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Certain Results for the 3-Variable Laguerre-Hermite Polynomials

Tabinda Nahid [1] *and Serkan Araci*[2,*]

[1]Department of Mathematics, Faculty of Science, Aligarh Muslim University, Aligarh, India
[2]Department of Economics, Faculty of Economics, Administrative and Social Sciences, Hasan Kalyoncu University, Gaziantep TR-27410, Turkey

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Abstract: This paper addresses the mathematical inspection of differential and integral equations for hybrid forms of special polynomials using generating functions. The study aims to find out the differential equations of 3-variable Laguerre-Hermite polynomials. The inclusion of the derivation of the Volterra integral equation of 3-variable Laguerre-Hermite polynomials brings a novelty to the existing literature. Using Mathematica, the surface plots and curves of the aforementioned polynomials are explored and their zeros are investigated.

Keywords: Laguerre-Hermite polynomials; Differential equations; Volterra integral equation.

1 Introduction and preliminaries

Generating functions are the main tool for defining and deriving the properties of special polynomials. They have various advantages; for example, they transform the problem on sequences to functions. The generating function for the 3-variable Laguerre-Hermite polynomials (3VLHP) ${}_LH_i(a,b,c)$ are given by [1]:

$$H(a,b,c,w) = C_0(aw)\exp(bw+cw^2) = \sum_{i=0}^{\infty} {}_LH_i(a,b,c)\frac{w^i}{i!}, \tag{1}$$

where $C_0(aw)$ is the Bessel-Tricomi function of order zero:

$$C_0(\alpha a) = \exp(-\alpha\hat{D}_a^{-1})\{1\}, \qquad \hat{D}_a^{-i}\{1\} := \frac{a^i}{i!}. \tag{2}$$

The Bessel-Tricomi function $C_i(a)$ of order a has the following series representation:

$$C_i(a) = \sum_{j=0}^{\infty} \frac{(-1)^j a^j}{j!(i+j)!}, \quad i \in \mathbb{N}_0. \tag{3}$$

Using relation (3), equation (1) has the following form:

$$H(a,b,c,w) = \exp((b-D_a^{-1})w+cw^2) = \sum_{i=0}^{\infty} {}_LH_i(a,b,c)\frac{w^i}{i!}. \tag{4}$$

The 3VLHP ${}_LH_i(a,b,c)$ possesses the following series expansion [1]:

$${}_LH_i(a,b,c) = i!\sum_{k=0}^{[\frac{i}{2}]} \frac{c^k L_{i-2k}(a,b)}{k!\,(i-2k)!}, \tag{5}$$

where $L_i(a,b)$ are 2-variable Laguerre polynomials, defined as [2]:

$$L_i(a,b) = i!\sum_{j=0}^{i} \frac{(-1)^j a^j b^{i-j}}{(j!)^2(i-j)!} \tag{6}$$

and satisfy the following property:

$$L_i(a,0) = \frac{(-1)^i a^i}{i!}. \tag{7}$$

The 3VLHP ${}_LH_i(a,b,c)$ satisfies the following differential equation:

$$\left(2c\frac{\partial^2}{\partial b^2} + (b-a)\frac{\partial}{\partial b} - i\right) {}_LH_i(a,b,c) = 0 \tag{8}$$

* Corresponding author e-mail: serkan.araci@hku.edu.tr

and possesses the differential recurrence relation:

$$\frac{\partial}{\partial b}{}_LH_i(a,b,c) = i\,{}_LH_{i-1}(a,b,c). \tag{9}$$

Remark 1.1. For $c = -\frac{1}{2}$, the 3VLHP ${}_LH_i(a,b,c)$ becomes the 2-variable Laguerre-Hermite polynomials (2VLHP) ${}_LH_i^*(a,b)$, defined as [3]:

$$G(a,b,c) = C_0(aw)\exp\left(bw - \frac{1}{2}w^2\right) = \sum_{i=0}^{\infty} {}_LH_i^*(a,b)\frac{w^i}{i!}, \tag{10}$$

The topic on differential and integral equations is a captivating area of research in various fields of science and engineering. The differential equations and other characterizations associated with Appell and q-Appell polynomials are considered in [4,5,6,7]. Using generating functions, linear and non-linear differential equations of special polynomials and numbers are derived, see [8,9,10,11,12]. Many authors have investigated the properties of 3VLHP ${}_LH_i(a,b,c)$ in numerous aspects, see for example [13,14]. Motivated by the works on differential and integral equations for the hybrid polynomials, the differential and integral equations for the 3VLHP are derived in this paper. The properties of the 3VLHP are explored using graphical representations.

