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On the Analytical and Numerical Study for Nonlinear Fredholm Integro-Differential Equations

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Abstract: In this paper, we investigate the existence of a unique solution to nonlinear Fredholm integro-differential equation. The exact solution of the proposed equation using the direct calculation method is given. We combine the finite difference method with the composite Simpson method to find the numerical solution of the equation. The error estimate of our scheme is discussed in this article. Finally, to illustrate the accuracy of the proposed method, we give five numerical examples and compare the exact solution with the numerical solution.

Keywords: Composite Simpson's method, Direct computation method, Finite difference method, Fredholm integro-differential equation

1 Introduction

Differential equations play an important role in many branches of modern mathematics and appear in various applications, including mechanics, engineering, mathematical physics, chemistry, biology, ...etc. Several researchers are interested in discussing different types of differential equations [1–9].

Numerical and analytical studies on different types of nonlinear integro-differential equations have been conducted, see [10–21].

Now, we consider a nonlinear Fredholm integro-differential equation of the following form:

$$u'(x) = f(x) + \int_a^b g(x,t,(u'(t)))dt, \quad u(a) = \alpha \qquad (1)$$

where $x,\ t \in [a,b]$, $-\infty < a < b < \infty$, $f(x)$ is known function, $u(x)$ is unknown function and g is a continuous function.

Many methods, such as a parametric iteration method [22], Chebyshev finite difference method [23], a combination of the finite difference method and the trapezoidal method [24] have been used to discuss Fredholm integro-differential equations. The Newton-type method [25] and [26] used compact finite difference formula.

In this work, the analytical and numerical solutions of equation (1) are investigated using the direct computation method [27] as well as the finite difference-Simpsons method [28–30] . Also, the existence of a unique solution is explored.

The present paper is organized, as follows: In Section 2, the existence of a unique solution will be discussed. In Section 3, the analysis and the derivation of analytical and numerical methods are presented. In Section 4, some examples are given and the exact solutions are compared to prove the applicability of the method. Section 5 is devoted to conclusion.

2 Existence of a unique solution

Before we start and prove the main results, we introduce the following assumptions: Consider the functional integro differential equation (1) with the following assumptions:
(i) $f : [a,b] \to R_+$ is continuous.
(ii) $g : [a,b] \times [a,b] \times R \to R_+$ is continuous and satisfies the Lipschitz condition

$$|g(x,t,z) - g(x,t,w)| \leqslant k_1(x,t)|z-w|,$$

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$$\sup_{t\in[a,b]}\int_a^b k_1(x,t)dx \le M.$$

(iii) $M \le 1$. Now, we introduce the following theorem for the existence of a unique positive continuous solution of the integro-differential equation (1).

Theorem 1. *Let the assumptions (i)-(iii) be satisfied, then the integro differential equation (1) has a unique positive continuous solution on $[a,b]$. Moreover, if the two functions f and g are monotonic, then the solution is monotonic*

Proof. Let $v(x) = u'(x)$, then equation (1) can be written as

$$v(x) = f(x) + \int_a^b g(x,t,(v(t)))dt, \tag{2}$$

and

$$u(x) = \alpha + \int_a^x v(t)dt. \tag{3}$$

Define the operator F by

$$Fv(x) = f(x) + \int_a^b g(x,t,v(t)))dt.$$

Let $x_1, x_2 \in [a,b]$ such that $|x_2 - x_1| < \delta$, then

$$\begin{aligned}\Big|Fv(x_2) - Fv(x_1)\Big| &= \Big|f(x_2) - f(x_1) + \int_a^b g(x_2,t,v(t))dt \\ &\quad - \int_a^b g(x_1,t,v(t))dt\Big| \\ &\leqslant \Big|f(x_2) - f(x_1)\Big| \\ &\quad + \int_a^b \Big|g(x_2,t,v(t)) - g(x_1,t,v(t))\Big|dt.\end{aligned}$$

