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Some Generalizations of Opial Type Inequalities

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Abstract: In this study, we establish some new n-th order integral inequalities of Opial type for differentiable functions. Furthermore, we extend our study by examining more general type of differentiable functions. Finally, we see that our results can cover the previous published studies.

Keywords: Hölder's inequality, Opial inequality.

1 Introduction

The differential equations with impulse perturbations lie in a special important position in the theory of differential equations. For example, integral inequalities are one of the important tools that investigate the qualitative characteristics of solutions of different kinds of equations such as difference equations, differential equations, partial differential equations, and impulsive differential equations; for example see [1–9].

Opial [10] established the following well-known integral inequality:

Theorem 1.*Suppose $f \in C^1[0,\lambda]$ satisfies $f(0)=f(\lambda)=0$ and $f(x)>0$ for all $x\in(0,\lambda)$. Then the integral inequality holds*

$$\int_0^{\lambda} |f(x)f'(x)|dx \leq \frac{\lambda}{4}\int_0^{\lambda}(f'(x))^2dx, \tag{1}$$

where this constant $\frac{\lambda}{4}$ is best possible.

Recently, Opial's inequality (1) and its generalizations extensions and discretizations, have become an important tool in establishing the existence and uniqueness of initial and boundary value problems for ordinary, partial differential equations and difference equations. Moreover, there are many papers in the literature which deals with the simple proofs, various generalizations and discrete analogues of Opial inequality and its generalizations for one and two independent variables functions [11, 12]. Let us recall the following interesting Opial type inequalities [13]:

Theorem 2.*Let $x:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be an absolutely continuous function such that $x'\in L_2[a,b]$.*

i) If $x(a)=x(b)=0$, then

$$\int_a^b |x(t)x'(t)|\,dt \leq \frac{b-a}{4}\int_a^b (x'(t))^2\,dt. \tag{2}$$

ii) If $x(a)=0\,(or\ x(b)=0)$, then

$$\int_a^b |x(t)x'(t)|\,dt \leq \frac{b-a}{2}\int_a^b (x'(t))^2\,dt. \tag{3}$$

Among the generaliztions of Opial's inequality (2) there exists a class of inequalities which instead of the first derivative involves the n-th derivative of the given function $x(t)$. The first result is due to Willett's paper published in 1968 [14]. In [15], Das improvements and further extensions which appeared one year later paved the way for many subsequent results of this type. For more details and applications, see [10–34].

The present paper aims to establish some new n-th order Opial type integral inequalities for differentiable functions by an extension of (2)–(3). Moreover, by taking some special cases, we obtain some results on Opial type inequalities which were obtained before in the literature. They provide some new estimates on such types of inequalities.

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2 The Main Results

First, we begin with the following main theorem:

Theorem 3. *Let $f,g\in C^{(n)}[a,b]$ such that $f^{(n)},g^{(n)}$ be absolutely continuous and $f^{(n)},g^{(n)}\in L_2[a,b]$. Then*

i. If $g^{(i)}(a)=0,\ i=0,1,2,...,n-1,\ n\geq 1$, then

$$\int_a^b \left|f^{(n)}(t)g(t)\right|dt \tag{4}$$
$$\leq \frac{1}{(n!)}\left(\int_a^b (t-a)^n\left|f^{(n)}(t)\right|^2dt\right)^{\frac{1}{2}} \times\left(\int_a^b (b-t)^n\left|g^{(n)}(t)\right|^2dt\right)^{\frac{1}{2}}$$
$$\leq \frac{1}{2(n!)}\left(\int_a^b (t-a)^n\left|f^{(n)}(t)\right|^2dt+\int_a^b (b-t)^n\left|g^{(n)}(t)\right|^2dt\right).$$

ii. If $g^{(i)}(b)=0,\ i=0,1,2,...,n-1,\ n\geq 1$, then

$$\int_a^b \left|f^{(n)}(t)g(t)\right|dt \tag{5}$$
$$\leq \frac{1}{(n!)}\left(\int_a^b (b-t)^n\left|f^{(n)}(t)\right|^2dt\right)^{\frac{1}{2}} \times\left(\int_a^b (t-a)^n\left|g^{(n)}(t)\right|^2dt\right)^{\frac{1}{2}}$$
$$\leq \frac{1}{2(n!)}\left(\int_a^b (b-t)^n\left|f^{(n)}(t)\right|^2dt+\int_a^b (t-a)^n\left|g^{(n)}(t)\right|^2dt\right).$$

