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New Perspective of Log-Convex Functions

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Abstract: In this paper, we consider some new perspectives of log-convex functions. We also investigate several properties of the log-convex functions and discuss their relations to convex functions. Optimality conditions are characterized by a class of variational inequalities. Several interesting results characterizing the log-convex functions were obtained. These results are significant developments of previously known results.

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1 Introduction and Preliminaries

Convex functions and convex sets have played an important and fundamental part in the development of various fields of pure and applied sciences. Convexity theory describes a broad spectrum of very interesting developments involving a link among various fields of mathematics, physics, economics and engineering sciences. Some of these developments have made mutually enriching contacts with other fields. Ideas explaining these concepts led to the developments of new and powerful techniques to solve a wide class of linear and nonlinear problems. Recently various extensions and generalizations of convex functions and convex sets investigated using innovative ideas and techniques. More accurate inequalities can be obtained using the log-convex functions rather than the convex functions. Closely related to the log-convex functions, we have the concept of exponentially convex(concave) functions, whose origin can be traced back to Bernstein [1]. The exponentially convex functions have important applications in information theory, big data analysis, machine learning and statistic. See, for example, [1–15] and the references therein.
Inspired by the ongoing research in this interesting, applicable and dynamic field, we again reconsider the concept of log-convex functions. We discuss the basic properties of the log-convex functions. It has been shown that the log-convex(concave) functions have distinctive properties. Several new concepts of log-convex functions have been introduced and investigated. We show that the local minimum of the log-convex functions is the global minimum. The difference (sum) of the log-convex function and affine log-convex function is a log-convex function. The optimal conditions of the differentiable log-convex functions can be characterized by a class of variational inequalities, which is an interesting outcome of our main results. The ideas and techniques of this paper may be starting point for further relevant research.

2 Preliminary Results

Let K be a nonempty closed set in a real Hilbert space H. We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the inner product and norm, respectively. Let $F : K \to R$ be a continuous function.

Definition 1. [8, 10].*The set K in H is said to be convex set, if*

$$u + t(v - u) \in K, \qquad \forall u, v \in K, t \in [0, 1].$$

Definition 2. [8–10] *A function F is said to be convex, if*

$$F((1-t)u + tv) \leq (1-t)F(u) + tF(v), \forall u, v \in K, t \in [0, 1].$$

Definition 3. [6, 8] *A strictly positive function F is said to be* log-convex, *if*

$$F((1-t)u + tv) \leq (F(u))^{1-t}(F(v))^{-t}, \forall u, v \in K, t \in [0, 1].$$

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Gamma and Beta functions are log-convex function. Every log-convex function is a convex function, but the converse is untrue. See [9–11] and the references therein.

We can rewrite Definition 3 in the following equivalent form as

Definition 4. *[12] A strictly positive function F is said to be* log-convex, *if*

$$\log F((1-t)u+tv)) \leq (1-t)\log F(u)+t\log F(v), \quad \forall u,v \in K, \quad t \in [0,1].$$

We use this equivalent concept (Definition 4) to discuss some new aspects of log-convex functions, which is objective of this paper.

If $\log F = e^{f(u)}$, we recover the concepts of the exponentially convex function, which are mainly due to Noor and Noor [13–15] as:

Definition 5. *[2] A positive function f is said to be exponentially convex function, if*

$$e^{f((1-t)u+tv)} \leq (1-t)e^{f(u)}+te^{f(v)}, \quad \forall u,v \in K, t \in [0,1].$$

We remark that Definition 5 can be rewritten in the following equivalent way, which is due to Antczak [3] and Avriel [4].

Definition 6. *A function f is said to be exponentially convex function, if*

$$f((1-t)a+tb) \leq \log[(1-t)e^{f(a)}+te^{f(b)}], \tag{1}$$

$$\forall a,b \in K, t \in [0,1]. \tag{2}$$

A function is called the exponentially concave function f, if $-f$ is exponentially convex function. For the applications and properties of exponentially convex functions, see [1–6].