2 Main results

In this section, using generating function (4), we derive the differential equations for 3VLHP ${}_LH_i(a,b,c)$.

Theorem 2.1. For each $M \in \mathbb{N}$, the following differential equation:

$$H^{(M)}(a,b,c,w) = \sum_{m=0}^{M} \alpha_m(M,a,b,c)w^m H(a,b,c,w), \tag{11}$$

has a solution $H = H(a,b,c,w) = \exp((b-D_a^{-1})w + cw^2)$, where $H^{(M)}(a,b,c,w) = \frac{d^M H(a,b,c,w)}{dw^M}$ and

$$\alpha_0(M+1,a,b,c) = \sum_{m=0}^{M}(b-D_a^{-1})^m \alpha_1(M-m,a,b,c) + (b-D_a^{-1})^{M+1}, \tag{12}$$

$$\alpha_{M-1}(M+1,a,b,c) = \sum_{m=0}^{M-1}(M-m)(2c)^m \alpha_{M-m}(M-m,a,b,c) + (b-D_a^{-1})\sum_{m=1}^{M}(2c)^{m-1}\alpha_{M-m}(M-m+1,a,b,c), \tag{13}$$

$$\alpha_M(M+1,a,b,c) = (b-D_a^{-1})\sum_{m=0}^{M}(2c)^m \alpha_{M-m}(M-m,a,b,c), \tag{14}$$

$$\alpha_{M+1}(M+1,a,b,c) = (2c)^{M+1} \tag{15}$$

and

$$\alpha_k(M+1,a,b,c) = (k+1)\sum_{m=0}^{M-k}(b-D_a^{-1})^m \alpha_{k+1}(M-m,a,b,c) + 2c\sum_{m=0}^{M-k+1}(b-D_a^{-1})^m \alpha_{k-1}(M-m,a,b,c), \quad 1 \le k \le M-2. \tag{16}$$

Proof. Differentiation of

$$H = H(a,b,c,w) = \exp((b-D_a^{-1})w + cw^2), \tag{17}$$

with respect to w gives

$$H^{(1)} = \frac{\partial}{\partial w}H(a,b,c,w) = ((b-D_a^{-1}) + 2cw)H. \tag{18}$$

Furthermore, differentiating the above-mentioned equation with respect to w, it gives

$$H^{(2)} = \left((2c + (b-D_a^{-1})^2) + 4c(b-D_a^{-1})w + 4c^2w^2\right)H. \tag{19}$$

Processing in the same manner up to M times, assertion (11) is obtained.

To find the coefficients $\alpha_k(M+1)$, differentiating equation (11) with respect to w, we get

$$H^{(M+1)} = \sum_{m=0}^{M-1}(m+1)w^m \alpha_{m+1}(M,a,b,c)H + (b-D_a^{-1})\sum_{m=0}^{M}\alpha_m(M,a,b,c)w^m H + 2c\sum_{m=1}^{M+1}\alpha_{M-m}(M,a,b,c)w^m H, \tag{20}$$

which, on replacing M by $M+1$, becomes

$$H^{(M+1)} = \sum_{m=0}^{M+1}\alpha_m(M+1,a,b,c)w^m H. \tag{21}$$

Equating the coefficients on both sides of equations (20) and (21), it follows that

$$\alpha_0(M+1,a,b,c) = \alpha_1(M,a,b,c) + (b-D_a^{-1})\alpha_0(M,a,b,c), \tag{22}$$

$$\alpha_1(M+1,a,b,c) = 2\alpha_2(M,a,b,c) + (b-D_a^{-1})\alpha_1(M,a,b,c) + 2c\alpha_0(M,a,b,c), \tag{23}$$

