z}))| = K(G) - |Z(G)| = z+1 \tag{1}$$

Now, from definition 3, the Wiener index of the graph $\Gamma(Q_{4z})$ of the group Q_{4z} is given as

$$\begin{aligned} W(\Gamma(Q_{4z})) &= \frac{1}{2}\sum_{k=1}^{z+1}\sum_{l=1}^{z+1} d(h_k,h_l) \\ &= \frac{1}{2}\sum_{k=1}^{z+1}[d(h_k,h_1)+d(h_k,h_2)+\ldots+d(h_k,h_{z+1})] \\ &= \frac{1}{2}[d(h_1,h_1)+d(h_2,h_1)+\ldots+d(h_{z+1},h_1)+ \\ &\quad d(h_1,h_2)+d(h_2,h_2)+\ldots+d(h_{z+1},h_2)+\ldots+ \\ &\quad d(h_1,h_{z+1})+d(h_2,h_{z+1})+\ldots+d(h_{z+1},h_{z+1})] \end{aligned}$$

On solving this, we get

$$W(\Gamma(Q_{4z})) = \frac{z(z+1)}{2} \tag{2}$$

*Example 4.*Let $G = Q_{4.2} = < c,d \mid c^4 = 1,\ c^2 = d^2,\ d^{-1}cd = c^{-1} >$ be the generalized quaternion group of order 8. Then the graph $\Gamma(Q_{4.2})$ of the group $Q_{4.2}$ is a complete graph with three vertices (by Proposition 1), which is given by

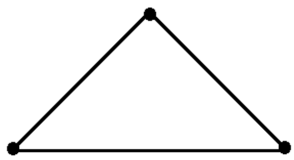

Fig. 2: Conjugacy class graph of $Q_{4.2}$

Then the Wiener index of the graph $\Gamma(Q_{4.2})$ is given as

$$W(\Gamma(Q_{4.2})) = \frac{2(2+1)}{2} = 3 \tag{3}$$

Theorem 2.*Let $G = Q_{4z} = < c,d \mid c^{2z} = 1,\ c^z = d^2,\ d^{-1}cd = c^{-1} >$ be the generalized quaternion group of order $4z$, where z is a positive even integer; $z \geq 2$. Then the first Zagreb index of the graph $\Gamma(Q_{4z})$ is*

$$M_1(\Gamma(Q_{4z})) = 2zW(\Gamma(Q_{4z})),$$

where $W(\Gamma(Q_{4z}))$ is the Wiener index of the graph $\Gamma(Q_{4z})$.

*Proof.*From equation (1), we find that the totality of vertices of the graph $\Gamma(Q_{4z})$ is

$$|V(\Gamma(Q_{4z}))| = K(G) - |Z(G)| = z+1 \tag{4}$$

Now, from definition 1, the First Zagreb index of the graph $\Gamma(Q_{4z})$ of the group Q_{4z} is given as

$$\begin{aligned} M_1(\Gamma(Q_{4z})) &= \sum_{k=1}^{z+1} deg(h_k)^2 \\ &= [deg(h_1)^2 + deg(h_2)^2 + \ldots + deg(h_{z+1})^2] \\ &= [z^2 + z^2 + \ldots + z^2] \\ &= z^2(z+1) \end{aligned}$$

i.e.,

$$M_1(\Gamma(Q_{4z})) = 2z\left(\frac{z(z+1)}{2}\right) \tag{5}$$

From equation (2), we get

$$M_1(\Gamma(Q_{4z})) = 2zW(\Gamma(Q_{4z}))$$

*Example 5.*Let $G = Q_{4.2} = < c,d \; |c^4 = 1, \; c^2 = d^2, \; d^{-1}cd = c^{-1} >$ be the generalized quaternion group of order 8. Then the graph $\Gamma(Q_{4.2})$ of the group $Q_{4.2}$ is a complete graph with three vertices (Proposition 1, Fig. 2). Then the first Zagreb index of the graph $\Gamma(Q_{4.2})$ is given as

$$M_1(\Gamma(Q_{4.2})) = 2.2.W(\Gamma(Q_{4.2}))$$

From equation (3), we get

$$M_1(\Gamma(Q_{4.2})) = 4.3 = 12. \tag{6}$$

Theorem 3.*Let $G = Q_{4z} = < c,d \; |c^{2z} = 1, \; c^z = d^2, \; d^{-1}cd = c^{-1} >$ be the generalized quaternion group of order $4z$, where z is a positive even integer; $z \geq 2$. Then the second Zagreb index of the graph $\Gamma(Q_{4z})$ is*

$$M_2(\Gamma(Q_{4z})) = \frac{1}{2}zM_1(\Gamma(Q_{4z})),$$

where $M_1(\Gamma(Q_{4z}))$ is the first Zagreb index of the graph $\Gamma(Q_{4z})$.