Definition 7. *A strictly positive function F is said to be affine* log-convex *function, if*

$$\log F((1-t)u+tv)) = (1-t)\log F(u)+t\log F(v), \quad \forall u,v \in K, t \in [0,1].$$

Definition 8. *A strictly positive function F on the convex set K is said to be quasi* log-convex, *if*

$$\log F((1-t)u+tv)) \leq \max\{\log F(u), \log F(v)\}, \quad \forall u,v \in K, t \in [0,1].$$

Definition 9. *A strictly positive function F on the convex set K is said to be* log-convex, *if*

$$\log F(u+t(v-u))) \leq (\log(F(u))^{1-t}(\log F(v))^{t}, \quad \forall u,v \in K, t \in [0,1],$$

where $F(\cdot) > 0$.

From the above-mentioned definitions, we have

$$\begin{aligned}\log F(u+t(v-u)) &\leq (\log(F(u))^{1-t}(\log F(v))^{t} \\ &\leq (1-t)\log F(u)+t\log F(v) \\ &\leq \max\{\log F(u), \log F(v)\}.\end{aligned}$$

This shows that every log-convex function is a quasi log-convex function. However, the converse is untrue.

Let $K = I = [a,b]$ be the interval. Now, we define the log-convex functions on I.

Definition 10. *Let $I = [a,b]$. Then F is* log-convex *function, if and only if,*

$$\begin{vmatrix} 1 & 1 & 1 \\ a & x & b \\ \log F(a) & \log F(x) & \log F(b) \end{vmatrix} \geq 0; \quad a \leq x \leq b.$$

One can easily show that the following is equivalent:

1. F is a log-convex function.
2. $\log F(x) \leq \log F(a) + \frac{\log F(b)-\log F(a)}{b-a}(x-a)$.
3. $\frac{\log F(x)-\log F(a)}{x-a} \leq \frac{\log F(b)-\log F(a)}{b-a}$.
4. $(b-x)\log F(a)+(a-b)\log F(x) + (x-a)\log F(b)) \geq 0$.
5. $\frac{\log F(a)}{(b-a)(a-x)} + \frac{\log F(x)}{(x-b)(a-x)} + \frac{\log F(b)}{(b-a)(x-b)} \leq 0$,

where $x = (1-t)a+tb \in [0,1]$.

3 Main Results

In this section, we consider some basic properties of log-convex functions.

Theorem 1. *Let F be a strictly* log-convex *function. Then any local minimum of F is a global minimum.*

Proof. Let the logconvex function F have a local minimum at $u \in K$. Assume the contrary, i. e. $F(v) < F(u)$ for some $v \in K$. Since F is a log-convex function,

$$\log F(u+t(v-u)) < t\log F(v)+(1-t)\log F(u), \quad \text{for} \quad 0 < t < 1.$$

Thus

$$\log F(u+t(v-u)) - \log F(u) < t[\log F(v) - \log F(u)] < 0,$$

from which it follows that

$$\log F(u+t(v-u)) < \log F(u),$$

for arbitrary small $t > 0$, contradicting the local minimum.

Theorem 2. *If the function F on the convex set K is* log-convex, *the level set*

$$L_{\alpha} = \{u \in K : \log F(u) \leq \alpha, \quad \alpha \in R\}$$

is a convex set.

*Proof.*Let $u,v \in L_\alpha$. Then $\log F(u) \le \alpha$ and $\log F(v) \le \alpha$. Now, $\forall t \in (0,1)$, $w = v+t(u-v) \in K$, since K is a convex set. Thus, by the log-convexity of F, we have

$$\begin{aligned}\log F(v+t(u-v)) &\le (1-t)\log F(v) + t\log F(u)\\ &\le (1-t)\alpha + t\alpha = \alpha,\end{aligned}$$

from which it follows that $v+t(u-v) \in L_\alpha$ Hence L_α is convex set.

Theorem 3.*A positive function F is a* log*-convex if and only if*

$$epi(F) = \{(u,\alpha) : u \in K : \log F(u) \le \alpha, \alpha \in R\}$$

is a convex set.