$$\alpha_{M-1}(M+1,a,b,c) = M\alpha_M(M,a,b,c) + (b-D_a^{-1})\alpha_{M-1}(M,a,b,c) + 2c\alpha_{M-2}(M,a,b,c), \tag{24}$$

$$\alpha_M(M+1,a,b,c) = (b-D_a^{-1})\alpha_M(M,a,b,c) + 2c\alpha_{M-1}(M,a,b,c), \tag{25}$$

$$\alpha_{M+1}(M+1,a,b,c) = 2c\alpha_M(M,a,b,c) \tag{26}$$

and

$$\begin{aligned}&\alpha_m(M+1,a,b,c)\\&= (m+1)\alpha_{m+1}(M,a,b,c) + (b-D_a^{-1})\alpha_m(M,a,b,c)\\&+2c\alpha_{m-1}(M,a,b,c), \qquad 2\le m\le M-2.\end{aligned} \tag{27}$$

In view of equations (11), (17) and (18), we find

$$\alpha_0(0,a,b,c)=1, \quad \alpha_0(1,a,b,c)=b-D_a^{-1} \text{ and } \alpha_1(1,a,b,c)=2c. \tag{28}$$

Moreover, in view of equations (11) and (19), we find

$$\begin{aligned}&\alpha_0(2,a,b,c) = 2c+(b-D_a^{-1})^2,\\&\alpha_1(2,a,b,c) = 4c(b-D_a^{-1}) \text{ and}\\&\alpha_2(2,a,b,c)=4c^2.\end{aligned} \tag{29}$$

Equation (22) gives

$$\begin{aligned}&\alpha_0(M+1,a,b,c)\\&=\alpha_1(M,a,b,c)+(b-D_a^{-1})\alpha_1(M-1,a,b,c)\\&+(b-D_a^{-1})^2\alpha_0(M-1,a,b,c)\\&=\alpha_1(M,a,b,c)+(b-D_a^{-1})\alpha_1(M-1,a,b,c)\\&+(b-D_a^{-1})^2\alpha_1(M-2,a,b,c)\\&+(b-D_a^{-1})^3\alpha_0(M-2,a,b,c)\\&=\cdots\\&=\sum_{m=0}^{M}(b-D_a^{-1})^m\alpha_1(M-m,a,b,c)\\&+(b-D_a^{-1})^M\alpha_0(1),\end{aligned}$$

which, on using relation (28), yields assertion (12).

From equation (24), we get

$$\begin{aligned}&\alpha_{M-1}(M+1,a,b,c)\\&=M\alpha_M(M,a,b,c)+(b-D_a^{-1})\alpha_{M-1}(M,a,b,c)\\&+2c\left((M-1)\alpha_{M-1}(M-1,a,b,c)+(b-D_a^{-1})\right.\\&\left.\alpha_{M-2}(M-1,a,b,c)+2c\alpha_{M-3}(M-1,a,b,c)\right)\\&=M\alpha_M(M,a,b,c)+2c(M-1)\alpha_{M-1}(M-1,a,b,c)\\&+(b-D_a^{-1})(\alpha_{M-1}(M,a,b,c)+2c\alpha_{M-2}(M-1,a,b,c))\\&+(2c)^2\alpha_{M-3}(M-1,a,b,c)\\&=\cdots\\&=\sum_{m=0}^{M-2}(M-m)(2c)^m\alpha_{M-m}(M-m,a,b,c)\\&+(b-D_a^{-1})\sum_{m=1}^{M-1}\alpha_{M-m}(M-m+1,a,b,c)\\&(2c)^{m-1}+(2c)^{M-1}\alpha_0(2),\end{aligned}$$

which, on using equations (28) and (29) and simplifying, becomes

$$\begin{aligned}&\alpha_{M-1}(M+1,a,b,c)\\&=\sum_{m=0}^{M-2}(M-m)(2c)^m\alpha_{M-m}(M-m,a,b,c)+(2c)^M\\&+(b-D_a^{-1})\sum_{m=1}^{M}(2c)^{m-1}\alpha_{M-m}(M-m+1,a,b,c),\end{aligned} \tag{30}$$

which, on rewriting $(2c)^M=(2c)^{M-1}\alpha_1(1,a,b,c)$, yields assertion (13).