iii. If $g^{(i)}(a)=g^{(i)}(b)=0,\ i=0,1,2,...,n-1,\ n\geq 1$. Then,

$$\int_a^b \left|f^{(n)}(t)g(t)\right|dt \tag{6}$$
$$\leq \frac{1}{(n!)}\left(\int_a^b P(t)\left|f^{(n)}(t)\right|^2dt\right)^{\frac{1}{2}} \times\left(\int_a^b Q(t)\left|g^{(n)}(t)\right|^2dt\right)^{\frac{1}{2}}$$
$$\leq \frac{1}{2(n!)}\left(\int_a^b P(t)\left|f^{(n)}(t)\right|^2dt+\int_a^b Q(t)\left|g^{(n)}(t)\right|^2dt\right),$$

where

$$P(t):=\begin{cases}(t-a)^n, a\leq t\leq \frac{a+b}{2}\\ (b-t)^n, \frac{a+b}{2}<t\leq b\end{cases},$$
$$Q(t):=\begin{cases}(t-\frac{a+b}{2})^n, a\leq t\leq \frac{a+b}{2}\\ (\frac{a+b}{2}-t)^n, \frac{a+b}{2}<t\leq b\end{cases}.$$

Proof.i. Since $g^{(i)}(a)=0$, we consider $g(t)=\frac{1}{(n-1)!}\int_a^t(t-s)^{n-1}g^{(n)}(s)ds$ for $t\in[a,b]$. Thus, we can write

$$\int_a^b \left|f^{(n)}(t)g(t)\right|dt$$
$$=\int_a^b (t-a)^{\frac{n}{2}}\left|f^{(n)}(t)\right|(t-a)^{-\frac{n}{2}} \times\left|\frac{1}{(n-1)!}\int_a^t(t-s)^{n-1}g^{(n)}(s)ds\right|dt.$$

Using Cauchy-Schwarz inequality, we have

$$\int_a^b \left|f^{(n)}(t)g(t)\right|dt$$
$$\leq\left(\int_a^b (t-a)^n\left|f^{(n)}(t)\right|^2dt\right)^{\frac{1}{2}} \times\left(\int_a^b (t-a)^{-n}\left|\frac{1}{(n-1)!}\int_a^t(t-s)^{n-1}g^{(n)}(s)ds\right|^2dt\right)^{\frac{1}{2}}$$
$$\leq \frac{1}{(n!)}\left(\int_a^b (t-a)^n\left|f^{(n)}(t)\right|^2dt\right)^{\frac{1}{2}} \times\left(\int_a^b (b-t)^n\left|g^{(n)}(t)\right|^2dt\right)^{\frac{1}{2}}.$$

The last parts are followed by the elementary inequality

$$\sqrt{\alpha\beta}\leq\frac{1}{2}(\alpha+\beta),\ \alpha,\beta\geq 0$$

we obtain that

$$\int_a^b \left|f^{(n)}(t)g(t)\right|dt$$
$$\leq \frac{1}{2(n!)}\left(\int_a^b (t-a)^n\left|f^{(n)}(t)\right|^2dt+\int_a^b (b-t)^n\left|g^{(n)}(t)\right|^2dt\right)$$

which proves inequality (4).

ii. Since $g^{(i)}(b)=0$, we consider $g(t)=-\frac{1}{(n-1)!}\int_t^b(s-t)^{n-1}g^{(n)}(s)ds$ for $t\in[a,b]$. Then, it follows that

$$\int_a^b \left|f^{(n)}(t)g(t)\right|dt$$
$$=\int_a^b (b-t)^{\frac{n}{2}}\left|f^{(n)}(t)\right|(b-t)^{-\frac{n}{2}} \times\left|\frac{1}{(n-1)!}\int_t^b(t-s)^{n-1}g^{(n)}(s)ds\right|dt.$$