*Proof.*Assume that F is log-convex function. Let (u,α), $(v,\beta) \in epi(F)$. Then it follows that $\log F(u) \le \alpha$ and $\log F(v) \le \beta$. Thus, $\forall t \in [0,1]$, $u,v \in K$, we have

$$\begin{aligned}\log F(u+t(v-u)) &\le (1-t)\log F(u) + t\log F(v)\\ &\le (1-t)\alpha + t\beta,\end{aligned}$$

which implies that

$$(u+t(v-u), (1-t)\alpha + t\beta) \in epi(F).$$

Thus $epi(F)$ is a convex set. Conversely, let $epi(F)$ be a convex set. Let $u,v \in K$. Then $(u, \log F(u)) \in epi(F)$ and $(v, \log F(v)) \in epi(F)$. Since $epi(F)$ is a convex set, we must have

$$(u+t(v-u), (1-t)\log F(u) + t\log F(v)) \in epi(F),$$

which implies that

$$\log F(u+t(v-u)) \le (1-t)\log F(u) + t\log F(u).$$

This shows that F is a log-convex function.

Theorem 4.*A positive function F is quasi* log*-convex if and only if the level set*

$$L_\alpha = \{u \in K, \alpha \in R : \log F(u) \le \alpha\}$$

is a convex set.

*Proof.*Let $u,v \in L_\alpha$. Then $u,v \in K$ and $\max(\log F(u), \log F(v)) \le \alpha$.
Now for $t \in (0,1), w = u+t(v-u) \in K$, we have to prove that $u+t(v-u) \in L_\alpha$. By the quasi log-convexity of F, we have

$$\log F(u+t(v-u)) \le \max(\log F(u), \log F(v)) \le \alpha,$$

which implies that $u+t(v-u) \in L_\alpha$, showing that the level set L_α is indeed a convex set.

Conversely, assume that L_α is a convex set. Then $\forall u,v \in L_\alpha, t \in [0,1]$, $u+t(v-u) \in L_\alpha$. Let $u,v \in L_\alpha$ for

$$\alpha = max(\log F(u), \log F(v) \quad \text{and} \quad \log F(v) \le \log F(u).$$

From the definition of the level set L_α, it follows that

$$\log F(u+t\,(v,u)) \le \max(\log F(u), \log F(v)) \le \alpha.$$

Thus F is a quasi log-convex function. This completes the proof.

Theorem 5.*Let F be a* log*-convex function. Let $\mu = \inf_{u \in K} F(u)$. Then the set $E = \{u \in K : \log F(u) = \mu\}$ is a convex set of K. If F is strictly* log*-convex, E is a singleton.*

*Proof.*Let $u,v \in E$. For $0 < t < 1$, let $w = u+t(v-u)$. Since F is a log-convex function, so

$$\begin{aligned}F(w) = \log F(u+t(v-u)) &\le (1-t)\log F(u) + t\log F(v)\\ &= t\mu + (1-t)\mu = \mu,\end{aligned}$$

which implies that to $w \in E$. and hence E is a convex set. For the second part, assume that $F(u) = F(v) = \mu$. Since K is a convex set, then for $0 < t < 1, u+t(v-u) \in K$. Furthermore, since F is strictly log-convex,

$$\begin{aligned}\log F(u+t(v-u)) &< (1-t)\log F(u) + t\log F(v)\\ &= (1-t)\mu + t\mu = \mu.\end{aligned}$$

This contradicts the fact that $\mu = \inf_{u \in K} F(u)$, so the result follows.

Theorem 6. *If F is a* log*-convex function such that $\log F(v) < \log F(u), \forall u,v \in K$, F is a strictly quasi* log*-convex function.*

*Proof.*By the log-convexity of the function F, $\forall u,v \in K, t \in [0,1]$, we have

$$\log F(u+t(v-u)) \le (1-t)\log F(u) + t\log F(v) < \log F(u),$$

since $\log F(v) < \log F(u)$, which shows that the function F is strictly quasi log-convex.

Now, we derive some properties of the differentiable log-convex functions.