This proves that $F : C[a,b] \to C[a,b]$.
Let $w, z \in C[a,b]$, then

$$\begin{aligned}\Big|Fw(x) - Fz(x)\Big| &= \Big|\int_a^b g(x,t,w(t))dt - \int_a^b g(x,t,z(t))dt\Big| \\ &\le \int_a^b \Big|g(x,t,w(t)) - g(x,t,z(t))\Big|dt \\ &\le \int_a^b k_1(x,t)\Big|w - z\Big|dt \\ &= \Big|w - z\Big|\int_a^b k_1(x,t)dt \\ &\le M\Big|w - z\Big|.\end{aligned}$$

Since $M \le 1$, then F is contraction. Then, using Banach Fixed Point theorem [31], the integral equation (2) has a unique solution $v \in C[a,b]$. Thus, based on equation (3), the integro-differential equation (1) possess a unique solution $u \in C[a,b]$.

Monotonicity
Here the monotonicity of the solution of equation (1) will be studied.

Lemma 1. *Let f and g be monotonic in the first argument and equation (1) has a solution. Then, this solution is monotonic.*

Proof. Let $v(x_1) = u'(x_1)$, $v(x_2) = u'(x_2)$.
Let f and g be nonincreasing in x.
Let $x_1, x_2 \in [a,b]$ such that $x_1 < x_2$, then

$$\begin{aligned}v(x_2) &= f(x_2) + \int_a^b g(x_2,t,v(t))dt \\ &\leqslant f(x_1) + \int_a^b g(x_2,t,v(t))dt \\ &\leqslant f(x_1) + \int_a^b g(x_1,t,v(t))dt = v(x_1),\end{aligned}$$

then,

$$v(x_2) \leqslant v(x_1).$$

Similarly, if f, g are nondecreasing in x and $x_1, x_2 \in [a,b]$ such that $x_1 < x_2$, we can prove that

$$v(x_2) \ge v(x_1).$$

3 Derivation of the analytical and numerical methods

Now, we introduce the analytical and numerical methods to solve this problem (1). In addition, we calculate the error estimate of the scheme.

3.1 The direct computation method

It is important to point out that this method will be applied only to the equations where the kernels are separable as

$$K(x,t) = \sum_{k=1}^{n} A_k(x)B_k(t). \tag{4}$$

We can use the direct computation method as follows:
Substituting (4) into (1) leads to

$$\begin{aligned}u'(x) &= f(x) + A_1(x)\int_a^b B_1(t)u'(t)dt \\ &\quad + A_2(x)\int_a^b B_2(t)u'(t)dt + \ldots + A_n(x)\int_a^b B_n(t)u'(t)dt.\end{aligned} \tag{5}$$

Each integral at the right side is equivalent to a constant. Then, (5) can be written as

$$u'(x) = f(x) + \alpha_1 A_1(x) + \alpha_2 A_2(x) + \ldots + \alpha_n A_n(x), \tag{6}$$

where

$$\alpha_i = \int_a^b B_i(t)u'(t)dt, \quad 1 \le i \le n. \tag{7}$$

Substituting (6) into (7), we obtain a system of n algebric equations which can be solved together to define α_i. Then, substituting the value of α_i into (6) and integrating the obtained result from 0 to x, we obtain the exact solution of nonlinear Fredholm integro-differential equation (1).

3.2 Finite difference method-composite Simpson's method

We divide the domain $[a,b]$ of (1) into finite points as $a = t_0 < t_1 < ... < t_{n-1} < t_n = b$. We use uniform step length $h = (b-a)/n$, such that $x_i = a + ih$, $i = 0,1,2,...,n$. We use composite Simpson's on the integral part and finite difference on the differential part of (1) to find the numerical solution.
The integral part of (1) can be approximated as:

$$\begin{aligned}\int_a^b k(x,t)u'(t)dt \simeq \frac{h}{3}\Bigg[&k(x,t_0)u'(t_0) \\ &+2\sum_{j=1}^{\frac{n}{2}-1} k(x,t_{2j})u'(t_{2j}) \\ &+4\sum_{j=1}^{\frac{n}{2}} k(x,t_{2j-1})u'(t_{2j-1}) \\ &+k(x,t_n)u'(t_n)\Bigg].\end{aligned}$$