From equation (25), we find

$$\begin{aligned}&\alpha_M(M+1,a,b,c)\\&=(b-D_a^{-1})\ \alpha_M(M,a,b,c)\\&+2c\left((b-D_a^{-1})\ \alpha_{M-1}(M-1,a,b,c)\right.\\&\left.+2c\ \alpha_{M-2}(M-1,a,b,c)\right)\\&=(b-D_a^{-1})\left(\alpha_M(M,a,b,c)+2c\ \alpha_{M-1}(M-1,a,b,c)\right.\\&\left.+(2c)^2\alpha_{M-2}(M-2,a,b,c)+(2c)^3\alpha_{M-3}(M-2,a,b,c)\right)\\&=\cdots\\&=(b-D_a^{-1})\sum_{m=0}^{M-1}(2c)^m\alpha_{M-m}(M-m,a,b,c)\\&+(2c)^M\alpha_0(1,a,b,c),\end{aligned}$$

which, on using relation (28), gives assertion (14).

From equation (26), we get

$$\begin{aligned}\alpha_{M+1}(M+1,a,b,c)&=2c\ \alpha_M(M,a,b,c)\\&=(2c)^2\alpha_{M-1}(M-1,a,b,c)\\&=\cdots\\&=(2c)^{M+1},\end{aligned}$$

which proves assertion (15).

From equation (23), we find

$$\begin{aligned}
&\alpha_1(M+1,a,b,c)\\
&=2\big(\alpha_2(M,a,b,c)+(b-D_a^{-1})\alpha_2(M-1,a,b,c)\big)\\
&\quad+2c\big(\alpha_0(M,a,b,c)+(b-D_a^{-1})\ \alpha_0(M-1,a,b,c)\big)\\
&\quad+(b-D_a^{-1})^2\alpha_1(M-1,a,b,c)\\
&=2\big(\alpha_2(M,a,b,c)+(b-D_a^{-1})\alpha_2(M-1,a,b,c)\\
&\quad+(b-D_a^{-1})^2\alpha_2(M-2,a,b,c)\big)+2c\big(\alpha_0(M,a,b,c)\\
&\quad+(b-D_a^{-1})\ \alpha_0(M-1,a,b,c)+(b-D_a^{-1})^2\\
&\quad\times\alpha_0(M-2,a,b,c)\big)+(b-D_a^{-1})^3\alpha_1(M-2,a,b,c)\\
&=\cdots\\
&=2\sum_{m=0}^{M-1}(b-D_a^{-1})^m\alpha_2(M-m,a,b,c)+2c\sum_{m=0}^{M-1}(b-D_a^{-1})^m\\
&\quad\times\alpha_0(M-m,a,b,c)+(b-D_a^{-1})^M\alpha_1(1,a,b,c),
\end{aligned}$$

which, on using relation (28), gives

$$\begin{aligned}
\alpha_1(M+1,a,b,c)&=2\sum_{m=0}^{M-1}(b-D_a^{-1})^m\alpha_2(M-m,a,b,c)\\
&\quad+2c\sum_{m=0}^{M}(b-D_a^{-1})^m\alpha_0(M-m,a,b,c).
\end{aligned} \tag{31}$$

Taking $m=2$ in equation (27), we find

$$\begin{aligned}
&\alpha_2(M+1,a,b,c)\\
&=3\alpha_3(M,a,b,c)+(b-D_a^{-1})\ \alpha_2(M,a,b,c)+2c\ \alpha_1(M,a,b,c)\\
&=3\big(\alpha_3(M,a,b,c)+(b-D_a^{-1})\ \alpha_3(M-1,a,b,c)\big)\\
&\quad+2c\big(\alpha_1(M,a,b,c)+(b-D_a^{-1})\ \alpha_1(M-1,a,b,c)\big)\\
&\quad+(b-D_a^{-1})^2\alpha_2(M-1,a,b,c)\big)\\
&=\cdots\\
&=3\sum_{m=0}^{M-2}(b-D_a^{-1})^m\alpha_3(M-m,a,b,c)\\
&\quad+2c\sum_{m=0}^{M-2}(b-D_a^{-1})^m\alpha_1(M-m,a,b,c)\\
&\quad+(b-D_a^{-1})^{M-1}\alpha_2(2,a,b,c),
\end{aligned}$$

which, in view of equations (28) and (29), becomes

$$\begin{aligned}
\alpha_2(M+1,a,b,c)&=3\sum_{m=0}^{M-2}(b-D_a^{-1})^m\alpha_3(M-m,a,b,c)\\
&\quad+2c\sum_{m=0}^{M-1}(b-D_a^{-1})^m\alpha_1(M-m,a,b,c).
\end{aligned} \tag{32}$$