Theorem 7.*Let F be a differentiable function on the convex set K. Then the function F is* log*-convex function if and only if*

$$\log F(v) - \log F(u) \ge \langle \frac{F'(u)}{F(u)}, v-u \rangle, \forall v,u \in K. \tag{3}$$

*Proof.*Let F be a log-convex function. Then

$$\log F(u+t(v-u)) \le (1-t)\log F(u) + t\log F(v), \quad \forall u,v \in K,$$

which can be written as

$$\log F(v) - \log F(u) \geq \{\frac{\log F(u+t(v-u)) - \log F(u)}{t}\}.$$

Taking the limit in the above-mentioned inequality as $t \to 0$, we have

$$\log F(v) - \log F(u) \geq \langle \frac{F'(u)}{F(u)}, v-u)\rangle,$$

which is (3), the required result.

Conversely, let (3) hold. Then $\forall u,v \in K, t \in [0,1]$, $v_t = u+t(v-u) \in K$, we have

$$\begin{aligned} logF(v) - \log F(v_t) &\geq \langle \frac{F'(v_t)}{F(v_t)}, v-v_t)\rangle \\ &= (1-t)\langle \frac{F'(v_t)}{F(v_t)}, v-u\rangle. \end{aligned} \quad (4)$$

Similarly, we have

$$\begin{aligned} \log F(u) - \log F(v_t) &\geq \langle \frac{F'(v_t)}{F(v_t)}, u-v_t)\rangle \\ &= -t\langle \frac{F'(v_t)}{F(v_t)}, v-u\rangle. \end{aligned} \quad (5)$$

Multiplying (4) by t and (5) by $(1-t)$ and adding the resultant, we have

$$\log F(u+t(v-u)) \leq (1-t)\log F(u) + t\log F(v),$$

showing that F is a log-convex function.

*Remark.*From (3), we have

$$F(v) \geq F(u)exp\{\langle \frac{F'(u)}{F(u)}, v-u\rangle\}, \quad u,v \in K.$$

Changing the role of u and v in the aforementioned inequality, we also have

$$F(u) \geq F(v)exp\{\langle \frac{F'(v)}{F(v)}, u-v\rangle\}, \quad u,v \in K.$$

Thus, we can obtain the following inequality

$$\begin{aligned} F(u)+F(v) \geq\ & F(v)exp\{\langle \frac{F'(v)}{F(v)}, u-v\rangle\}, \\ & +F(u)exp\{\langle \frac{F'(u)}{F(u)}, v-u\rangle\} \quad u,v \in K. \end{aligned}$$

Theorem 7 enables us to introduce the concept of the log-monotone operators, which appears to be a new one.

Definition 11.*The differential $F'(.)$ is said to be* log-*monotone, if*

$$\langle \frac{F'(u)}{F(u)} - \frac{F'(v)}{F(v)}, u-v\rangle \geq 0, \quad \forall u,v \in H.$$

Definition 12.*The differential $F'(.)$ is said to be* log-*pseudo-monotone, if*

$$\langle \frac{F'(u)}{F(u)}, v-u\rangle \geq 0, \quad \Rightarrow \langle \frac{F'(v)}{F(v)}, v-u\rangle \geq 0, \quad \forall u,v \in H.$$

Accordingly, it follows that log-monotonicity implies log-pseudo-monotonicity, but the converse is untrue.

Theorem 8. *Let F be differentiable* log-*convex function on the convex set K. Then (3) holds if and only if $F'(.)$ satisfies*

$$\langle \frac{F'(u)}{F(u)} - \frac{F'(v)}{F(v)}, u-v\rangle \geq 0, \forall u,v \in K. \quad (6)$$

*Proof.*Let F be a log-convex function on the convex set K. Then, from Theorem 3.1, we have

$$\log F(v) - \log F(u) \geq \langle \frac{F'(u)}{F(u)}, v-u\rangle, \quad \forall u,v \in K. \quad (7)$$

Changing the role of u and v in (7), we have

$$\log F(u) - \log F(v) \geq \langle \frac{F'(v)}{F(v)}, u-v)\rangle, \quad \forall u,v \in K. \quad (8)$$

Adding (7) and (8), we have

$$\langle \frac{F'(u)}{F(u)} - \frac{F'(v)}{F(v)}, u-v\rangle \geq 0,$$

which shows that F' is a log-monotone.
Conversely, from (6), we have

$$\langle \frac{F'(v)}{F(v)}, u-v\rangle \leq \langle \frac{F'(u)}{F(u)}, u-v)\rangle. \quad (9)$$