*Proof.*From proposition 1, we find that the graph $\Gamma(Q_{4z})$ of the group Q_{4z} is K_{z+1} with $z+1$ vertices and is a complete graph. We know that the totality of edges for a complete graph K_w with w vertices is $\frac{w(w-1)}{2}$, so in this case the number of edges of the graph $\Gamma(Q_{4z})$ of the group Q_{4z} is obtained by replacing w by $z+1$, which is found to be $\frac{z(z+1)}{2}$ and in this case, the degree of each vertex is z.
Now, from definition 2, the second Zagreb index of the graph $\Gamma(Q_{4z})$ of the group Q_{4z} is given as

$$\begin{aligned} M_2(\Gamma(Q_{4z})) &= \sum_{h_k h_l \in E(\Gamma(Q_{4z}))} deg(h_k)deg(h_l) \\ &= \sum_{p=1}^{\frac{z(z+1)}{2}} e_p \\ &= [e_1 + e_2 + ... + e_{\frac{z(z+1)}{2}}] \\ &= z^2\left(\frac{z(z+1)}{2}\right) \\ &= \frac{1}{2}z\left(2z\left(\frac{z(z+1)}{2}\right)\right) \end{aligned}$$

From equation (5), we get

$$M_2(\Gamma(Q_{4z})) = \frac{1}{2}zM_1(\Gamma(Q_{4z}))$$

*Example 6.*Let $G = Q_{4.2} = < c,d \; |c^4 = 1, \; c^2 = d^2, \; d^{-1}cd = c^{-1} >$ be the generalized quaternion group of order 8. Then the graph $\Gamma(Q_{4.2})$ of the group $Q_{4.2}$ is a complete graph with three vertices (Proposition 1, Fig. 2). Then the second Zagreb index of the graph $\Gamma(Q_{4.2})$ is given as

$$M_2(\Gamma(Q_{4.2})) = \frac{1}{2}.2.M_1(\Gamma(Q_{4.2}))$$

From equation (6), we get

$$M_2(\Gamma(Q_{4.2})) = 12. \tag{7}$$

Next, we consider z to be odd. In this case, as we discuss above that the Wiener and the first and second Zagreb indices are not possible to obtain as the graph $\Gamma(Q_{4z})$ is disconnected. So we obtained the same for the maximum clique Γ_M^{cq} of the graph $\Gamma(Q_{4z})$. Thus, we have the following results:

Theorem 4.*Let $G = Q_{4z} = < c,d \; |c^{2z} = 1, \; c^z = d^2, \; d^{-1}cd = c^{-1} >$ be the generalized quaternion group of order $4z$, where z is an odd positive integer; $z \geq 2$. Then the Wiener index of the maximum clique Γ_M^{cq} of the graph $\Gamma(Q_{4z})$ is*

$$W(\Gamma_M^{cq}) = \frac{(z-1)(z-2)}{2}.$$

*Proof.*From remark 2.1, we find that the maximum clique of the graph $\Gamma(Q_{4z})$ of the group G is K_{z-1} with $z-1$ vertices and is a complete graph. Then the totality of vertices of the maximum clique of the graph $\Gamma(Q_{4z})$ is

$$|V(\Gamma_M^{cq})| = |V(\Gamma(Q_{4z}))| - 2 = \; (K(G) - |Z(G)|) - 2 = \; z - 1 \tag{8}$$

Now, from definition 3, the Wiener index of the maximum clique of the graph $\Gamma(Q_{4z})$ of the group G is given as

$$\begin{aligned} W(\Gamma_M^{cq}) &= \frac{1}{2}\sum_{k=1}^{z-1}\sum_{l=1}^{z-1} d(h_k,h_l) \\ &= \frac{1}{2}\sum_{k=1}^{z-1}[d(h_k,h_1) + d(h_k,h_2) + ... + d(h_k,h_{z-1})] \\ &= \frac{1}{2}[d(h_1,h_1) + d(h_2,h_1) + ... + d(h_{z-1},h_1) + \\ &\quad d(h_1,h_2) + d(h_2,h_2) + ... + d(h_{z-1},h_2) + ... + \\ &\quad d(h_1,h_{z-1}) + d(h_2,h_{z-1}) + ... + d(h_{z-1},h_{z-1})] \end{aligned}$$