Consequently, for $m=3$ in equation (27), we find

$$\begin{aligned}
\alpha_3(M+1,a,b,c)&=4\sum_{m=0}^{M-3}(b-D_a^{-1})^m\alpha_4(M-m,a,b,c)\\
&\quad+2c\sum_{m=0}^{M-2}(b-D_a^{-1})^m\alpha_2(M-m,a,b,c).
\end{aligned} \tag{33}$$

Continuing the same process, we get for $1\le k\le M-2$

$$\begin{aligned}
&\alpha_k(M+1,a,b,c)\\
&\quad=(k+1)\sum_{m=0}^{M-k}(b-D_a^{-1})^m\alpha_{k+1}(M-m,a,b,c)\\
&\quad+2c\sum_{m=0}^{M-k+1}(b-D_a^{-1})^m\alpha_{k-1}(M-m,a,b,c),
\end{aligned}$$

which gives assertion (16).

In view of Remark 1.1, the following consequence is deduced:

Corollary 2.1. For each $M\in\mathbb{N}$, the following differential equation

$$G^{(M)}(a,b,w)=\sum_{m=0}^{M}\beta_m(M,a,b)w^mG(a,b,w), \tag{34}$$

has a solution $G=G(a,b,w)=\exp((b-D_a^{-1})w-\frac{1}{2}w^2)$, where $G^{(M)}(a,b,w)=\frac{d^MG(a,b,w)}{dw^M}$ and

$$\begin{aligned}
&\beta_0(M+1,a,b)\\
&\quad=\sum_{m=0}^{M}(b-D_a^{-1})^m\beta_1(M-m,a,b)+(b-D_a^{-1})^{M+1},
\end{aligned} \tag{35}$$

$$\begin{aligned}
\beta_{M-1}(M+1,a,b)&=\sum_{m=0}^{M-1}(M-m)(-1)^m\beta_{M-m}(M-m,a,b)\\
&\quad+(b-D_a^{-1})\sum_{m=1}^{M}(-1)^{m-1}\beta_{M-m}(M-m+1,a,b),
\end{aligned} \tag{36}$$

$$\beta_M(M+1,a,b)=(b-D_a^{-1})\sum_{m=0}^{M}(-1)^m\beta_{M-m}(M-m,a,b), \tag{37}$$

$$\beta_{M+1}(M+1,a,b)=(-1)^{M+1} \tag{38}$$

and

$$\begin{aligned}
\beta_k(M+1,a,b)&=(k+1)\sum_{m=0}^{M-k}(b-D_a^{-1})^m\beta_{k+1}(M-m,a,b)\\
&\quad-\sum_{m=0}^{M-k+1}(b-D_a^{-1})^m\beta_{k-1}(M-m,a,b),\quad 1\le k\le M-2.
\end{aligned} \tag{39}$$

Next, we find out the integral equation for 3VLHP ${}_LH_n(x,y,z)$. Integral equations come out in various problems of engineering and science. The integral equation for the hybrid polynomials was investigated by several researchers, see [15,16,17]. Numerous science and engineering problems are expressible as differential and integral equations, which have special functions as their solutions.