Since K is an convex set, $\forall u,v \in K, \quad t \in [0,1]$ $v_t = u + t(v-u) \in K$.
Taking $v = v_t$ in (9), we have

$$\begin{aligned} \langle \frac{F'(v_t)}{F(v_t)}, u-v_t\rangle &\leq \langle \frac{F'(u)}{F(u)}, u-v_t\rangle \\ &= -t\langle \frac{F'(u)}{F(u)}, v-u\rangle, \end{aligned}$$

which implies that

$$\langle \frac{F'(v_t)}{F(v_t)}, v-u\rangle \geq \langle \frac{F'(u)}{F(u)}, v-u\rangle. \quad (10)$$

Consider the auxiliary function

$$\xi(t) = \log F(u+t(v-u)),$$

from which, we have

$$\xi(1) = \log F(v), \quad \xi(0) = \log F(u).$$

Then, from (10), we have

$$\xi'(t) = \langle \frac{F'(v_t)}{F(v_t)}, v-u\rangle \geq \langle \frac{F'(u)}{F(u)}, v-u\rangle. \quad (11)$$

Integrating (11) between 0 and 1, we have

$$\xi(1)-\xi(0)=\int_0^1 \xi'(t)dt \geq \langle \frac{F'(u)}{F(u)}, v-u\rangle.$$

Thus it follows that

$$\log F(v)-\log F(u) \geq \langle \frac{F'(u)}{F(u)}, v-u\rangle,$$

which is the required (3).

Now, we give a necessary condition for log-pseudoconvex function.

Theorem 9. *Let F' be a* log-*pseudomonotone. Then F is a* log-*pseudoconvex function.*

Proof. Let F' be a log-pseudomonotone. Then,

$$\langle \frac{F'(u)}{F(u)}, v-u\rangle \geq 0, \forall u,v \in K,$$

implies that

$$\langle \frac{F'(v)}{F(v)}, v-u\rangle \geq 0. \tag{12}$$

Since K is a convex set, $\forall u,v \in K$, $t\in[0,1]$, $v_t = u+t(v-u) \in K$.
Taking $v = v_t$ in (12), we have

$$\langle e^{F(v_t)}F'(v_t), v-u\rangle \geq 0. \tag{13}$$

Consider the auxiliary function

$$\xi(t) = \log F(u+t(v-u)) = \log F(v_t), \quad \forall u,v \in K, t\in[0,1],$$

which is differentiable. Then, using (13), we have

$$\xi'(t) = \langle \frac{F'(v_t)}{F(v_t)}, v-u)\rangle \geq 0.$$

Integrating the above-mentioned relation between 0 to 1, we have

$$\xi(1)-\xi(0)=\int_0^1 \xi'(t)dt \geq 0.$$

That is,

$$\log F(v)-\log F(u) \geq 0,$$

showing that F is a log-pseudoconvex function.

Definition 13. *The function F is said to be sharply* log-*pseudoconvex, if there exists a constant $\mu > 0$ such that*

$$\begin{aligned}\langle \frac{F'(u)}{F(u)}, v-u\rangle &\geq 0 \\ &\Rightarrow \\ F(v) &\geq \log F(v+t(u-v)), \quad \forall u,v \in K, t\in[0,1].\end{aligned}$$

Theorem 10. *Let F be a sharply* log-*pseudoconvex function on K. Then*

$$\langle \frac{F'(v)}{F(v)}, v-u\rangle \geq 0, \quad \forall u,v \in K.$$

Proof. Let F be a sharply log-pseudoconvex function on K. Then

$$\log F(v) \geq \log F(v+t(u-v)), \quad \forall u,v \in K, t\in[0,1].$$

from which we have

$$\begin{aligned}0 &\leq \lim_{t\to 0}\{\frac{\log F(v+t(u-v)) - \log F(v)}{t}\} \\ &= \langle \frac{F'(v)}{F(v)}, v-u\rangle,\end{aligned}$$

the required result.