By taking $u_i' = u'(x_i)$, $k(x_i,t_j) = k_{i,j}$, $i = 0,1,2,...,n$, then (1) can be written as

$$\begin{aligned}u_i' \simeq \frac{h}{3}\Bigg[&k_{i,0}u_0' + 2\sum_{j=1}^{\frac{n}{2}-1} k_{i,2j}u_{2j}' \\ &+4\sum_{j=1}^{\frac{n}{2}} k_{i,2j-1}u_{2j-1}' + k_{i,n}u_n'\Bigg].\end{aligned} \tag{8}$$

We use forward difference to approximate the derivative part of (8) as

$$u_i' \simeq \frac{u_{i+1}-u_i}{h}, \qquad i = 0.$$

We use central difference to approximate the derivative part of (8) as

$$u_i' \simeq \frac{u_{i+1}-u_{i-1}}{2h}, \qquad i = 1,2,...,n-1.$$

At the end point n we use second Backward finite difference

$$u_i' \simeq \frac{3u_n - 4u_{n-1} + u_{n-2}}{2h}, \quad i = n.$$

Substituting by u_i' into (8) we have for $i = 1,2,...,n-1$.

$$\begin{aligned}\frac{u_{i+1}-u_{i-1}}{2h} \simeq f_i + \frac{h}{3}\Bigg[&k_{i,0}(\frac{u_1-u_0}{h}) \\ &+2\sum_{j=1}^{\frac{n}{2}-1} k_{i,2j}\frac{u_{2j+1}-u_{2j-1}}{2h} \\ &+4\sum_{j=1}^{\frac{n}{2}} k_{i,2j-1}\frac{u_{2j}-u_{2j-2}}{2h} \\ &+k_{i,n}(\frac{3u_n-4u_{n-1}+u_{n-2}}{2h})\Bigg].\end{aligned} \tag{9}$$

$$\begin{aligned}\frac{3u_n-4u_{n-1}+u_{n-2}}{2h} \simeq f_n + \frac{h}{3}\Bigg[&k_{n,0}(\frac{u_1-u_0}{h}) \\ &+2\sum_{j=1}^{\frac{n}{2}-1} k_{n,2j}\frac{u_{2j+1}-u_{2j-1}}{2h} \\ &+4\sum_{j=1}^{\frac{n}{2}} k_{n,2j-1}\frac{u_{2j}-u_{2j-2}}{2h} \\ &+k_{n,n}(\frac{3u_n-4u_{n-1}+u_{n-2}}{2h})\Bigg].\end{aligned} \tag{10}$$

3.3 Error estimation

Suppose that ρ_1, ρ_2, ρ_3, $\rho_4 \in (a,b)$ such that the errors e_1 of forward difference, e_2 of central difference, e_3 of second backward difference approximation and e_4 of composite Simpson's rule respectively are given by $\frac{h}{2}u^{(2)}(\rho_4)$, $\frac{h^2}{6}u^{(3)}(\rho_1)$, $\frac{h^2}{3}u^{(3)}(\rho_3)$ and $\frac{(b-a)}{180}h^4u^{(4)}(\rho_2)$. Then, we obtain the error estimation for (1) by

$$e \le \left|\frac{(b-a)^2}{2n^2}M + \frac{(b-a)}{n}M + \frac{(b-a)^5}{90n^4}M\right|, \tag{11}$$

where $M = max\{u^{(3)}(\rho_1), u^{(3)}(\rho_3), u^{(4)}(\rho_2), u^{(2)}(\rho_4)\}$ and N is the number of subinterval.
From (9) and (10), the exact solution for $i = 1,2,3,\ldots,n-1$.