Rewrite differential equation (8) as:

$$2c\frac{\partial^2}{\partial b^2}{}_LH_i(a,b,c)+(b-a)\frac{\partial}{\partial b}{}_LH_i(a,b,c) - i\,{}_LH_i(a,b,c)=0. \tag{40}$$

In view of equations (5), (7) and (9), it follows that

$${}_LH_i(a,0,c)=i!\sum_{k=0}^{[\frac{i}{2}]}\frac{c^k(-a)^{i-2k}}{k!\,((i-2k)!)^2}:=\mathbb{P}_i(a,c), \tag{41}$$

$$\frac{\partial}{\partial b}{}_LH_i(a,0,c)=i!\sum_{k=0}^{[\frac{i-1}{2}]}\frac{c^k(-a)^{i-2k-1}}{k!\,((i-2k-1)!)^2}:=\mathbb{R}_i(a,c). \tag{42}$$

Using relation (9), it follows that

$$\frac{\partial^2}{\partial b^2}{}_LH_i(a,b,c)=i(i-1)\,{}_LH_{i-2}(a,b,c). \tag{43}$$

Integrating equation (43) and using initial condition (42), we have

$$\frac{\partial}{\partial b}{}_LH_i(a,b,c)=\int_0^b i(i-1)\,{}_LH_{i-2}(a,\xi,c)d\xi+\mathbb{R}_i(a,c), \tag{44}$$

which, on further integrating and in view of initial condition (41), gives

$${}_LH_i(a,b,c)=\int_0^b i(i-1)\,(b-\xi)\,{}_LH_{i-2}(a,\xi,c)d\xi + b\mathbb{R}_i(a,c)+\mathbb{P}_i(a,c). \tag{45}$$

Using equations (43) – (45) in differential equation (40), we get

$$2ci(i-1)\,{}_LH_{i-2}(a,b,c)+i(i-1)(b-a)\int_0^b {}_LH_{i-2}(a,\xi,c)d\xi + \mathbb{R}_i(a,c)-i^2(i-1)\int_0^b(b-\xi)\,{}_LH_{i-2}(a,\xi,c)d\xi + b\mathbb{R}_i(a,c)+\mathbb{P}_i(a,c)=0,$$

which, on replacing i by $i+2$, gives the following integral equation for the 3VLHP ${}_LH_i(a,b,c)$:

$$(i+2)(i+1)\Big(2c\,{}_LH_i(a,b,c)+\int_0^b\big((b-a) - (i+2)(b-\xi)\big)\,{}_LH_i(a,\xi,c)d\xi\Big) + (b+1)\mathbb{R}_{i+2}(a,c)+\mathbb{P}_{i+2}(a,c)=0.$$

Remark 2.1.

Taking $c=-\frac{1}{2}$ and proceeding similarly as above, the integral equation associated with 2-variable Laguerre-Hermite polynomials ${}_LH_i^*(a,b)$ can be obtained.

In the forthcoming section, we use Mathematica to draw the surface plots and curves of the 3VLHP ${}_LH_i(a,b,c)$. In addition, zeros of the polynomials are explored to be located in a remarkably symmetrical way.

3 Graphical approach

This section aims to present the benefit of using graphical and numerical aspects to support theoretical prediction and to discover new interesting pattern of the zeros of the polynomials. The softwares Mathematica is used to show the behaviour of the polynomials by plotting the graphs for special values of indices. The investigation in this manner will provide a new approach to examine several interesting properties of hybrid special polynomials, see [18,6].

Using relations (5) and (6), we get the following explicit series representation of 3VLHP ${}_LH_i(a,b,c)$:

$${}_LH_i(a,b,c)=i!\sum_{k=0}^{[\frac{i}{2}]}\sum_{j=0}^{i-2k}\frac{(-1)^j a^j b^{i-2k-j}c^k}{k!\,(j!)^2\,(i-2k-j)!}. \tag{46}$$

For $i=30$ and $c=2$, we display the surface and contour plot of 3VLHP ${}_LH_i(a,b,c)$, see Figure 1 and 2.