On solving this, we get

$$W(\Gamma_M^{cq}) = \frac{(z-1)(z-2)}{2} \tag{9}$$

*Example 7.*Let $G = Q_{4.3} = < c,d \; |c^6 = 1, \; c^3 = d^2, \; d^{-1}cd = c^{-1} >$ be the generalized quaternion group of order 12. Then the maximum clique $\Gamma_M^{cq}(Q_{4.3})$ of the graph $\Gamma(Q_{4.3})$ of the group $Q_{4.3}$ is a complete graph with two vertices (Remark 1), which is given by

Fig. 3: Maximum clique of $\Gamma(Q_{4.3})$

Then the Wiener index of the maximum clique of the graph $\Gamma(Q_{4.3})$ is given as

$$W(\Gamma_M^{cq}(Q_{4.3})) = \frac{(3-1)(3-2)}{2} = 1 \tag{10}$$

Theorem 5.*Let* $G = Q_{4z} = < c,d\ |c^{2z} = 1,\ c^z = d^2,\ d^{-1}cd = c^{-1} >$ *be the generalized quaternion group of order* $4z$*, where* z *is an odd positive integer;* $z \geq 2$*. Then the first Zagreb index of the maximum clique* Γ_M^{cq} *of the graph* $\Gamma(Q_{4z})$ *is*

$$M_1(\Gamma_M^{cq}) = 2(z-2)W(\Gamma_M^{cq}),$$

where $W(\Gamma_M^{cq})$ *is the Wiener index of the maximum clique* $\Gamma_M^{cq}(Q_{4z})$ *of the graph* $\Gamma(Q_{4z})$ *of the group* (Q_{4z})*.*

*Proof.*From equation (8), we find that the totality of vertices of the maximum clique Γ_M^{cq} of the graph $\Gamma(Q_{4z})$ is

$$|V(\Gamma_M^{cq})| = |V(\Gamma(Q_{4z}))| - 2 = (K(G) - |Z(G)|) - 2 = z-1$$

Now, from definition 1, the First Zagreb index of the maximum clique Γ_M^{cq} of the graph $\Gamma(Q_{4z})$ of the group Q_{4z} is given as

$$\begin{aligned} M_1(\Gamma_M^{cq}) &= \sum_{k=1}^{z-1} deg(h_k)^2 \\ &= [deg(h_1)^2 + deg(h_2)^2 + \ldots + deg(h_{z-1})^2] \\ &= [(z-2)^2 + (z-2)^2 + \ldots + (z-2)^2] \\ &= (z-2)^2(z-1) \end{aligned}$$

i.e.,

$$M_1(\Gamma_M^{cq}) = 2(z-2)\left(\frac{(z-1)(z-2)}{2}\right) \tag{11}$$

From equation (9), we get

$$M_1(\Gamma_M^{cq}) = 2(z-2)W(\Gamma_M^{cq}).$$

*Example 8.*Let $G = Q_{4.3} = < c,d\ |c^6 = 1,\ c^3 = d^2,\ d^{-1}cd = c^{-1} >$ be the generalized quaternion group of order 12. Then the graph $\Gamma(Q_{4.3})$ of the group $Q_{4.3}$ is a complete graph with two vertices (Remark 1, Fig. 3). Then the first Zagreb index of the maximum clique of the graph $\Gamma(Q_{4.3})$ is given as

$$M_1(\Gamma(Q_{4.3})) = 2(3-2)W(\Gamma_M^{cq})$$

From equation (10), we get

$$M_1(\Gamma(Q_{4.3})) = 2.1 = 2. \tag{12}$$

Theorem 6.*Let* $G = Q_{4z} = < c,d\ |c^{2z} = 1,\ c^z = d^2,\ d^{-1}cd = c^{-1} >$ *be the generalized quaternion group of order* $4z$*, where* z *is an odd positive integer;* $z \geq 2$*. Then the second Zagreb index of the maximum clique* Γ_M^{cq} *of the graph* $\Gamma(Q_{4z})$ *is*

$$M_2(\Gamma_{(Q_{4z})M}^{cq}) = \frac{1}{2}(z-2)M_1(\Gamma_M^{cq}),$$

where $M_1(\Gamma_M^{cq})$ *is the first Zagreb index of the maximum clique* $\Gamma_M^{cq}(Q_{4z})$ *of the graph* $\Gamma(Q_{4z})$ *of the group* (Q_{4z})*.*