Definition 14. *A function F is said to be a* log-*pseudoconvex function, if there exists a strictly positive bifunction $B(.,.)$ such that*

$$\begin{aligned}\log F(v) &< \log F(u) \\ &\Rightarrow \\ \log F(u+t(v-u)) &< \log F(u) + t(t-1)B(v,u), \\ &\quad \forall u,v \in K, t\in[0,1].\end{aligned}$$

Theorem 11. *If the function F is* log-*convex function such that $\log F(v) < \log F(u)$, the function F is* log-*pseudo convex.*

Proof. Since $\log F(v) < \log F(u)$ and F is log-convex function, then $\forall u,v \in K$, $t\in[0,1]$, we have

$$\begin{aligned}\log F(u+t\, l(v,u)) &\leq \log F(u) + t(\log F(v) - \log F(u)) \\ &< \log F(u) + t(1-t)(\log F(v) - \log F(u)) \\ &= \log F(u) + t(t-1)(\log F(u) - \log F(v)) \\ &< \log F(u) + t(t-1)B(u,v),\end{aligned}$$

where $B(u,v) = \log F(u) - \log F(v) > 0$, the required result. This shows that the function F is log-convex function.

Now, we show that the difference of log-convex function and affine log-convex function is a log-convex function.

Theorem 12. *Let f be a affine* log-*convex function. Then F is a* log-*convex function, if and only if, $g = F - f$ is a* log-*convex function.*

Proof. Let f be an affine log-convex function. Then

$$\log f((1-t)u+tv) = (1-t)\log f(u) + t\log f(v), \\ \forall u,v \in K, \quad t\in[0,1]. \tag{14}$$

From the log-convexity of F, we have

$$\log F((1-t)u+tv) \leq (1-t)\log F(u) + t\log F(v), \\ \forall u,v \in K, \quad t\in[0,1]. \tag{15}$$

From (14) and (15), we have

$$\begin{aligned}&\log F((1-t)u+tv) - \log f((1-t)u+tv) \\ &\leq (1-t)(\log F(u) - \log f(u)) \\ &\quad + t(\log F(v) - \log f(v)),\end{aligned} \tag{16}$$

from which it follows that

$$\begin{aligned} &\log g((1-t)u+tv) \\ &= \log F((1-t)u+tv) - \log f((1-t)u+tv) \\ \le\ &(1-t)(\log F(u) - \log f(u)) \\ &+t(\log F(v) - \log f(v)), \end{aligned}$$

which shows that $g = F - f$ is a log-convex function.
The inverse implication is obvious.

We now discuss the optimality condition for the differentiable log-convex functions, which is the main motivation of our next result.

Theorem 13. *Let F be a differentiable* log-*convex function. Then $u \in K$ is a minimum of the function F, if and only if, $u \in K$ satisfies the inequality*

$$\langle \frac{F'(u)}{F(u)}, v-u\rangle \ge 0, \forall u,v \in K. \tag{17}$$

Proof. Let $u \in K$ be a minimum of the function F. Then

$$F(u) \le F(v), \forall v \in K.$$

From which, we have

$$\log F(u) \le \log F(v), \forall v \in K. \tag{18}$$

Since K is a convex set, $\forall u,v \in K, \quad t \in [0,1]$,

$$v_t = (1-t)u + tv \in K.$$

Taking $v = v_t$ in (18), we have

$$\begin{aligned} 0 &\le \lim_{t\to 0}\{\frac{\log F(u+t(v-u)) - \log F(u)}{t}\} \\ &= \langle \frac{F'(u)}{F(u)}, v-u\rangle. \end{aligned} \tag{19}$$

Since F is differentiable log-convex function,

$$\log F(u+t(v-u)) \le \log F(u) + t(\log F(v) - \log F(u)), \quad u,v \in K, t \in [0,1].$$

Using (19), we have

$$\begin{aligned} \log F(v) - \log F(u) &\ge \lim_{t\to 0}\{\frac{\log F(u+t(v-u)) - \log F(u)}{t}\} \\ &= \langle \frac{F'(u)}{F(u)}, v-u\rangle \ge 0. \end{aligned}$$

Thus, it follows that

$$\log F(v) - \log F(u) \ge 0,$$

from which , we have

$$F(u) \le F(v), \quad \forall v \in K.$$

This shows that $u \in K$ is the minimum of the differentiable log-convex function.