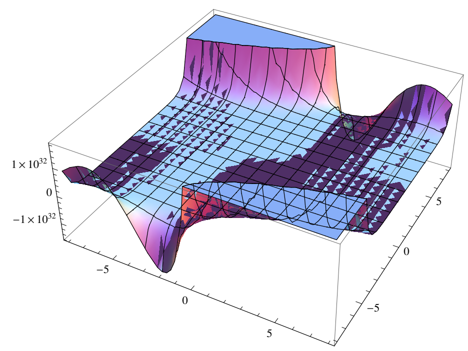

Figure 1: Surface plot of ${}_LH_{30}(a,b,2)$

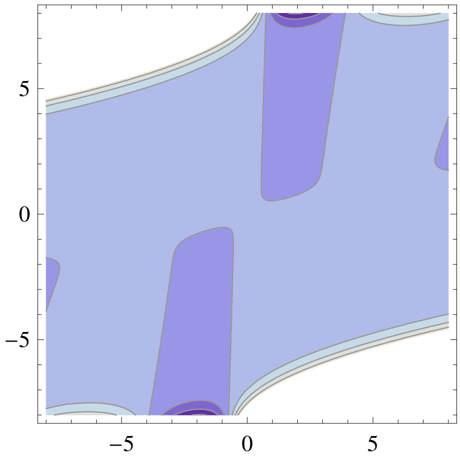

Figure 2: Contour plot of ${}_LH_{30}(a,b,2)$

For $i = 30$ and $c = -2$, the surface and contour plot of 3VLHP ${}_LH_i(a,b,c)$ are drawn, see Figure 3 and 4.

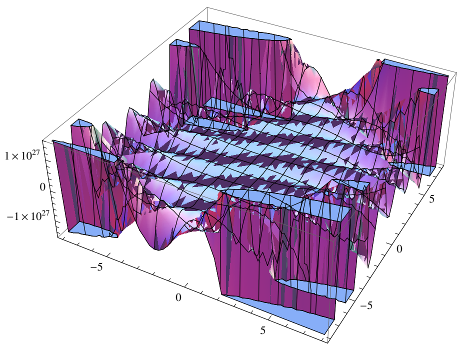

Figure 3: Surface plot of ${}_LH_{30}(a,b,-2)$

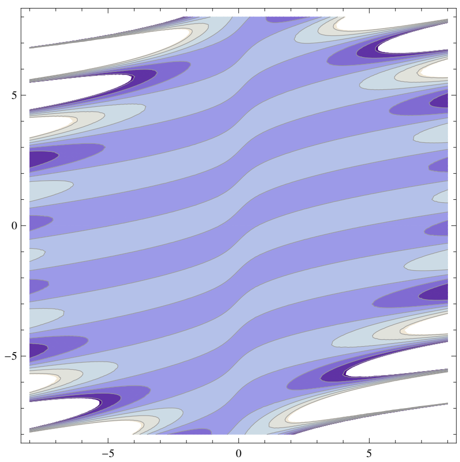

Figure 4: Contour plot of ${}_LH_{30}(a,b,-2)$

These graphs show the distinct behaviour of the 3VLHP ${}_LH_i(a,b,c)$. Both contour and surface plots help visualize the response surface. A surface plot displays a 3-dimensional view of the surface defined by a function of two variables, while a contour plot provides a 2-dimensional view of the surface.

Next, for $i = 30$ and $b = c = 2$, the graph of ${}_LH_i(a)$ is drawn, see Figure 5. For $i = 30$ and $b = 2; c = -2$, the graph of ${}_LH_i(a)$ is drawn, see Figure 6.

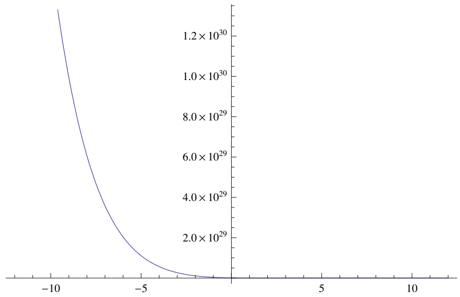

Figure 5: Curve of ${}_LH_{30}(a,2,2)$

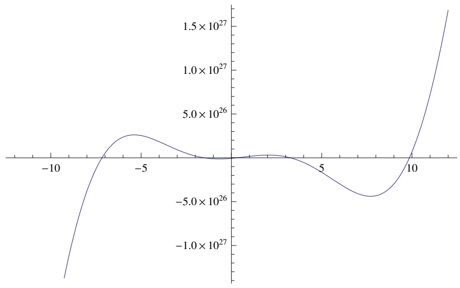

Figure 6: Curve of ${}_LH_{30}(a,2,-2)$

Using Mathematica, the real and complex roots of the special polynomials related to 3VLHP ${}_LH_i(a,b,c)$ are explored. The distribution of the zeros of ${}_LH_i(a)$ are observed in the complex plane. For $i = 30$ and $b = c = 2$, zeros of ${}_LH_i(a)$ are drawn, see Figure 7 and for $i = 30$ and $b = c = -2$, zeros of ${}_LH_i(a)$ are plotted, see Figure 8.