*Proof.*From Remark 1, we find that the maximum clique Γ_M^{cq} of the graph $\Gamma(Q_{4z})$ of the group Q_{4z} is K_{z-1} with $z-1$ vertices and is a complete graph. We know that the totality of edges for a complete graph K_w with w vertices is $\frac{w(w-1)}{2}$, so in this case, the number of edges of the maximum clique Γ_M^{cq} of the graph $\Gamma(Q_{4z})$ of the group Q_{4z} is obtained by replacing w by $z-1$, which is found to be $\frac{(z-1)(z-2)}{2}$ and in this case, the degree of each vertex is $z-2$.
Now, from definition 2, the second Zagreb index of the maximum clique Γ_M^{cq} of the graph $\Gamma(Q_{4z})$ of the group Q_{4z} is given as

$$\begin{aligned} M_2(\Gamma_M^{cq}) &= \sum_{h_k h_l \in E(\Gamma_M^{cq})} deg(h_k)deg(h_l) \\ &= \sum_{p=1}^{\frac{(z-1)(z-2)}{2}} e_p \\ &= [e_1 + e_2 + \ldots + e_{\frac{(z-1)(z-2)}{2}}] \\ &= (z-2)^2\left(\frac{(z-1)(z-2)}{2}\right) \\ &= \frac{1}{2}(z-2)\left(2(z-2)\left(\frac{(z-1)(z-2)}{2}\right)\right) \end{aligned}$$

From equation (11), we get

$$M_2(\Gamma_M^{cq}) = \frac{1}{2}(z-2)M_1(\Gamma_M^{cq})$$

*Example 9.*Let $G = Q_{4.3} = < c,d\ |c^6 = 1,\ c^3 = d^2,\ d^{-1}cd = c^{-1} >$ be the generalized quaternion group of order 12. Then the graph $\Gamma(Q_{4.3})$ of the group $Q_{4.3}$ is a complete graph with two vertices (Remark 1, Fig. 3). Then the second Zagreb index of the maximum clique of the graph $\Gamma(Q_{4.3})$ is given as

$$M_2(\Gamma_M^{cq}(Q_{4.3})) = \frac{1}{2}(3-2)M_1(\Gamma_M^{cq})$$

From equation (12), we get

$$M_2(\Gamma_M^{cq}(Q_{4.3})) = \frac{1}{2}.2 = 1. \tag{13}$$

Now, for the quasi-dihedral group QD_{2^z} of order 2^z, it is possible to find the Wiener and the first and second Zagreb indices for the graph $\Gamma(QD_{2^z})$ as it is connected for all z (by Proposition 2). Therefore, we have the following results:

Theorem 7. *Let* $G = QD_{2^z} = < c,d \,| c^{2^{(z-1)}} = d^2 = 1,\ dcd^{-1} = c^{2^{(z-2)}-1} >$ *be the quasi-dihedral group of order* 2^z*, where* z *is a positive integer;* $z \geq 4$*. Then the Wiener index of the graph* $\Gamma(QD_{2^z})$ *of the group* QD_{2^z} *is*

$$W(\Gamma(QD_{2^z})) = 2^{(z-3)}(2^{(z-2)}+1).$$

Proof. From proposition 2, we find that the graph $\Gamma(QD_{2^z})$ of the group QD_{2^z} is $K_{2^{(z-2)}+1}$ with $(2^{(z-2)}+1)$ vertices and is a complete graph. Then the totality of vertices of the graph $\Gamma(QD_{2^z})$ is

$$|V(\Gamma(QD_{2^z}))| = K(G) - |Z(G)| = 2^{(z-2)}+1 \tag{14}$$

Now, from definition 3, the Wiener index of the graph $\Gamma(QD_{2^z})$ of the group QD_{2^z} is given as