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Using Cauchy-Schwarz inequality, to the right side again, we obtain

$$\int_a^b \left|f^{(n)}(t)\,g(t)\right|dt$$
$$\leq \frac{1}{(n!)}\left(\int_a^b (b-t)^n \left|f^{(n)}(t)\right|^2 dt\right)^{\frac{1}{2}}$$
$$\times\left(\int_a^b (t-a)^n \left|g^{(n)}(t)\right|^2 dt\right)^{\frac{1}{2}}.$$

Using arithmetic-geometric mean inequality, we have

$$\int_a^b \left|f^{(n)}(t)\,g(t)\right|dt$$
$$\leq \frac{1}{2(n!)}\left(\int_a^b (b-t)^n \left|f^{(n)}(t)\right|^2 dt\right.$$
$$\left.+\int_a^b (t-a)^n \left|g^{(n)}(t)\right|^2 dt\right)$$

which proves inequality (5).

iii. If we write the inequality (4) on the interval $\left[a,\frac{a+b}{2}\right]$, we have

$$\int_a^{\frac{a+b}{2}} \left|f^{(n)}(t)\,g(t)\right|dt \tag{7}$$
$$\leq \frac{1}{(n!)}\left(\int_a^{\frac{a+b}{2}} P(t)\left|f^{(n)}(t)\right|^2 dt\right)^{\frac{1}{2}}$$
$$\times\left(\int_a^{\frac{a+b}{2}} Q(t)\left|g^{(n)}(t)\right|^2 dt\right)^{\frac{1}{2}}$$

and if we write inequality (5) on the interval $\left[\frac{a+b}{2},b\right]$, we have

$$\int_{\frac{a+b}{2}}^b \left|f^{(n)}(t)\,g(t)\right|dt \tag{8}$$
$$\leq \frac{1}{(n!)}\left(\int_{\frac{a+b}{2}}^b P(t)\left|f^{(n)}(t)\right|^2 dt\right)^{\frac{1}{2}}$$
$$\times\left(\int_{\frac{a+b}{2}}^b Q(t)\left|g^{(n)}(t)\right|^2 dt\right)^{\frac{1}{2}}.$$

If we add inequalities (7) and (8), and using

$$\alpha\beta+\gamma\delta \leq (\alpha^2+\gamma^2)^{\frac{1}{2}}(\beta^2+\delta^2)^{\frac{1}{2}},\quad \alpha,\beta,\gamma,\delta\geq 0$$

as well as the elementary inequality

$$\sqrt{\alpha\beta}\leq\frac{1}{2}(\alpha+\beta),\quad \alpha,\beta\geq 0$$

we get

$$\int_a^b \left|f^{(n)}(t)\,g(t)\right|dt$$
$$\leq \frac{1}{(n!)}\left(\int_a^{\frac{a+b}{2}} P(t)\left|f^{(n)}(t)\right|^2 dt\right)^{\frac{1}{2}}$$
$$\times\left(\int_a^{\frac{a+b}{2}} Q(t)\left|g^{(n)}(t)\right|^2 dt\right)^{\frac{1}{2}}$$
$$+\frac{1}{(n!)}\left(\int_{\frac{a+b}{2}}^b P(t)\left|f^{(n)}(t)\right|^2 dt\right)^{\frac{1}{2}}$$
$$\times\left(\int_{\frac{a+b}{2}}^b Q(t)\left|g^{(n)}(t)\right|^2 dt\right)^{\frac{1}{2}}$$
$$\leq \frac{1}{2(n!)}\left(\int_a^b P(t)\left|f^{(n)}(t)\right|^2 dt+\int_a^b Q(t)\left|g^{(n)}(t)\right|^2 dt\right).$$

which completes the proof of the inequality (6).

*Remark.*For $n=1$, Theorem 3 reduces to Theorem 2 which is proved by Dragomir in [16].