$$\begin{aligned}\frac{u_{i+1}-u_{i-1}}{2h} + \frac{h^2}{6}u^{(3)}(\rho_1) = f_i + \frac{h}{3}\Bigg[&k_{i,0}(\frac{u_1-u_0}{h}) \\ &+2\sum_{j=1}^{\frac{n}{2}-1} k_{i,2j}\frac{u_{2j+1}-u_{2j-1}}{2h} \\ &+4\sum_{j=1}^{\frac{n}{2}} k_{i,2j-1}\frac{u_{2j}-u_{2j-2}}{2h} \\ &+k_{i,n}(\frac{3u_n-4u_{n-1}+u_{n-2}}{2h})\Bigg] \\ &+\frac{h}{2}u^{(2)}(\rho_4) + \frac{h^2}{6}u^{(3)}(\rho_1) \\ &+\frac{h^2}{3}u^{(3)}(\rho_3) + \frac{(b-a)}{180}h^4u^{(4)}(\rho_2),\end{aligned} \tag{12}$$

and for $i=n$

$$\frac{3u_n-4u_{n-1}+u_{n-2}}{2h}+\frac{h^2}{3}u^{(3)}(\rho_3)=f_n+\frac{h}{3}\Big[k_{n,0}(\frac{u_1-u_0}{h})+2\sum_{j=1}^{\frac{n}{2}-1}k_{n,2j}\frac{u_{2j+1}-u_{2j-1}}{2h}+4\sum_{j=1}^{\frac{n}{2}}k_{n,2j-1}\frac{u_{2j}-u_{2j-2}}{2h}+k_{n,n}(\frac{3u_n-4u_{n-1}+u_{n-2}}{2h})\Big]+\frac{h}{2}u^{(2)}(\rho_4)+\frac{h^2}{6}u^{(3)}(\rho_1)+\frac{h^2}{3}u^{(3)}(\rho_3)+\frac{(b-a)}{180}h^4u^{(4)}(\rho_2), \tag{13}$$

where $\rho_1,\rho_2,\rho_3,\rho_4\in(a,b)$.
Substracting (9), (10) from (12), (13), we obtained the error terms as follows:

$$\begin{aligned}e&=\Big|\frac{h^2}{6}u^{(3)}(\rho_1)+\frac{h^2}{3}u^{(3)}(\rho_3)-2\frac{h}{2}u^{(2)}(\rho_4)-2\frac{h^2}{6}u^{(3)}(\rho_1)-2\frac{h^2}{3}u^{(3)}(\rho_3)-2\frac{(b-a)}{180}h^4u^{(4)}(\rho_2)\Big|,\\&=\Big|-\frac{h^2}{6}u^{(3)}(\rho_1)-\frac{h^2}{3}u^{(3)}(\rho_3)-hu^{(2)}(\rho_4)-\frac{(b-a)}{90}h^4u^{(4)}(\rho_2)\Big|,\\&=\Big|-(\frac{h^2}{6}u^{(3)}(\rho_1)+\frac{h^2}{3}u^{(3)}(\rho_3)+hu^{(2)}(\rho_4)+\frac{(b-a)}{90}h^4u^{(4)}(\rho_2))\Big|,\\&=\Big|\frac{h^2}{6}u^{(3)}(\rho_1)+\frac{h^2}{3}u^{(3)}(\rho_3)+hu^{(2)}(\rho_4)+\frac{(b-a)}{90}h^4u^{(4)}(\rho_2)\Big|.\end{aligned}$$

Let $M_1=u^{(3)}(\rho_1)$, $M_2=u^{(3)}(\rho_3)$, $M_3=u^{(2)}(\rho_4)$, and $M_4=u^{(4)}(\rho_2)$, then

$$e=\Big|\frac{h^2}{6}M_1+\frac{h^2}{3}M_2+hM_3+\frac{(b-a)}{90}h^4M_4\Big|,$$

if we take $M=max\{M_1,M_2,M_3,M_4\}$, then we have

$$\begin{aligned}e&\leq\Big|\frac{h^2}{6}M+\frac{h^2}{3}M+hM+\frac{(b-a)}{90}h^4M\Big|\\&=\Big|\frac{h^2}{2}M+hM+\frac{(b-a)}{90}h^4M\Big|.\end{aligned}\tag{14}$$

Substituting $h=\frac{b-a}{n}$ in (14) we get

$$e\leq\Big|\frac{(b-a)^2}{2n^2}M+\frac{(b-a)}{n}M+\frac{(b-a)^5}{90n^4}M\Big|.$$

It is the error estimation.

4 Applications

In this section, we apply the existence theorem to some examples of nonlinear Fredhom integro-differential equations, and use the direct calculation method as well as the finite difference composite Simpson method to solve them analytically and numerically. The results obtained are listed in Table 1-5. All results of these examples were performed using Mathematica.