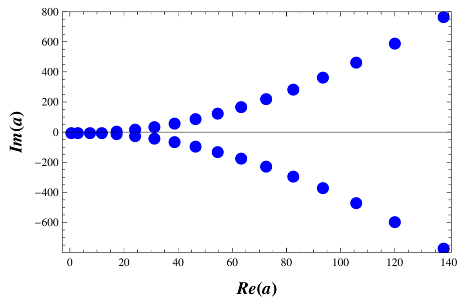

Figure 7: Zeros of ${}_LH_{30}(a,2,2)$

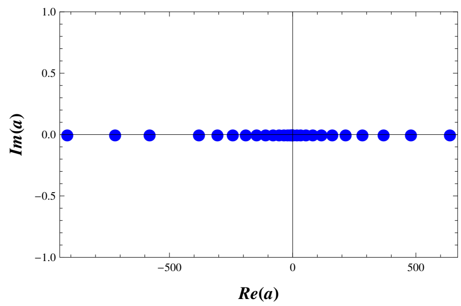

Figure 8: Zeros of ${}_LH_{30}(a,-2,-2)$

Similarly, for $i = 20$ and $b = c = 2$, zeros of ${}_LH_i(a)$ are drawn, see Figure 9 and for $i = 20$ and $b = c = -2$, zeros of ${}_LH_i(a)$ are plotted, see Figure 10.

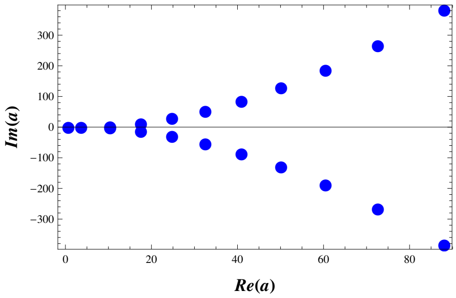

Figure 9: Zeros of $_LH_{20}(a,2,2)$

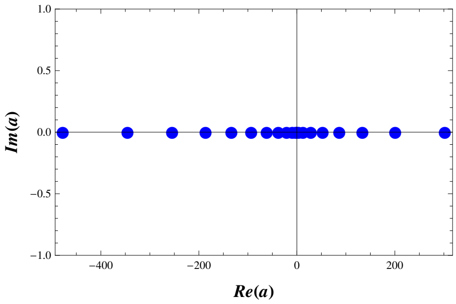

Figure 10: Zeros of $_LH_{20}(a,-2,-2)$

The investigation regarding surface plots, contour plots, shapes and scattering of zeros of the special polynomials will be tremendously beneficial for the researchers to understand the behaviour of the polynomials and location of their zeros in the complex plane.

Conflict of Interest. The authors declare that they have no conflict of interest.

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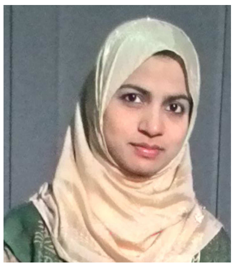

Tabinda Nahid received her Ph.D. degree in Mathematics from Aligarh Muslim University, India in the year 2019. Her research interest include special functions, operational methods, fractional calculus, differential equations and quantum calculus. She has published around 13 research papers in the international journals.

Serkan Araci was born in Hatay, Turkey, on October 1, 1988. He has published over than 150 papers in reputed international journals. His research interests include theory of special functions, p-adic analysis, Analytic Numbers theory, q-series and q-polynomials, etc. Currently, he works as an Assistant Professor at Hasan Kalyoncu University, Gaziantep, Turkey. On the other hand, Araci is an editor and a referee for several international journals. For details, one can visit the link: https://www.researchgate.net/profile/Serkan_Araci