Corollary 1.*Let $f\in C^{(n)}[a,b]$ such that $f^{(n)}$ be absolutely continuous and $f^{(n)}\in L_2[a,b]$. Then*

i. If $f^{(i)}(a)=0$(or $f^{(i)}(b)=0$), $i=0,1,2,...,n-1$, $n\geq 1$, then

$$\int_a^b \left|f^{(n)}(t)\,f(t)\right|dt \leq \frac{(b-a)^n}{2^n(n!)}\int_a^b \left|f^{(n)}(t)\right|^2 dt. \tag{9}$$

ii. If $f^{(i)}(a)=f^{(i)}(b)=0$, $i=0,1,2,...,n-1$, $n\geq 1$, then

$$\int_a^b \left|f^{(n)}(t)\,f(t)\right|dt \leq \frac{(b-a)^n}{2^{2n}(n!)}\int_a^b \left|f^{(n)}(t)\right|^2 dt. \tag{10}$$

Proof.i. If we take $g=f$ in Theorem3, we have

$$\int_a^b \left|f^{(n)}(t)\,f(t)\right|dt$$
$$\leq \frac{1}{2(n!)}\left(\int_a^b \left[(t-a)^n+(b-t)^n\right]\left|f^{(n)}(t)\right|^2 dt\right).$$

Thus, taking $r(t)=(t-a)^n+(b-t)^n$ for $t\in[a,b]$, it follows that $r'(t)=0$, then $\max r(t)=\frac{(b-a)^n}{2^{n-1}}$ which completes the proof.

ii. The statement (ii) is followed by (iii) of Theorem3 for $g=f$ and it is noticeable that

$$\int_a^b \left|f^{(n)}(t)\,f(t)\right|dt \leq \frac{1}{2(n!)}\int_a^b (P(t)+Q(t))\left|f^{(n)}(t)\right|^2 dt,$$

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where

$$P(t) := \left\{ \begin{array}{l} (t-a)^n, a \le t \le \frac{a+b}{2} \\ (b-t)^n, \frac{a+b}{2} < t \le b \end{array} \right\},$$

$$Q(t) := \left\{ \begin{array}{l} \left(\frac{a+b}{2}-t\right)^n, a \le t \le \frac{a+b}{2} \\ \left(t-\frac{a+b}{2}\right)^n, \frac{a+b}{2} < t \le b \end{array} \right\}.$$

Then, it follows that

$$(t-a)^n + \left(\frac{a+b}{2}-t\right)^n \le \frac{(b-a)^n}{2^{2n-1}} \text{for } t \in \left[a, \frac{a+b}{2}\right]$$

$$(b-t)^n + \left(t-\frac{a+b}{2}\right)^n \le \frac{(b-a)^n}{2^{2n-1}} \text{for } t \in \left[\frac{a+b}{2}, b\right].$$

Then, we have

$$\int_a^b \left| f^{(n)}(t) f(t) \right| dt$$
$$\le \frac{1}{2(n!)} \int_a^{\frac{a+b}{2}} \frac{(b-a)^n}{2^{2n-1}} \left| f^{(n)}(t) \right|^2 dt$$
$$+ \frac{1}{2(n!)} \int_{\frac{a+b}{2}}^b \frac{(b-a)^n}{2^{2n-1}} \left| f^{(n)}(t) \right|^2 dt$$

which completes the proof.

*Remark.*i) If we take $n=1$ in Corollary 1, the inequality (9) reduces to the inequality (3).

ii) If we take $n=1$ in Corollary 1, the inequality (10) reduces to the inequality (2).

*Remark.*In [15], Das proved the following Opial type inequality:

$$\int_a^b \left| f^{(n)}(t) f(t) \right| dt \le \frac{(b-a)^n}{2(n!)} \left(\frac{n}{2n-1}\right)^{\frac{1}{2}} \int_a^b \left| f^{(n)}(t) \right|^2 dt.$$

In Corollary 1, the constant of inequality (10) is better than that of the above-mentioned inequality, i.e

$$\frac{1}{2^{2n}(n!)} \le \frac{1}{2(n!)} \left(\frac{n}{2n-1}\right)^{\frac{1}{2}} \text{ for } n \ge 1.$$