Example 4.1 Consider the equation:

$$u'(x)=-1+\frac{1}{e}-\cosh(x)+x\sinh(1)+\int_0^1(x-t)u'(t)dt,\quad u(0)=1. \tag{15}$$

First, we prove that this example has a unique solution, as follows:

$$\begin{aligned}\Big|g(x,t,v(t))-g(x,t,w(t)\Big|&=\Big|(x-t)v(t)-(x-t)w(t)\Big|\\&=\Big|(x-t)\Big|\Big|(v(t)-w(t))\Big|,\end{aligned}$$

where $k_1(x,t)=(x-t)$.
Since $\underbrace{sup}_{t\in[0,1]}\int_0^1 k_1(x,t)dx=\underbrace{sup}_{t\in[0,1]}\int_0^1(x-t)dx\leq 1$, then from Theorem 1, we can deduce that example 4.1 has a unique solution.
Therefore, we can use the direct computation method to find the exact solution of this example. (15) can be writen as

$$u'(x)=-1+\frac{1}{e}-\cosh(x)+x\sinh(1)+x\alpha-\beta, \tag{16}$$

where

$$\alpha=\int_0^1 u'(t)dt,\quad \beta=\int_0^1 tu'(t)dt. \tag{17}$$

Substituting (16) into (17) and integrating the right side, we obtain α and β as

$$\begin{aligned}\alpha&=\frac{1}{e}+\frac{1}{2}(-2+\alpha-2\beta-\sinh(1)),\\\beta&=\frac{8+e^2+2e\alpha-3e(3+\beta)}{6e}.\end{aligned}\tag{18}$$

Solving (18) for α and β we obtain

$$(\alpha,\beta) = \Big(-\frac{-2+2e^2+9e\, sinh(1)}{13e}, -\frac{-12+13e-e^2+2e\, sinh(1)}{13e}\Big). \tag{19}$$

Substituting (19) into (16) and integrating the resulting equation from 0 to x, we obtain the exact solution

$$u(x) = 1-\sinh(x). \tag{20}$$

Now, we introduce the approximate and the exact solution in Table 1 with $n = 10$ and we provide Figure 1 below to show that the proposed numerical method is effective.

Table 1: The exact and approximate solution of example 4.1

x_i	Approximate solution	Exact solution	Absolute error
0.2	0.798645	0.798664	1.9129 E-5
0.4	0.589204	0.589248	4.3296 E-5
0.6	0.363288	0.363346	5.8509 E-5
0.8	0.111844	0.111894	4.9678 E-5

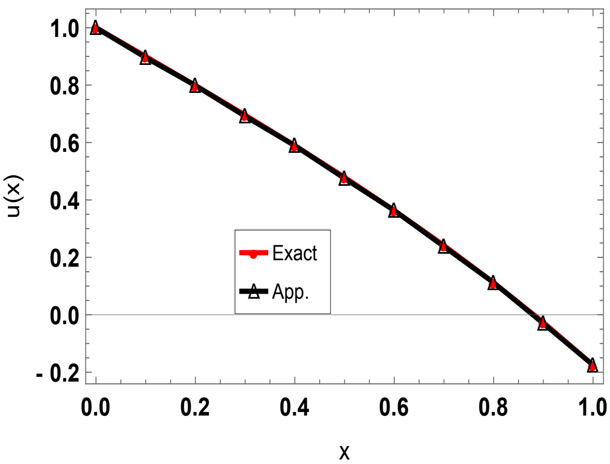

Fig. 1: Comparison between the approximate and exact solutions of example 4.1

Example 4.2 Consider the equation:

$$u'(x) = e^x - \frac{1}{300}e^{12+x}(-1+e^{12}) + \frac{1}{50}\int_2^4 e^x(u'(t))^6 dt, \quad u(2) = e^2. \tag{21}$$