$$\begin{aligned} W(\Gamma(QD_{2^z})) &= \frac{1}{2}\sum_{k=1}^{(2^{(z-2)}+1)}\sum_{l=1}^{(2^{(z-2)}+1)} d(h_k,h_l) \\ &= \frac{1}{2}\sum_{k=1}^{(2^{(z-2)}+1)} [d(h_k,h_1)+d(h_k,h_2)+...+ \\ &\quad d(h_k,h_{2^{(z-2)}+1})] \\ &= \frac{1}{2}[d(h_1,h_1)+d(h_2,h_1)+...+d(h_{2^{(z-2)}+1},h_1) \\ &\quad +d(h_1,h_2)+d(h_2,h_2)+...+d(h_{2^{(z-2)}+1},h_2) \\ &\quad +...+d(h_1,h_{2^{(z-2)}+1})+d(h_2,h_{2^{(z-2)}+1})+... \\ &\quad +d(h_{2^{(z-2)}+1},h_{2^{(z-2)}+1})] \end{aligned}$$

On solving this, we get

$$W(\Gamma(QD_{2^z})) = 2^{(z-3)}(2^{(z-2)}+1) \tag{15}$$

Example 10. Let $G = QD_{2^4} = < c,d \,| c^{2^{(3)}} = d^2 = 1,\ dcd^{-1} = c^{2^{(2)}-1} >$ be the quasi-dihedral group of order 2^4. Then the graph $(\Gamma(QD_{2^4}))$ of the group QD_{2^4} is a complete graph with five vertices (Proposition 2), which is given by

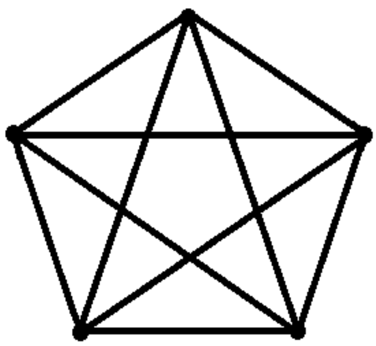

Fig. 4: Conjugacy class graph of QD_{2^4}

Then the Wiener index of the graph $\Gamma(Q_{4,3})$ is given as

$$W(\Gamma(QD_{2^4})) = 2.(4+1) = 10 \tag{16}$$

Theorem 8. *Let* $G = QD_{2^z} = < c,d \,| c^{2^{(z-1)}} = d^2 = 1,\ dcd^{-1} = c^{2^{(z-2)}-1} >$ *be the quasi-dihedral group of order* 2^z*, where* z *is a positive integer;* $z \geq 4$*. Then the first Zagreb index of the graph* $\Gamma(QD_{2^z})$ *of the group* QD_{2^z} *is*

$$M_1(\Gamma(QD_{2^z})) = 2^{(z-1)}W(\Gamma(QD_{2^z})),$$

where $W(\Gamma(QD_{2^z}))$ *is the Wiener index of the graph* $\Gamma(QD_{2^z})$ *of the group* QD_{2^z}*.*

Proof. From equation (14), we find that the totality of vertices of the graph $\Gamma(QD_{2^z})$ is

$$|V(\Gamma(QD_{2^z}))| = K(G) - |Z(G)| = 2^{(z-2)}+1$$

Now, from definition 1, the First Zagreb index of the graph $\Gamma(QD_{2^z})$ of the group QD_{2^z} is given as

$$\begin{aligned} M_1(\Gamma(QD_{2^z})) &= \sum_{k=1}^{(2^{(z-2)}+1)} deg(h_k)^2 \\ &= [deg(h_1)^2+deg(h_2)^2+...+deg(h_{2^{z-2}+1})^2] \\ &= [(2^{(z-2)})^2+(2^{(z-2)})^2+...+(2^{(z-2)})^2] \\ &= (2^{(z-2)})^2(2^{(z-2)}+1) \end{aligned}$$

i.e.,

$$M_1(\Gamma(QD_{2^z})) = 2^{(z-1)}(2^{(z-3)}(2^{(z-2)}+1)) \tag{17}$$

From equation (15), we get

$$M_1(\Gamma(QD_{2^z})) = 2^{(z-1)}W(\Gamma(QD_{2^z}))$$

Example 11. Let $G = QD_{2^4} = < c,d \,| c^{2^{(3)}} = d^2 = 1,\ dcd^{-1} = c^{2^{(2)}-1} >$ be the quasi-dihedral group of order 2^4. Then the graph $\Gamma(QD_{2^4})$ of the group QD_{2^4} is a complete graph with five vertices (Proposition 2, Fig. 4). Then the first Zagreb index of the graph $\Gamma(QD_{2^4})$ is given as