Corollary 2.*Let $f \in C^{(n)}[a,b]$ such that $f^{(n)}$ is absolutely continuous and $f^{(n)} \in L_2[a,b]$ and $h \in L_2[a,b]$. Then*

$$\int_a^b |f(t) h(t)| dt$$
$$\le \frac{1}{(n!)} \left(\int_a^b P(t) \left| f^{(n)}(t) \right|^2 dt \right)^{\frac{1}{2}} \left(\int_a^b Q(t) |h(t)|^2 dt \right)^{\frac{1}{2}}$$
$$\le \frac{1}{2(n!)} \left(\int_a^b P(t) \left| f^{(n)}(t) \right|^2 dt + \int_a^b Q(t) |h(t)|^2 dt \right). \quad (11)$$

*Proof.*In (6), if we take $g(t) := \int_a^t \int_a^{t_1} ... \int_a^{t_{n-1}} h(s) ds dt_{n-1} ... dt_1$, $t \in [a,b]$, such that $g^{(n)}(t) = h(t)$, it follows that

$$\int_a^b \left| f^{(n)}(t) \int_a^t \int_a^{t_1} ... \int_a^{t_{n-1}} h(s) ds dt_{n-1} ... dt_1 \right| dt$$
$$\le \frac{1}{(n!)} \left(\int_a^b P(t) \left| f^{(n)}(t) \right|^2 dt \right)^{\frac{1}{2}} \left(\int_a^b Q(t) |h(t)|^2 dt \right)^{\frac{1}{2}}$$
$$\le \frac{1}{2(n!)} \left(\int_a^b P(t) \left| f^{(n)}(t) \right|^2 dt + \int_a^b Q(t) |h(t)|^2 dt \right) \quad (12)$$

Furthermore, by the modulus properties and using change order of the integration, and then using n times integrating by parts, we have

$$\int_a^b \left| f^{(n)}(t) \int_a^t \int_a^{t_1} ... \int_a^{t_{n-1}} h(s) ds dt_{n-1} ... dt_1 \right| dt \quad (13)$$
$$\ge \left| \int_a^b \int_a^t \int_a^{t_1} ... \int_a^{t_{n-1}} f^{(n)}(t) h(s) ds dt_{n-1} ... dt_1 dt \right|$$
$$= \left| \int_a^b h(s) \int_s^b \int_{t_{n-1}}^b ... \int_{t_1}^b f^{(n)}(t) dt dt_1 ... dt_{n-1} ds \right|$$
$$= \left| \int_a^b f(s) h(s) ds \right|.$$

Employing (12) and (13), we obtain the desired result (11).

Corollary 3.*Let $g \in C^{(n)}[a,b]$ such that $g^{(n)}, h$ are absolutely continuous. If $g^{(i)}(a) = g^{(i)}(b) = 0$, $i = 0,1,2,...,n-1$, $n \ge 1$, and $h, g^{(n)} \in L_2[a,b]$. Then*

$$\int_a^b |h(t) g(t)| dt$$
$$\le \frac{1}{(n!)} \left(\int_a^b P(t) |h(t)|^2 dt \right)^{\frac{1}{2}} \left(\int_a^b Q(t) \left| g^{(n)}(t) \right|^2 dt \right)^{\frac{1}{2}}$$
$$\le \frac{1}{2(n!)} \left(\int_a^b P(t) |h(t)|^2 dt + \int_a^b Q(t) \left| g^{(n)}(t) \right|^2 dt \right). \quad (14)$$

*Proof.*If we select f as below in (6),

$$f(t) = \int_a^t \int_a^{t_1} ... \int_a^{t_{n-1}} h(s) ds dt_{n-1} ... dt_1$$

$$f^{(n)}(t) = h(t).$$

Thus, the proof is clear.

Proposition 1.*Assume that $h \in C^{(n)}[a,b]$ such that $h^{(n)}$ is absolutely continuous with $h^{(n-1)}(a) = h^{(n-1)}(b) = 0$ for $n \ge 2$, and $h^{(n)} \in L_2[a,b]$. Let $w : [a,b] \longrightarrow \mathbb{C}$ such that $w \in L_2[a,b]$, then*

$$\left| \int_a^b \frac{w(t) + w(a+b-t)}{2} h(t) dt - \frac{h(a)+h(b)}{2} \int_a^b w(t) dt \right|$$
$$\le \frac{1}{2(n!)} \left(\int_a^b P(t) |w(t)|^2 dt \right)^{\frac{1}{2}} \quad (15)$$
$$\times \left(\int_a^b Q(t) \left| h^{(n)}(t) + (-1)^n h^{(n)}(a+b-t) \right|^2 dt \right)^{\frac{1}{2}}.$$