Now, we prove that this example has a unique solution: we have

$$\Big|g(x,t,v(t)) - g(x,t,w(t)\Big| = \Big|\frac{1}{50}e^x v(t) - \frac{1}{50}e^x w(t)\Big| = \Big|\frac{1}{50}e^x\Big|\Big|(v(t)-w(t))\Big|,$$

where $k_1(x,t) = \frac{1}{50}e^x$, since $\underbrace{sup}_{t\in[2,4]} \int_2^4 k_1(x,t)dx = \underbrace{sup}_{t\in[2,4]} \int_2^4 \frac{1}{50}e^x dx \leq 1$, then from the existence Theorem 1 we can deduce that example 4.2 has a unique solution.
We use the direct computation method to find the exact solution of this example. (21) can be written as

$$u'(x) = e^x - \frac{1}{300}e^{12+x}(-1+e^{12}) + \frac{1}{50}e^x\alpha, \tag{22}$$

where

$$\alpha = \int_2^4 (u'(t))^6 dt. \tag{23}$$

Substituting (22) into (23), integrating the right side and solving the resulting equation we obtain

$$\alpha = \frac{1}{6}(-e^{12}+e^{24}). \tag{24}$$

Substituting (24) into (22) and integrating the resulting equation from 0 to x, we obtain the exact solution

$$u(x) = e^x. \tag{25}$$

Now, we find the approximate and the exact solutions in Table 2 at $n = 10$ and we introduce Figure 2 below for the solutions.

Table 2: The exact and approximate solution of example 4.2

x_i	Approximate solution	Exact solution	Absolute error
2.4	11.0232	11.0232	4.08732 E-7
2.8	16.4446	16.4446	7.40178 E-7
3.2	24.5325	24.5325	1.02885 E-6
3.6	36.5982	36.5982	7.77374 E-7

Example 4.3

Consider the eaquation:

$$u'(x) = -0.66\sin(x) + \int_{-0.5}^{1} \sin(x)u'(t)dt, \quad u(-0.5) = cos(-0.5). \tag{26}$$

To prove that the example has a unique solution we have

$$\Big|g(x,t,v(t)) - g(x,t,w(t)\Big| = |\sin(x)v(t) - \sin(x)w(t)| = \Big|\sin(x)\Big|\Big|(v(t)-w(t))\Big|,$$

where $k_1(x,t) = \sin(x)$, since $\underbrace{sup}_{t\in[-0.5,1]} \int_{-0.5}^{1} k_1(x,t)dx = \underbrace{sup}_{t\in[-0.5,1]} \int_{-0.5}^{1} \sin(x)dx \leq 1$.

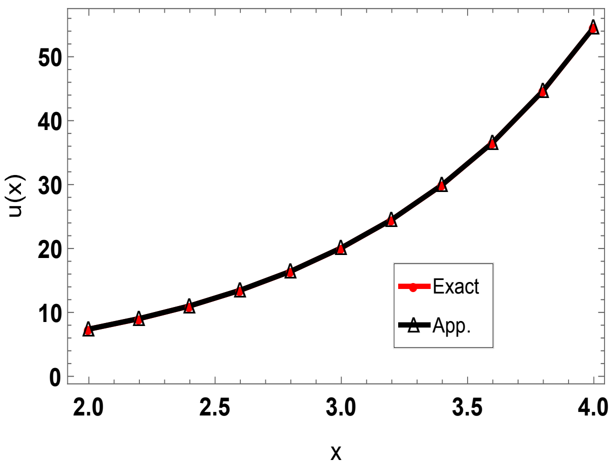

Fig. 2: Comparison between the approximate and exact solutions of example 4.2

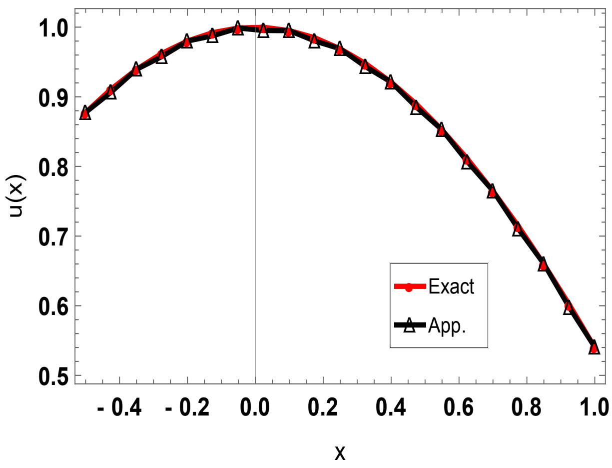

Fig. 3: Comparison between the approximate and exact solutions of example 4.3

Using the existence Theorem 1, we can say that example 4.3 has a unique solution.