*Remark.*The inequality of the type (17) is called the log-variational inequality and appears to be a new one. For the applications, formulations, numerical methods and other aspects of variational inequalities, see Noor [16, 17].

We remark that if a strictly positive function F is a log-convex function, we have

$$\begin{aligned} &\log F((1-t)u+tv) + \log F(tu+(1-t)v) \\ &\qquad \le \log F(u) + \log F(v), \\ &\forall u,v \in K, t \in [0,1], \end{aligned} \tag{20}$$

which is called the Wright log-convex function.
From (20), we have

$$\begin{aligned} &\log F((1-t)u+tv)F(tu+(1-t)v) \\ &= \log F((1-t)u+tv) + \log F(tu+(1-t)v) \\ &\le \log F(u) + \log F(v) \\ &= \log F(u)F(v), \quad \forall u,v \in K, t \in [0,1]. \end{aligned}$$

This implies that

$$F((1-t)u+tv)F(tu+(1-t)v) \le F(u)F(v), \quad \forall u,v \in K, t \in [0,1],$$

which shows that a strictly positive function F is a multiplicative Wright convex function. It is an interesting problem to address the properties and applications of the Wright log-convex functions.

4 Conclusion

In this paper, we have investigated some new aspects of log-convex function. Several new classes of log-convex functions have been introduced. We have shown that the minimum of the differentiable log-convex functions can be characterized by a new class of variational inequalities, which is called the log-variational inequality. One can explore the applications of the log-variational inequalities. This may stimulate further research.

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Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this article

References

[1] S. N. Bernstein, Sur les fonctions absolument monotones, *Acta Math.* **52**, 1-66(1929).
[2] G. Alirezaei and R. Mazhar, On exponentially concave functions and their impact in information theory, *J. Inform. Theory Appl.* **9(5)**, 265-274(2018).
[3] T. Antczak, On (p,r)-invex sets and functions, *J. Math. Anal. Appl.* **263**, 355-379(2001).
[4] M. Avriel, r-Convex functions. *Math. Program.,* **2**, 309-323(1972).
[5] S.Pal and T. K. Wong, On exponentially concave functions and a new information geometry, *Annals. Prob. 46(2)*, 1070-1113(2018).
[6] Y. X. Zhao, S. Y. Wang and L. Coladas Uria, Characterizations of r-Convex Func-tions, *J Optim. Theory Appl.* **145**, 186-195(2010).
[7] M. U. Awan, M. A. Noor and K. I. Noor, Hermite-Hadamard inequalities for exponentially convex functions, *Appl. Math. Inform. Sci.,* **12(2)**, 405-409(2018).
[8] G. Cristescu, L. Lupsa, *Non-Connected Convexities and Applications,* Kluwer Academic Publisher, Dordrechet, 2002.
[9] C. F. Niculescu and L. E. Persson, *Convex Functions and Their Applications.* Springer-Verlag, New York, 2018
[10] C. F. Niculescu and L. E. Persson, *Convex Functions and Their Applications.* Springer-Verlag, New York, 2018
[11] J. Pecaric, F. Proschan and Y. L. Tong, *Convex Functions, Partial Orderings and Statistical Applications,* Academic Press, New York, 1992
[12] M. A. Noor. On Hermite-Hadamard integral inequalities for product of two nonconvex functions, *J. Adv. Math. Stud.* **2(1)**, 53-62(2009).
[13] M. A. Noor and K. I. Noor, Strongly exponentially convex functions, *U.P.B. Bull Sci. Appl. Math. Series A,* **81(4)**. 75-84(2019).
[14] M. A. Noor and K. I. Noor, Strongly exponentially convex functions and their properties, *J. Adv. Math. Stud.,* **9(2)**, 177-185(2019).
[15] M. A. Noor and K. I. Noor, On generalized strongly convex functions involving bifunction, *Appl. Math. Inform. Sci.,* **13(3)**, 411-416(2019).
[16] M. A. Noor, New approximation schemes for general variational inequalities, *J. Math. Anal. Appl.* **251**, 217-229(2000).
[17] M. A. Noor, Some developments in general variational inequalities, *Appl. Math. Comput.* **152**, 199-277 (2004).

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