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Moreover, if w is symetrical namely $w(a+b-t)=w(t)$ for all $t\in[a,b]$, then

$$\left|\int_a^b w(t)h(t)dt-\frac{h(a)+h(b)}{2}\int_a^b w(t)dt\right| \tag{16}$$
$$\leq \frac{1}{2(n!)}\left(\int_a^b P(t)\left|w(t)\right|^2dt\right)^{\frac{1}{2}}$$
$$\left(\int_a^b Q(t)\left|h^{(n)}(t)+(-1)^n h^{(n)}(a+b-t)\right|^2dt\right)^{\frac{1}{2}}.$$

*Proof.*Define the function $g:[a,b]\longrightarrow\mathbb{C}$ as:

$$g(t)=\frac{h(t)+h(a+b-t)}{2}-\frac{h(a)+h(b)}{2},\ t\in[a,b]$$

so that

$$g^{(n)}(t)=\frac{h^{(n)}(t)+(-1)^n h^{(n)}(a+b-t)}{2}.$$

Thus, we have $g^{(i)}(a)=g^{(i)}(b)=0,\ i=0,1,2,...,n-1$, $n\geq 1$. If we select f as:

$$f(t)=\int_a^t\int_a^{t_1}...\int_a^{t_{n-1}} w(s)dsdt_{n-1}...dt_1,$$
$$f^{(n)}(t)=w(t).$$

Substituting this into (6), we get

$$\int_a^b\left|w(t)\left[\frac{h(t)+h(a+b-t)}{2}-\frac{h(a)+h(b)}{2}\right]\right|dt$$
$$\leq\frac{1}{(n!)}\left(\int_a^b P(t)\left|w(t)\right|^2dt\right)^{\frac{1}{2}}$$
$$\times\left(\int_a^b Q(t)\left|\frac{h^{(n)}(t)+(-1)^n h^{(n)}(a+b-t)}{2}\right|^2dt\right)^{\frac{1}{2}}$$
$$\leq\frac{1}{2(n!)}\left(\int_a^b P(t)\left|w(t)\right|^2dt\right)^{\frac{1}{2}}$$
$$\times\left(\int_a^b Q(t)\left|h^{(n)}(t)+(-1)^n h^{(n)}(a+b-t)\right|^2dt\right)^{\frac{1}{2}}.$$

Change the variable $u=a+b-t, t\in[a,b]$, then

$$\left|\int_a^b \frac{w(t)+w(a+b-t)}{2}h(t)dt-\frac{h(a)+h(b)}{2}\int_a^b w(t)dt\right| \tag{17}$$
$$=\left|\int_a^b w(t)h(t)dt-\frac{h(a)+h(b)}{2}\int_a^b w(t)dt\right|$$
$$\leq\frac{1}{2(n!)}\left(\int_a^b P(t)\left|w(t)\right|^2dt\right)^{\frac{1}{2}}$$
$$\times\left(\int_a^b Q(t)\left|h^{(n)}(t)+(-1)^n h^{(n)}(a+b-t)\right|^2dt\right)^{\frac{1}{2}}.$$

Since w is symmetrical, we get

$$\int_a^b w(t)h(a+b-t)dt=\int_a^b w(a+b-t)h(t)dt=\int_a^b w(t)h(t)dt.$$

Then by the modulus property, we have

$$\int_a^b\left|\left[\frac{h(t)+h(a+b-t)}{2}-\frac{h(a)+h(b)}{2}\right]w(t)dt\right| \tag{18}$$
$$\geq\left|\frac{1}{2}\int_a^b[w(t)h(t)+w(t)h(a+b-t)]dt-\frac{h(a)+h(b)}{2}\int_a^b w(t)dt\right|$$
$$=\left|\frac{1}{2}\int_a^b[w(t)h(t)+w(t)h(a+b-t)]dt-\frac{h(a)+h(b)}{2}\int_a^b w(t)dt\right|$$
$$=\left|\int_a^b w(t)h(t)dt-\frac{h(a)+h(b)}{2}\int_a^b w(t)dt\right|.$$

Using the inequalities (17) and (18), we get the desired result (15).