The exact solution for this example can be found using the direct computation method. Now, we can write (26) as

$$u'(x) = -0.66\sin(x) + \alpha\sin(x), \tag{27}$$

where

$$\alpha = \int_{-0.5}^{1} (u'(t))dt. \tag{28}$$

To find the value of α, we substitute (27) into (28). Then, we obtain

$$\alpha = -0.33728. \tag{29}$$

Substituting (29) into (27) and integrating the resulting equation from 0 to x, we get the exact solution for our example

$$u(x) = \cos(x). \tag{30}$$

In Table 3, we introduce the values of approximate and the exact solutions with $n = 10$. We introduce Figure 3 below for the approximate and the exact solutions.

Table 3: The exact and approximate solution of example 6.3

x_i	Approximate solution	Exact solution	Absolute error
-0.35	0.939373	0.939373	2.22045 E-16
-0.2	0.980067	0.980067	1.11022 E-16
-0.05	0.99875	0.99875	0.00000
0.1	0.995004	0.995004	0.00000
0.25	0.968912	0.968912	1.11022 E-16
0.4	0.921061	0.921061	2.22045E-16
0.55	0.852525	0.852525	2.22045 E-16
0.7	0.764842	0.764842	1.11022 E-16
0.85	0.659983	0.659983	0.00000

Example 4.4
Consider the equation:

$$u'(x) = -5.7\ e^x + 0.5\int_0^1 e^x (u'(t))^4 dt, \quad u(0) = 1. \tag{31}$$

First, we prove that this example has a unique solution:

$$\begin{aligned}\left|g(x,t,v(t)) - g(x,t,w(t)\right| &= \left|0.5e^x v(t) - 0.5e^x w(t)\right| \\ &= \left|0.5e^x\right|\left|(v(t) - w(t))\right|,\end{aligned}$$

where $k_1(x,t) = 0.5e^x$, since $\underbrace{sup}_{t\in[0,1]} \int_0^1 k_1(x,t)dx = \underbrace{sup}_{t\in[0,1]} \int_0^1 0.5e^x dx \le 1$, then from the existence theorem, we can deduce that example 4.4 has a unique solution.
Then, we use the direct computation method to find the exact solution of this example. (31) can be written as

$$u'(x) = -5.7\ e^x + 0.5\alpha e^x, \tag{32}$$

where

$$\alpha = \int_0^1 (u'(t))^4 dt. \tag{33}$$

To find the value of α, we substitute (32) into (33), integrate the right side, and solve the resulting equation. Then, we obtain

$$\alpha = 13.3995. \tag{34}$$

Substituting (34) into (32) and integrating the resulting equation from 0 to x, we obtain the exact solution

$$u(x) = e^x. \tag{35}$$

The approximate and the exact solutions are introduced in Table 4 with $n = 10$. We introduce Figure 4 below for the approximate and the exact solutions.

Table 4: The exact and approximate solution of example 4.4

x_i	Approximate solution	Exact solution	Absolute error
0.2	1.2214	1.2214	0.00000
0.4	1.49182	1.49182	0.00000
0.6	1.82212	1.82212	2.22045 E-16
0.8	2.22554	2.22554	4.44089 E-16