$$M_1(\Gamma(QD_{2^4})) = 2^3W(\Gamma(QD_{2^4}))$$

From equation (16) we get,

$$M_1(\Gamma(QD_{2^4})) = 8.10 = 80. \tag{18}$$

Theorem 9. *Let* $G = QD_{2^z} = < c,d \,| c^{2^{(z-1)}} = d^2 = 1,\ dcd^{-1} = c^{2^{(z-2)}-1} >$ *be the quasi-dihedral group of order* 2^z*, where* z *is a positive integer;* $z \geq 4$*. Then the second Zagreb index of the graph* $\Gamma(QD_{2^z})$ *of the group* QD_{2^z} *is*

$$M_2(\Gamma(QD_{2^z})) = 2^{(z-3)}M_1(\Gamma(QD_{2^z})),$$

where $M_1(\Gamma(QD_{2^z}))$ *is the first Zagreb index of the graph* $\Gamma(QD_{2^z})$ *of the group* QD_{2^z}*.*

*Proof.*From proposition 2, we find that the graph $\Gamma(QD_{2^z})$ of the group QD_{2^z} is $K_{2^{(z-2)}+1}$ with $(2^{(z-2)}+1)$ vertices and is a complete graph. We know that the totality of edges for a complete graph K_w with w vertices is $\frac{w(w-1)}{2}$, so in this case, the number of edges of the graph $\Gamma(QD_{2^z})$ of the group QD_{2^z} is obtained by replacing w by $2^{(z-2)}+1$, which is found to be $\frac{2^{(z-2)}(2^{(z-2)}+1)}{2}$ and in this case the degree of each vertex is $2^{(z-2)}$.
Now, from definition 2, the second Zagreb index of the graph $\Gamma(QD_{2^z})$ of the group QD_{2^z} is given as

$$\begin{aligned} M_2(\Gamma(QD_{2^z})) &= \sum_{h_k h_l \in E(\Gamma(QD_{2^z}))} deg(h_k)deg(h_l) \\ &= \sum_{p=1}^{\frac{2^{(z-2)}(2^{(z-2)}+1)}{2}} e_p \\ &= [e_1+e_2+\ldots+e_{\frac{2^{(z-2)}(2^{(z-2)}+1)}{2}}] \\ &= 2^{(z-3)}(2^{(z-2)})^2(2^{(z-2)}+1) \\ &= 2^{(z-3)}(2^{(z-1)}(2^{(z-3)}(2^{(z-2)}+1))) \end{aligned}$$

From equation (17), we get

$$M_2(\Gamma(QD_{2^z})) = 2^{(z-3)}M_1(\Gamma(QD_{2^z}))$$

*Example 12.*Let
$G = QD_{2^4} = <c,d \mid c^{2^{(3)}} = d^2 = 1,\ dcd^{-1} = c^{2^{(2)}-1}>$ be the quasi-dihedral group of order 2^4. Then the graph $\Gamma(QD_{2^4})$ of the group QD_{2^4} is a complete graph with five vertices (Proposition 2, Fig. (4)). Then the second Zagreb index of the graph $\Gamma(QD_{2^4})$ is given as

$$M_2(\Gamma(QD_{2^4})) = 2.M_1(\Gamma(QD_{2^z}))$$

From equation (18) we get,

$$M_2(\Gamma(QD_{2^4})) = 2.80 = 160.$$

4 Conclusion

For the generalized quaternion groups of order $4z$, where $z \geq 2$ and $z \in \mathbb{N}$, we have obtained the generalized Wiener index and the first and second Zagreb indices for the graph $\Gamma(Q_{4z})$ when z is even and for the maximum clique Γ_M^{cq} of the graph $\Gamma(Q_{4z})$ when z is odd. We also obtained the generalized Wiener index and the first and second Zagreb indices for the graph $\Gamma(QD_{2^z})$ of the quasi-dihedral groups. In each of the two cases for generalized quaternion groups, and for quasi-dihedral groups we notice that the first Zagreb index is obtained by using the Wiener index and the second Zagreb index is obtained by using the first Zagreb index.

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Conflict of Interest

The authors declare that they have no conflict of interest.

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