Corollary 4.*With the assumptions of Proposition 1 and if $h^{(2k+1)}$ is Lipschitzian with constant $L>0$, namely $\left|h^{(2k+1)}(t)-h^{(2k+1)}(s)\right|\leq L|t-s|$ for any $t,s\in[a,b]$, $k=0,1,..,n$, then*

$$\left|\int_a^b \frac{w(t)+w(a+b-t)}{2}h(t)dt-\frac{h(a)+h(b)}{2}\int_a^b w(t)dt\right|$$
$$\leq L\frac{\sqrt{2}(b-a)^n}{8(n!)}\left(\int_a^b P(t)\left|w(t)\right|^2dt\right)^{\frac{1}{2}}. \tag{19}$$

*Proof.*Consider the function $g:[a,b]\longrightarrow\mathbb{C}$ defined by

$$g(t)=\frac{h(t)+h(a+b-t)}{2}-\frac{h(a)+h(b)}{2},\ t\in[a,b],$$
$$g^{(n)}(t)=\frac{h^{(n)}(t)+(-1)^n h^{(n)}(a+b-t)}{2}.$$

Thus, we have $g^{(i)}(a)=g^{(i)}(b)=0,\ i=0,1,2,...,n-1$, $n\geq 1$. If we select f as below in (6),

$$f(t)=\int_a^t\int_a^{t_1}...\int_a^{t_{n-1}} w(s)dsdt_{n-1}...dt_1,$$
$$f^{(n)}(t)=w(t).$$

Then, we get

$$\int_a^b w(t)\left[\frac{h(t)+h(a+b-t)}{2}-\frac{h(a)+h(b)}{2}\right]dt \tag{20}$$
$$\leq\frac{1}{(n!)}\left(\int_a^b P(t)\left|w(t)\right|^2dt\right)^{\frac{1}{2}}$$
$$\times\left(\int_a^b Q(t)\frac{\left|h^{(2k+1)}(t)-h^{(2k+1)}(a+b-t)\right|^2}{2}dt\right)^{\frac{1}{2}}$$
$$=\frac{1}{2(n!)}\left(\int_a^b P(t)\left|w(t)\right|^2dt\right)^{\frac{1}{2}}$$
$$\times\left(\int_a^b Q(t)\left|h^{(2k+1)}(t)-h^{(2k+1)}(a+b-t)\right|^2dt\right)^{\frac{1}{2}}.$$

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Calculating the last integral of the inequality, we have

$$\left(\int_a^b Q(t)\left|h^{(2k+1)}(t)-h^{(2k+1)}(a+b-t)\right|^2 dt\right)^{\frac{1}{2}} \tag{21}$$
$$\leq \left(\int_a^b \left|\frac{a+b}{2}-t\right| L^2 |t-a-b+t)|^2 dt\right)^{\frac{1}{2}}$$
$$= \left(\int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2}-t\right) L^2(2t-a-b)^2 dt\right.$$
$$\left. + \int_{\frac{a+b}{2}}^b \left(t-\frac{a+b}{2}\right) L^2(2t-a-b)^2 dt\right)^{\frac{1}{2}}.$$

Change the variable $u=\frac{a+b}{2}-t$, we obtain

$$\left(\int_a^b Q(t)\left|h^{(2k+1)}(t)-h^{(2k+1)}(a+b-t)\right|^2 dt\right)^{\frac{1}{2}} \tag{22}$$
$$\leq \frac{\sqrt{2}}{4}(b-a)^2 L.$$

It follows that

$$\int_a^b w(t)\left[\frac{h(t)+h(a+b-t)}{2}-\frac{h(a)+h(b)}{2}\right] dt \tag{23}$$
$$\leq \frac{\sqrt{2}}{8(n!)}(b-a)^2 L\left(\int_a^b P(t)\,|w(t)|^2 dt\right)^{\frac{1}{2}},$$

which completes the proof.