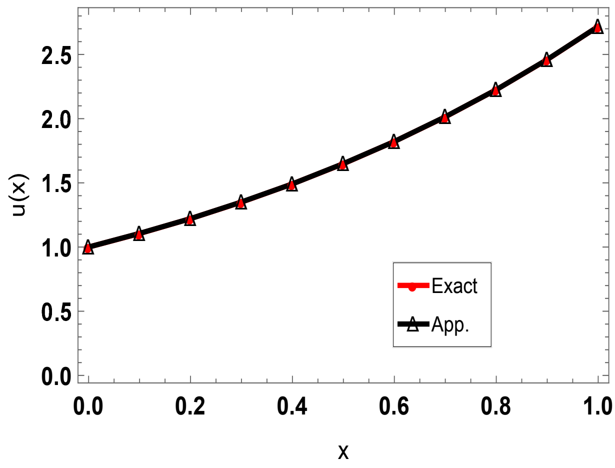

Fig. 4: Comparison between the approximate and exact solutions of example 4.4

Example 4.5
Consider the equation :

$$u'(x) = \cos(x) - \cos(x)\sin(1) + \int_0^1 \cos(x)u'(t)dt, \quad u(0) = 0. \tag{36}$$

First, we prove that this example has a unique solution:
We have

$$\Big|g(x,t,v(t)) - g(x,t,w(t)\Big| = \Big|\cos(x)v(t) - \cos(x)w(t)\Big| = \Big|\cos(x)\Big|\Big|(v(t) - w(t))\Big|,$$

where $k_1(x,t) = \cos(x)$, since $\underbrace{sup}_{t\in[0,1]} \int_0^1 k_1(x,t)dx = \underbrace{sup}_{t\in[0,1]} \int_0^1 \cos(x)dx \le 1$, then from the existence theorem we can deduce that example 4.5 has a unique solution.
Then, we use the direct computation method to find the exact solution of this example. (36) can be written as

$$u'(x) = \cos(x) - \cos(x)\sin(1) + \alpha\cos(x), \tag{37}$$

where

$$\alpha = \int_0^1 u'(t)dt. \tag{38}$$

To find the value of α, we substitute (37) into (38), integrate the right side, and solve the resulting equation. Then, we obtain

$$\alpha = \sin(1). \tag{39}$$

Substituting (39) into (37) and integrating the resulting equation from 0 to x, we obtain the exact solution

$$u(x) = \sin(x). \tag{40}$$

Now, we find the numerical solution of this example.
We introduce the approximate and the exact solution in Table 5 with $n = 10$, and we introduce Figure 5 below to show that the method is effective.

Table 5: The exact and approximate solution of example 4.5

x_i	Approximate solution	Exact solution	Absolute error
0.2	0.198669	0.198669	8.3267 E-17
0.4	0.389418	0.389418	1.6653 E-17
0.6	0.564642	0.564642	1.1102 E-16
0.8	0.717356	0.717356	1.1102 E-16

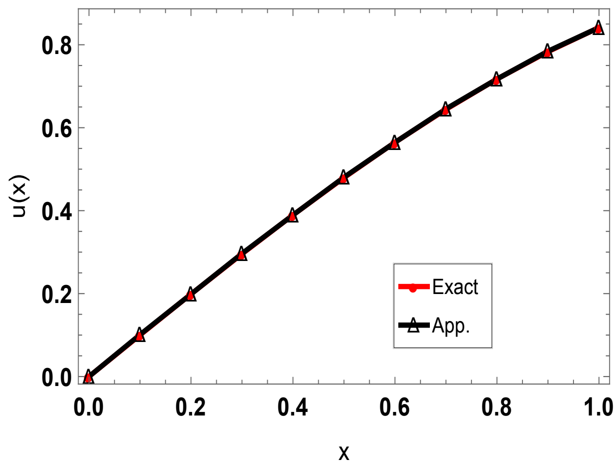

Fig. 5: Comparison between the approximate and exact solutions of example 4.5

Based on discussions of the numerical examples above as an application to the proposed methods in this paper, We can say that we have provided an accurate numerical study of the proposed equation using finite difference method-composite Simpson's method.

5 Conclusion

The analytical and numerical solutions of nonlinear Fredholm integro-differential equations have been discussed using direct calculation method and finite difference-Simpson method. In addition, we have examined the existence of a unique solution of the proposed system, and obtained the estimated error of the scheme. Five numerical examples have been presented and compared with the exact solution to show the accuracy of the proposed method.

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Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this article

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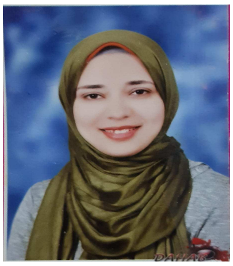

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