Theorem 4(Fejer Inequality). *Consider the integral* $\int_a^b h(x)w(x)dx$, *where* h *is a convex function in the interval* (a,b) *and* w *is a positive function in the same interval such that*

$$w(x)=w(a+b-x),\ \textit{for any } x\in[a,b]$$

i.e., $y=w(x)$ *is a symmetric curve with respect to the straight line which contains the point* $\left(\frac{1}{2}(a+b),0\right)$ *and is normal to the* x*-axis. Under those conditions the following inequalities are valid:*

$$h\left(\frac{a+b}{2}\right)\leq \frac{1}{\int_a^b w(x)dt}\int_a^b h(x)w(x)dx \leq \frac{h(a)+h(b)}{2}. \tag{24}$$

If h *is concave on* (a,b), *then the inequalities reverse in* (24).

If $w\equiv 1$, (24) *becomes the well known Hermit Hadamard inequality*

$$h\left(\frac{a+b}{2}\right)\leq \frac{1}{(b-a)}\int_a^b h(x)dx \leq \frac{h(a)+h(b)}{2}. \tag{25}$$

We have the following reverse of Fejer inequality:

Corollary 5. *Let* $h:[a,b]\longrightarrow\mathbb{R}$ *be a convex function and* $w:[a,b]\longrightarrow(0,\infty)$ *be continuous, symetrical on* $[a,b]$ *and such that* $h^{(n-1)}(a)=h^{(n-1)}(b)=0$ *for* $n\geq 2$, *and* $h^{(n)}\in L_2[a,b]$. *Then*

$$0\leq \frac{h(a)+h(b)}{2}-\frac{1}{\int_a^b w(t)dt}\int_a^b h(t)w(t)dt$$
$$\leq \left(\int_a^b P(t)\,|w(t)|^2 dt\right)^{\frac{1}{2}} \tag{26}$$
$$\times\left(\int_a^b Q(t)\left|h^{(n)}(t)+(-1)^n h^{(n)}(a+b-t)\right|^2 dt\right)^{\frac{1}{2}}.$$

Furthermore, if $h^{(2k+1)}$ *is L-Lipschitzian* $k=0,1,..,n$, *then*

$$0\leq \frac{h(a)+h(b)}{2}-\frac{1}{\int_a^b w(t)dt}\int_a^b h(t)w(t)dt$$
$$\leq \frac{\sqrt{2}(b-a)^2}{8\,(n!)}L\left(\int_a^b P(t)\,|w(t)|^2 dt\right)^{\frac{1}{2}}. \tag{27}$$

Proof. Let us take the following two functions in (6),

$$f(t)=\int_a^t\int_a^{t_1}...\int_a^{t_{n-1}} w(s)ds dt_{n-1}...dt_1,$$
$$f^{(n)}(t)=w(t),$$
$$g(t)=\frac{h(t)+h(a+b-t)}{2}-\frac{h(a)+h(b)}{2},\ t\in[a,b]$$
$$g^{(n)}(t)=\frac{h^{(n)}(t)+(-1)^n h^{(n)}(a+b-t)}{2}.$$

Since w is symetrical and $h^{(n)}$ is L-Lipschitzian, we have

$$\left|\int_a^b w(t)\left[\frac{h(t)+h(a+b-t)}{2}-\frac{h(a)+h(b)}{2}\right] dt\right| \tag{28}$$
$$=\left|\int_a^b w(t)h(t)dt-\frac{h(a)+h(b)}{2}\int_a^b w(t)dt\right|$$
$$\leq \frac{1}{2\,(n!)}\left(\int_a^b P(t)\,|w(t)|^2 dt\right)^{\frac{1}{2}}$$
$$\times\left(\int_a^b Q(t)\left|h^{(n)}(t)+(-1)^n h^{(n)}(a+b-t)\right|^2 dt\right)^{\frac{1}{2}}.$$

Using the L-Lipschitzian of $h^{(2k+1)}, k=0,1,..,n$ as well as the Fejer inequality in (28), we get the desired results (26) and (27).

3 Conclusion

In this paper, we established some new n-th order Opial type integral inequalities for differentiable functions by an extension of (2)–(3). Moreover, taking some special cases, we obtained some results on Opial type inequalities which were obtained before in the literature. They provide some new estimates on such types of inequalities.

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Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this article

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